Brownian Motion of Charged Particles in Crossed Electric and Magnetic Fields

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(Received 30 August 1965)

A new method for evaluating the Brownian motion of charged particles in crossed electric and magnetic fields is presented. In the problems considered, a steady B_z field and radial electric field are embedded in a viscous medium. Calculations are made for the three-dimensional mean-square displacement of particles migrating away from the cylindrical axis. The results are applied to the theory of sheaths to uncover the manner in which the wall charge affects the familiar Debye displacement. The resulting formula is independent of the presence of a magnetic field parallel to the wall. Relaxing the electric conditions yields the Brownian motion of a neutral particle in a neutral viscous fluid.

I. INTRODUCTION AND SUMMARY OF RESULTS

THE theory of Brownian motion[1] divides into two classes of endeavor, these being the nonphenomenological and phenomenological approaches respectively. The former class stems from a statistical-mechanics point of view and was initiated with Einstein's[2] and Smoluchowski's[3] classical treatments of the problem. The more detailed kinetic analyses[4–10] which are centered about refined calculations of the diffusion coefficient from the Boltzmann equation are also part of this first class. More recently Lebowitz and Rubin[11] treat the problem of Brownian motion in terms of a joint distribution function for the fluid and the Brownian particle. An equation for this distribution is derived from the Liouville equation.

In the simpler but more phenomonological approach initiated by Langevin,[12] the starting point is Newton's second law in which the interaction between the Brownian particle and host appears as a dissipative term. Such formulations may be found in many "modern physics" texts[13] where they are often loosely presented, as are the original works.[14]

The problem of the Brownian motion of charged particles in electric and magnetic fields is greatly obscured by the Larmor effects and only lower order estimates of the diffusion coefficient are available.[15]

In this paper we wish to present a new method of calculating the Brownian motion of charged particles in crossed electric and magnetic fields which is closely related to the phenomenological approach mentioned above. However, results are readily obtained which render considerable insight into the problem at hand.

The method consists of first: obtaining an exact equation (within the framework of a Langevin formulation) for r^2, the square of the absolute value of the radius vector from the origin to the Brownian particle; second: obtaining from this equation an exact equation for the time average, $\langle r^2\rangle$; third: solving this equation in appropriate limits.

The method is best suited for motion in a plane which is normal to a steady magnetic field. The plane so defined includes a radial electric field. For problems involving only radial fields (e.g., Coulomb and Newtonian) the method is easily extended to three dimensions.

In the first configuration a wire of constant charge density, which is oriented parallel to a steady magnetic field, creates an electric field which decays as the inverse radius. These fields are embedded in a viscous medium characterized in the force equation by a constant collision frequency. Evaluating the time average of the mean-square displacement (in a plane normal to the magnetic field) after many collisions have taken place indicates that the sole effect of the electric field is to cause the square of a characteristic electric velocity to be added to the square of the thermal speed as it appears in the Chapman-Cowling[4,5] estimate of the diffusion coefficient. The mean-square displacement grows linearly with time and the particle diffuses "outward" in the same manner as in the Einstein[2] result. If the wire charge is opposite in sign to the test charge, the diffusion is still outward granted that the line charge density is less than the critical value $\rho_{\text{crit}}=2\pi\epsilon_0 kT/q$, where ϵ_0 is the permitivity of free

[1] See (a) *Selected Papers on Noise and Stochastic Processes*, edited by N. Wax (Dover Publications Inc., New York, 1954); (b) G. L. deHaas-Lorentz, *Die Brownische Bewegung and Einige Verwandte Erscheinungen* (Braunschweig, 1913).

[2] *Investigations on the Theory of the Brownian Movement*, edited with notes by R. Fürth (Dover Publications Inc., New York, 1956).

[3] M. V. Smoluchowski, Ann. Physik **21**, 756 (1906).

[4] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1953).

[5] R. L. Liboff, Phys. Fluids **5**, 963, (1962).

[6] M. N. Rosenbluth and A. N. Kaufman, Phys. Rev. **109**, 1 (1958).

[7] J. P. Wright, Phys. Fluids **3**, 607 (1960); **41**, 1341 (1961).

[8] J. B. Taylor, Phys. Rev. Letters **6**, 262 (1961).

[9] J. B. Taylor, Phys. Fluids **4**, 1142 (1961).

[10] J. B. Taylor, Nucl. Fusion **2**, 477 (1962).

[11] J. Lebowitz and E. Rubin, Phys. Rev. **131**, 2381 (1963).

[12] P. Langevin, Compt. Rend. **146**, 530 (1908).

[13] In such analyses authors indiscriminately interchange differentiating and time averaging operations. Examples include: H. Semat, *Introduction to Atomic and Nuclear Physics* (Holt, Rinehart and Winston, Inc., New York, 1962); A. D. French, *Principles of Modern Physics* (John Wiley & Sons Inc., New York, 1958).

[14] See Ref. 1b.

[15] For recent experimental results see O. D. Olsen and H. M. Skarsgard, Can. J. Phys. **43**, 855 (1965).

space, k is Boltzmann's constant, T is the temperature, and q is the charge of the test particle. In either case of like or oppositely charged test particle and line source, the presence of the magnetic field reduces the rate at which the mean-square displacement grows by the factor $[1+(\Omega/\nu)^2]^{-1}$, where Ω is the Larmor frequency and ν is the collision frequency between the Brownian particle and the host. In both cases an expression for the three-dimensional mean-square displacement is easily obtained by adding the "longitudinal" diffusion to the above described "transverse" diffusion.

In the second configuration the cylindrical geometry is filled with a viscous medium of uniform charge density and a steady magnetic field is again oriented parallel to the axis of symmetry. The charged medium generates a field which grows linearly with the radius. When the medium and test particle are of like charge the particle diffuses outward, exponentially in time, for arbitrarily small charge density and arbitrarily large magnetic field. The presence of the magnetic field reduces the mean-square displacement by the factor

$$[1+(\Omega/\nu)^2]\exp\{-\omega_p{}^2\Omega^2 t/\nu(\nu^2+\Omega^2)\},$$

where ω_p is the plasma frequency of the medium[16] and t is the time. If the medium and Brownian particle are of opