VII. CONCLUDING REMARK

In this article, an equation for the normalized density matrix is set up. If one reduces the equation, one would obtain a hierarchy of equations for the reduced density matrices. The solution of the set of equations would give us all the information we need. In this article, we have adopted a more practical use of the equation; that is, it is used to set up the hierarchy of equations for the thermal average of quantities with which the quantity of interest is expressed. The process has been illustrated in the example of calculating the spontaneous magnetization of the Heisenberg ferromagnet at low temperatures and high external fields. The calculation has been done by taking care not to neglect any term which may contribute to the order we are considering.

It is hoped that the method will be extended to higher temperatures and lower fields and become applicable even to the discussions of the critical phenomena, and also to other problems than ferromagnetism.

In the above analysis, we have treated nonlinear differential equations. It was the case even when we calculated the spontaneous magnetization in the ideal spin wave approximation in Sec. IV. This nonlinearity of the differential equations makes the solution difficult. However this difficulty may be said to be a technical one but not an essential one like the divergence difficulty in the method of the two-time Green's function.

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Electrodynamics of Superconducting Alloys in High Magnetic Fields*

KAZUMI MAKI†
The Enrico Fermi Institute for Nuclear Studies, The University of Chicago, Chicago, Illinois
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The surface impedance of the superconducting alloys in high magnetic fields is explicitly calculated by restricting consideration to the critical region where the order parameter is small. It is shown that we have an expression for the complex conductivity equivalent to that for superconductors containing paramagnetic impurity, as long as we are concerned with gapless region. A brief discussion is given on the complex conductivity in the weak-field region where the low-lying excitations associated with each flux line play a dominant role.

1. INTRODUCTION

In a series of papers[1] (which we shall refer to hereafter as I) we have investigated the equilibrium properties of superconducting alloys in strong magnetic fields. In the dirty limit where the electronic mean free path is much shorter than the coherence length and in a high magnetic field ($H_{c2}-H_0\ll H_{c2}$, where H_0 is the external field and H_{c2} is the upper critical field), we were able to solve the general Gor'kov equations by expanding them in powers of $\Delta(\mathbf{r})$, the position-dependent order parameter.

The technique developed there is so general that one can apply it to the calculation of the transport coefficients which describe the nonequilibrium properties of superconducting alloys in the critical regions. The purpose of this paper is to present the calculation of the electromagnetic conductivity of superconducting alloys in strong magnetic fields as an illustration of the method.

Using the formalism developed by Abrikosov and Gor'kov[2] (AG) we can express the complex conductivity in terms of two Green's functions. In the high-field region where the ordering parameter is small, we can expand these two Green's functions in powers of $\Delta(\mathbf{r})$ and calculate the conductivity explicitly. It turns out that the conductivity is essentially local and a function of $|\Delta(\mathbf{r})|^2$.

We shall consider some limiting cases of special interest. We also discuss briefly the conductivity in the weak-field region where the low-lying excitations associated with each flux line play the essential role in the dissipation mechanism of the system. In the following we adopt units such that $\hbar=k_B=c=1$.

2. EQUATION FOR CURRENT

In this section we shall calculate the current induced by an oscillating electromagnetic field which is superposed on a static magnetic field H_0. According to AG

* This work is supported by the U. S. Atomic Energy Commission.
† On leave of absence from Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan. Present address: Department of Physics, University of California, San Diego, La Jolla, California.

[1] K. Maki, Physics 1, 21 (1964); 1, 127 (1964).

[2] A. A. Abrikosov and L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. 35, 1558 (1958); 36, 319 (1958) [English transls.: Soviet Phys.—JETP 8, 1090 (1959); 9, 220 (1965)]. For a detailed exposition of the method see A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, in *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963).

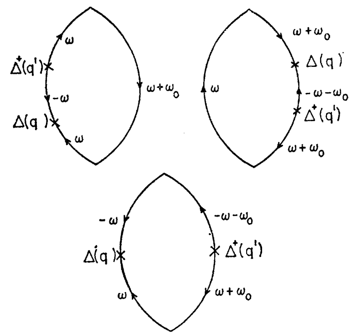

FIG. 1. Diagrams corresponding to Eq. (10).

the frequency-dependent current is given as

$$\mathbf{j}(\mathbf{r},\omega)=Q(\mathbf{r},\omega)\mathbf{A}(\mathbf{r},\omega)\,, \tag{1}$$

where $Q(\mathbf{r},\omega)$ is obtained through an analytical continuation of the quantity $Q'(\mathbf{r},\omega_0)$, where[2]

$$Q'(\mathbf{r},\omega_0)=\frac{Ne^2}{m}-\frac{2e^2}{m^2}2\pi T\sum_{n=-\infty}^{\infty}\int d^3r'\times\{\nabla_1 G_\omega(\mathbf{r},\mathbf{r}')\nabla_1 G_{\omega+\omega_0}(\mathbf{r}',\mathbf{r})-\nabla_1 F_\omega(\mathbf{r},\mathbf{r}')\nabla_1 F_{\omega+\omega_0}{}^\dagger(\mathbf{r}',\mathbf{r})\}\,. \tag{2}$$

Here $\omega=2\pi T(n+\frac{1}{2})$, where n is an integer, and T is the temperature.

The two Green's functions G_ω and F_ω are given by

$$G_\omega(\mathbf{r},\mathbf{r}')=i\int_0^{1/T}e^{i\omega\tau}\langle T_\tau\psi_\uparrow(\mathbf{r},0)\psi_\uparrow{}^\dagger(\mathbf{r},'\tau)\rangle d\omega\,,$$
$$F_\omega(\mathbf{r},\mathbf{r}')=i\int_0^{1/T}e^{i\omega\tau}\langle T_\tau\psi_\uparrow(\mathbf{r},0)\psi_\downarrow(\mathbf{r},'\tau)\rangle d\omega\,, \tag{3}$$

where $\langle T_\tau\cdots\rangle$ means the average of the time-ordered product over Gibbs' ensemble.

In the following we shall restrict ourselves to the critical region where H_0 is slightly smaller than the upper critical field H_{c2}. In this field region $\Delta(\mathbf{r})$ is small[1] and we expect we can calculate $Q'(\mathbf{r},\omega_0)$ by expanding it in powers of $\Delta(\mathbf{r})$. In order to carry out such a calculation, it is convenient to begin with the case of an impurity-free metal in the absence of an external field.

In this case the general Gor'kov equations are solved in powers of $\Delta(\mathbf{r})$ and we have[3]

$$G_\omega(\mathbf{r},\mathbf{r}')=G_\omega{}^0(\mathbf{r}-\mathbf{r}')-\int\int d^3s d^3l\, G_\omega{}^0(\mathbf{r}-\mathbf{s})\Delta(\mathbf{s})\times G_{-\omega}{}^0(\mathbf{s}-\mathbf{l})\Delta^+(\mathbf{l})G_\omega{}^0(\mathbf{l}-\mathbf{s})\,, \tag{4}$$

$$F_\omega(\mathbf{r},\mathbf{r}')=\int d^3s\, G_\omega{}^0(\mathbf{r}-\mathbf{s})\Delta(\mathbf{s})G_{-\omega}{}^0(\mathbf{s}-\mathbf{r}')\,,$$

where $G_\omega{}^0(\mathbf{r}-\mathbf{r}')$ is Green's function of the electron in the normal metal and is given by

$$G_\omega{}^0(\mathbf{r}-\mathbf{r}')=\frac{1}{(2\pi)^3}\int d^3p\frac{e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}}{i\omega-\xi}\,,\qquad \xi=p^2/2m-\mu\,, \tag{5}$$

and $\Delta(\mathbf{r})$ is given by

$$\Delta(\mathbf{r})=|g|\langle T_\tau\psi_\uparrow(\mathbf{r},t)\psi_\downarrow(\mathbf{r},t)\rangle\,. \tag{6}$$

Substituting Eq. (4) into Eq. (2) we find that

$$\begin{aligned}Q'(\mathbf{r},\omega_0)=&\left(\frac{Ne^2}{m}-\frac{2e^2}{m^2}2\pi T\sum_n\int d^3r'\nabla_1 G_\omega{}^0(\mathbf{r}-\mathbf{r}')\cdot\nabla_1 G_{-\omega}{}^0(\mathbf{r}-\mathbf{r}')\right)\\&+\frac{2e^2}{m^2}2\pi T\sum_n\int\int\int d^3r'd^3sd^3l\{\nabla_1 G_\omega{}^0(\mathbf{r}-\mathbf{s})\Delta(\mathbf{s})G_{-\omega}{}^0(\mathbf{s}-\mathbf{l})\\&\times\Delta^+(\mathbf{l})G_\omega{}^0(\mathbf{l}-\mathbf{r}')\nabla_1 G_{\omega+\omega_0}(\mathbf{r}-\mathbf{r}')+\nabla_1 G_\omega{}^0(\mathbf{r}-\mathbf{r}')\nabla_1 G_{\omega+\omega_0}{}^0(\mathbf{r}-\mathbf{s})\Delta(\mathbf{s})G_{-\omega-\omega_0}{}^0(\mathbf{s}-\mathbf{l})\Delta^+(\mathbf{l})G_{\omega+\omega_0}{}^0(\mathbf{l}-\mathbf{r}')\\&-\nabla_1 G_\omega{}^0(\mathbf{r}-\mathbf{s})\Delta(\mathbf{s})G_{-\omega}{}^0(\mathbf{s}-\mathbf{r}')\nabla_1 G_{\omega+\omega_0}{}^0(\mathbf{l}-\mathbf{r})\Delta^+(\mathbf{l})G_{-\omega-\omega_0}{}^0(\mathbf{r}'-\mathbf{l})\}\,.\end{aligned} \tag{7}$$

In the above expression the terms in the first bracket give the normal conductivity. We draw the diagrams corresponding to the remaining terms in Eq. (7) in Fig. 1.

In the presence of impurity atoms the electron suffers scattering from these atoms. Such effects are easily introduced into the theory by a simple renormalization procedure; one makes the following set of replacements[1]:

$$\omega\to\tilde{\omega}=\eta_\omega\omega=(1+1/2\tau|\omega|)\omega\,,$$
$$\Delta_\mathbf{q}\to\tilde{\Delta}_\mathbf{q}=\eta_{\omega q}\Delta_\mathbf{q}=\left\{1-\frac{1}{2\tau|\tilde{\omega}|}\left(1-\frac{\tau\tau_{\mathrm{tr}}}{3}v^2\mathbf{q}^2\right)\right\}^{-1}\Delta_\mathbf{q}\,. \tag{8}$$

Here τ is the lifetime of the electron, v is the Fermi velocity, τ_{tr} is the transport lifetime, and $\mathbf{q}$ is the external momentum associated with each vertex.

[3] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. 36, 1918 (1959); 37, 1407 (1959) [English transls.: Soviet Phys.—JETP 9, 1364 (1959); 10, 998 (1960)].

Similarly we take account of the effect of the external field by replacing $\mathbf{q}$ by $\mathbf{q}\pm 2e\mathbf{A}$, depending upon whether it operates on Δ or Δ^+, where $\mathbf{A}$ is the vector potential. After these replacements we obtain

$$Q'(\mathbf{r},\omega_0)=\sigma\omega_0+I(\omega_0)\,, \tag{9}$$

where $\sigma=\tau_{\mathrm{tr}}e^2N/m$, the conductivity of the normal metal, and

$$\begin{aligned}I(\omega_0)=&\frac{2e^2}{m^2}2\pi T\sum_n\int\int\int d^3r'd^3sd^3l\int\int\int\int\frac{d^3pd^3qd^3q'd^3k}{(2\pi)^{3\times4}}p_1{}^2\\&\times\bigg\{\exp[i\mathbf{q}\cdot(\mathbf{s}-\mathbf{r})-i\mathbf{q}'\cdot(\mathbf{l}-\mathbf{r})+i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]\bigg(\frac{1}{i\tilde{\omega}-\xi_p}\eta_{\omega,\mathbf{q}}\Delta_{\mathbf{q}}\frac{1}{i\tilde{\omega}+\xi_p+\mathbf{v}\cdot\mathbf{q}}\\&\times\eta_{\omega,q'}{}^+\Delta_{q'}{}^+\frac{1}{i\tilde{\omega}-\xi_p-\mathbf{v}\cdot(\mathbf{q}-\mathbf{q}')}\frac{1}{i\tilde{\omega}_+-\xi_p-\mathbf{v}\cdot(\mathbf{k}+\mathbf{q}-\mathbf{q}')}+\frac{1}{i\tilde{\omega}-\xi_p}\frac{1}{i\tilde{\omega}_+-\xi_p-\mathbf{v}\cdot\mathbf{k}}\\&\times\eta_{\omega_+,q}\Delta_{\mathbf{q}}\frac{1}{i\tilde{\omega}_++\xi_p-\mathbf{v}\cdot(\mathbf{k}+\mathbf{q})}\eta_{\omega_+,q'}{}^\dagger\Delta_{q'}{}^\dagger\frac{1}{i\tilde{\omega}_+-\xi_p+\mathbf{v}\cdot(\mathbf{k}+\mathbf{q}-\mathbf{q}')}\\&-\frac{1}{i\tilde{\omega}-\xi_p}\eta_{\omega,q}\Delta_{\mathbf{q}}\frac{1}{i\tilde{\omega}+\xi_p+\mathbf{v}\cdot\mathbf{q}}\frac{1}{i\tilde{\omega}_++\xi_p+\mathbf{v}\cdot(\mathbf{q}+\mathbf{k})}\eta_{\omega_+,q'}\Delta_{q'}{}^\dagger\frac{1}{i\tilde{\omega}_+-\xi_p-\mathbf{v}\cdot(\mathbf{q}+\mathbf{k}-\mathbf{q}')}\bigg)\bigg\}\,.\end{aligned} \tag{10}$$

Here

$$\xi_p=p^2/2m-\mu,\quad \omega_+=\omega+\omega_0,\quad \mathbf{v}=\mathbf{p}/m\,,$$

and

$$\eta_{\omega,q}{}^+=\left\{1-\frac{1}{2\tau|\tilde{\omega}|}\left(1-\frac{\tau\tau_{\mathrm{tr}}}{3}v^2(\mathbf{q}+2e\mathbf{A})^2\right)\right\}^{-1}.$$

After replacing $p_1{}^2d^3p/(2\pi)^3$ by $\frac{1}{3}(mp_0{}^3/2\pi^2)d\xi d\Omega$ we carry out the integration over ξ and obtain

$$\begin{aligned}I(\omega_0)=\frac{\tau Ne^2}{mc}2\pi T\sum_{n=-\infty}^{\infty}\bigg\{&\frac{1}{2}\frac{\omega\omega_+}{|\omega||\omega_+|}\bigg(\frac{1}{|\omega|+Q_1}\Delta(\mathbf{r}_1)\frac{1}{|\omega|+Q_2}\Delta^+(\mathbf{r}_2)+\frac{1}{|\omega_+|+Q_1}\Delta(\mathbf{r}_1)\frac{1}{|\omega_+|+Q_2}\Delta^+(\mathbf{r}_2)\bigg)\\&+\frac{1}{|\omega|+Q_1}\Delta(\mathbf{r}_1)\frac{1}{|\omega_+|+Q_2}\Delta^+(\mathbf{r}_2)\bigg\}\bigg|_{\mathbf{r}_1=\mathbf{r}_2=\mathbf{r}}\,,\end{aligned} \tag{11}$$

where $Q_i=\frac{1}{6}\tau_{\mathrm{tr}}v^2(\nabla_i/i+(-1)^i2eA(\mathbf{r}_i))^2$ (for $i=1,2$) and we have made use of the relation

$$\int\frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{r}}\Delta_{\mathbf{q}}=\Delta(\mathbf{r})\,. \tag{12}$$

In the above derivation we neglect the terms of higher order in τvq. In the present case we are interested in the critical region where $\Delta(\mathbf{r})$ is small. In this region $\Delta(\mathbf{r})$ satisfies with good approximation[1]

$$\tfrac{1}{6}\tau_{\mathrm{tr}}v^2(\nabla/i-2eA)^2\Delta(\mathbf{r})=\alpha\Delta(\mathbf{r})\,, \tag{13}$$

where $\alpha=\frac{1}{3}\tau_{\mathrm{tr}}v^2eH_0$ and H_0 is the external field. Thus Eq. (11) reduces to

$$\begin{aligned}I(\omega_0)=\frac{\tau Ne^2}{m}2\pi T\sum_n\bigg\{&\frac{1}{2}\frac{\omega\omega_+}{|\omega||\omega_+|}\\&\times\bigg(\frac{|\Delta(\mathbf{r})|^2}{(|\omega|+\alpha)^2}+\frac{|\Delta(\mathbf{r})|^2}{(|\omega_+|+\alpha)^2}\bigg)\\&+\frac{|\Delta(\mathbf{r})|^2}{(|\omega|+\alpha)(|\omega_+|+\alpha)}\bigg\}\,.\end{aligned} \tag{14}$$

In the above treatment we discard the renormalization at the vertex associated with the current operator (p-wave vertex) for simplicity. This renormalization amounts[2] to replacing τ before the summation sign in Eq. (14) by τ_{tr}.

The above summation is expressed in terms of the poly-gamma function:

$$\begin{aligned}I(\omega_0)=\sigma(|\Delta(\mathbf{r})|^2/2\pi T)\{&\psi'(\tfrac{1}{2}+\omega_0/2\pi T+\alpha/2\pi T)\\&+(2\pi T/\omega_0+2\pi T/(\omega_0+2\alpha))\\&\times(\psi(\tfrac{1}{2}+\omega_0/2\pi T+\alpha/2\pi T)-\psi(\tfrac{1}{2}+\alpha/2\pi T))\}\,,\end{aligned} \tag{15}$$

where $\psi'(z)$ and $\psi(z)=\Gamma'(z)/\Gamma(z)$ are tri- and di-gamma functions, respectively.

The quantity $Q(\mathbf{r},\omega)$ is given as (after replacing ω_0 by $i\omega$)

$$\begin{aligned}Q(\mathbf{r},\omega)=\sigma\{&i\omega+(|\Delta(\mathbf{r})|^2/2\pi T)(\psi'(\tfrac{1}{2}+i\omega/2\pi T+\rho))\\&+[2\pi T/i\omega+2\pi T/(i\omega+2\alpha)]\\&\times(\psi(\tfrac{1}{2}+i\omega/2\pi T+\rho)-\psi(\tfrac{1}{2}+\rho))\}\,,\end{aligned} \tag{16}$$

where $\rho=\alpha/2\pi T$ and σ is the conductivity of the normal metal.

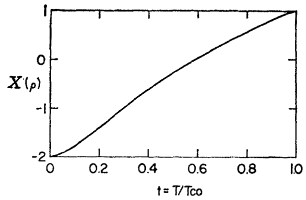

FIG. 2. The temperature dependence of $X(\rho)$.

It is interesting to note that the equivalent expression for $Q(\mathbf{r},\omega)$ is obtained for gapless superconductors containing paramagnetic impurities.[4] It is easy to show generally that the other transport coefficients (more precisely, current-current correlation functions) take an equivalent form in the two cases, insofar as we are concerned with the gapless region.

3. SURFACE IMPEDANCE

In the following we shall consider some limiting expressions of Eq. (16) which are of special interest.

(a) Low-frequency limit ($\omega \ll \pi T_{c0}$). Expanding Eq. (16) in powers of ω, we obtain

$$\begin{aligned} Q(\mathbf{r}_1\omega) = \sigma i\omega [1 + (|\Delta(\mathbf{r})|^2/2\pi T) \\ \times(\tfrac{3}{2}\psi''(\tfrac{1}{2}+\rho) + \psi'(\tfrac{1}{2}+\rho)/2\rho)] \\ + (2\sigma|\Delta(\mathbf{r})|^2/2\pi T)\psi'(\tfrac{1}{2}+\rho), \quad (17) \end{aligned}$$

where $\psi''(\frac{1}{2}+\rho) = (d^2/d\rho^2)\psi(\frac{1}{2}+\rho)$ and $\psi'(\frac{1}{2}+\rho) = (d/d\rho) \times\psi(\frac{1}{2}+\rho)$.

The space average of the imaginary part of Q is given by

$$\begin{aligned} \langle \mathrm{Im}Q\rangle_{\mathrm{av}} = \omega\sigma\{1 + (\langle|\Delta(\mathbf{r})|^2\rangle_{\mathrm{av}}/(2\pi T)^2) \\ \times(\tfrac{3}{2}\psi''(\tfrac{1}{2}+\rho) + (1/2\rho)\psi'(\tfrac{1}{2}+\rho))\}, \quad (18) \end{aligned}$$

$$= \omega\sigma\left\{1 - \frac{2\pi^5}{7\zeta(3)} \frac{\kappa^2 M}{H_{c2}(T)} X(\rho)\right\}, \quad (19)$$

where

$$X(\rho) = 1 + 3\rho(d/d\rho)(\ln g(\rho)), \quad g(\rho) = \sum_{n=0}^{\infty}(n+\tfrac{1}{2}+\rho)^{-2},$$

and

$$\kappa = \frac{3m}{2\pi^2 e\tau_{\mathrm{tr}}}\left(\frac{2\pi m}{p_0^5} 7\zeta(3)\right)^{1/2},$$

the Ginzburg-Landau parameter.[3] The magnetization M is given as[1]

$$-4\pi M = \frac{e\tau_{\mathrm{tr}} N}{mT} g(\rho)\langle|\Delta(\mathbf{r})|^2\rangle_{\mathrm{av}} = \frac{H_{12} - H_0}{(2\kappa_2^2(T) - 1)\beta},$$

where

$$\beta = 1.16 \quad (20)$$

and κ_2 is the temperature-dependent parameter.[1] On the other hand, the space average of the real part of Q is given by

$$\begin{aligned} \langle \mathrm{Re}Q(\mathbf{r}_1\omega)\rangle_{\mathrm{av}} = 2\sigma(\langle|\Delta(r)|^2\rangle_{\mathrm{av}}/2\pi T)\psi'(\tfrac{1}{2}+\rho) \\ = 4eM, \end{aligned} \quad (21)$$

as it should be.

The asymptotic behavior of $X(\rho)$ is given as

$$\begin{aligned} X(\rho) &= 1 - \frac{84}{\pi^2}\zeta(3)\rho + \left(\frac{540}{\pi^2}\zeta(4) - \frac{2352}{\pi^4}\zeta^2(3)\right)\rho^2, \quad \text{for } \rho \ll 1 \\ &= -2 + 1/2\rho^2 - 37/120\rho^4, \quad \text{for } \rho \gg 1. \end{aligned} \quad (22)$$

The temperature dependence of $X(\rho)$ is depicted in Fig. 2. At $T = 0°\mathrm{K}$ we have

$$\langle \mathrm{Im}Q(\mathbf{r},\omega)\rangle_{\mathrm{av}} = \omega\sigma\left(1 - \frac{\pi^4\kappa^2}{14\zeta(3)} \frac{(1 - H_0/H_{12})}{(2\kappa_2^2(0) - 1)\beta}\right), \quad (23)$$

where[1] $\kappa_2^2(0) = [\pi^4/168\zeta(3)]\kappa^2$. In particular, in the limit $\kappa \to \infty$ we obtain

$$\langle \mathrm{Im}Q\rangle_{\mathrm{av}} = \omega\sigma(1 - (6/\beta)(1 - H_0/H_{c2})). \quad (24)$$

The coefficient of $\omega\sigma$ may be interpreted as the density of states at zero frequency.

One sees from Fig. 2 that $\langle \mathrm{Im}Q\rangle_{\mathrm{av}}$ increases as H_0 increases at lower temperature, while it decreases at higher temperature.

It is possible that the above expansions break down at higher temperature since the singularity of $Q(\mathbf{r},\omega)$ in the complex ω plane approaches the origin along the imaginary axis as the temperature increases, and the radius of convergence of $Q(\mathbf{r},\omega)$ at the origin becomes smaller and smaller.

On the other hand, at lower temperature the above expansion converges as long as $\omega/\pi T_{c0} < 1$, where T_{c0} is the transition temperature in the absence of magnetic field.

(b) High-frequency region ($\omega \geq \pi T_{c0}$).

In this frequency region we have

$$\begin{aligned} Q(\mathbf{r},\omega) &= i\omega\sigma\{1 - (2|\Delta(\mathbf{r})|^2/\omega^2) \\ &\quad\times[\tfrac{1}{2} + (\ln(\omega/2\pi T) - \psi(\tfrac{1}{2}+\rho)) + \tfrac{1}{2}\pi i]\} \\ &= i\omega\sigma\{1 - (2|\Delta(\mathbf{r})|^2/\omega^2) \\ &\quad\times[\ln(2\omega/\Delta_{00}) + \tfrac{1}{2} + \tfrac{1}{2}\pi i]\}, \end{aligned} \quad (25)$$

where Δ_{00} is the ordering parameter at $T = 0°\mathrm{K}$ in the absence of the magnetic field and is given by $\Delta_{00} = \pi T_{c0}/\gamma$. In the above we made use of the identity[1,5]

$$\ln T/T_{c0} + \psi(\tfrac{1}{2}+\rho) - \psi(\tfrac{1}{2}) = 0. \quad (26)$$

It is interesting to note that Eq. (25) holds[2] even in the case $H_0 = 0$.

(c) At the absolute zero of temperature.

Putting $T = 0°\mathrm{K}$ in Eq. (16) we have

$$Q(\mathbf{r},\omega) = \sigma\{i\omega + (|\Delta(r)|^2/\alpha)j(\omega/\alpha)\}, \quad (27)$$

[4] S. Skalski, O. Betbeder-Matibet, and P. R. Weiss, Phys. Rev. **136**, A1500 (1964).

[5] P. G. de Gennes, Phys. Condensed Matter **3**, 6 (1965).

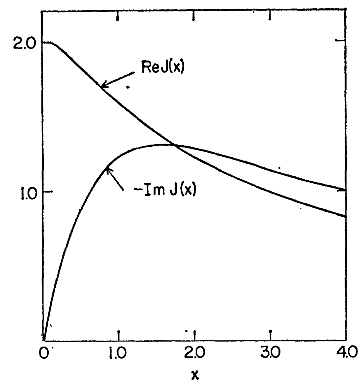

FIG. 3. The frequency dependence of the real and the imaginary part of $j(\omega/\alpha)$.

where

$$j(x)=1/(ix+1)+(1/ix+[1/(ix+2)]\ln(1+ix)). \quad (28)$$

$Rj(x)$ as well as $\mathrm{Im}\,j(x)$ are plotted in Fig. 3.

The surface impedance of the bulk specimen is obtained from Eq. (1) as[2,6]

$$Z=i\omega(4\pi/Q(\omega))^{1/2},$$

where (29)

$$Q(\omega)=\langle Q(\mathbf{r},\omega)\rangle_{\mathrm{av}}.$$

Here we approximate $Q(\mathbf{r},\omega)$ by its space average for simplicity. The error involved in such approximations can be estimated by using a method similar to one due to Fisher,[7] and we find it is of the order of $(\beta-1)\cong 0.1$.

Equation (27) together with Eq. (15) completely determines the surface impedance of a bulk superconductor of the second kind provided the external field is close to H_{c2} ($H_{c2}-H_0<H_{c2}$).

4. CONCLUDING REMARKS

In the above sections we have seen that the complex conductivity of superconducting alloys in high magnetic fields can be calculated explicitly if we restrict ourselves to the critical (or gapless) region. The validity of the above approach relies heavily upon the existence of a gapless region where the analytical properties of the quantity $Q(\mathbf{r},\omega)$ become simple (i.e., the singularities of Q are poles on the imaginary axis, which is in contrast to the branch point on the real axis corresponding to the BCS state). The above method is easily extended to the calculation of other transport coefficients such as the thermal conductivity, the attenuation coefficient of ultrasound, etc.

[6] We correct here the wrong sign of Z in Ref. 2.
[7] G. Fisher, J. Math. Phys. **5**, 1158 (1964).

In the weak-field region, on the other hand, the situation is not so clear. It is well known[8] that the low-lying excitations associated with each flux line play the dominant role in the dissipation mechanism. We shall briefly indicate how to formulate the problem in this region. It is shown that the absorptive part of $Q(\mathbf{r},\omega)$ is expressed in terms of

$$N(\mathbf{r},\omega)=\frac{2\pi^2}{mp_0}\mathrm{Im}\left\{\int_{-\infty}^{\infty}\langle T_\tau\psi_\uparrow(\mathbf{r},0)\psi_\uparrow^\dagger(\mathbf{r},t)\rangle e^{-i\omega t}dt\right\},$$

the density of states. In particular, in the low-frequency limit we have

$$\mathrm{Im}Q(\mathbf{r},\omega)=\sigma\omega(N^2(\mathbf{r},0)+O(T/T_{c0})^2),\quad \text{for } T\ll T_{10}; \quad (30)$$

that is, we can express $\mathrm{Im}Q(\mathbf{r},\omega)$ in terms of $N(\mathbf{r},0)$ at low temperatures.

We know that $N(\mathbf{r},0)=1$ at the center of a flux line where $\Delta(\mathbf{r})$ vanishes, and $N(\mathbf{r},0)$ decreases as we go away from the center of the flux line. The space average of $\mathrm{Im}Q(\mathbf{r},\omega)$ given by

$$\begin{aligned}\langle\mathrm{Im}Q(\mathbf{r},\omega)\rangle_{\mathrm{av}}&=\sigma\omega\langle N^2(\mathbf{r},0)\rangle_{\mathrm{av}}\\&=\sigma\omega n\langle\langle N^2(\mathbf{r},0)\rangle\rangle,\end{aligned} \quad (31)$$

where n is the density of flux lines and $\langle\langle N^2(\mathbf{r},0)\rangle\rangle$ is the average of $N^2(\mathbf{r},0)$ on a single flux line. It is interesting to note that the coefficient of the linear term in temperature in the expression of the specific heat can be written as

$$\tfrac{1}{3}mp_0n\langle\langle N(\mathbf{r},0)\rangle\rangle.$$

On the other hand we expect that $\langle\mathrm{Re}Q(\mathbf{r},\omega)\rangle_{\mathrm{av}}$ is almost equal to 1 in a superconductor containing paramagnetic impurities,[4] and the only difference is that the parameter $1/\tau_s=\Gamma_s$ is replaced by $\frac{1}{3}\tau_{\mathrm{tr}}v^2eB$, where B is the magnetic induction.[1]

A detailed study of available experimental data[9] seems to show that the reactive part of $\sigma(\omega)$ changes more slowly than that predicted by our theory. One possible explanation of this discrepancy may be that $|\Delta(\mathbf{r})|^2$ in the surface region where the main electromagnetic absorption takes place changes somewhat more gradually than $\langle|\Delta(\mathbf{r})|^2\rangle_{\mathrm{av}}$. I would like to thank Dr. P. C. Hohenberg for calling my attention to the above discrepancy.

[8] C. Caroli, P. G. de Gennes, and J. Matricon, Phys. Letters **9**, 307 (1964).
[9] B. Rosenblum and M. Cardona, Phys. Rev. Letters **12**, 657 (1964).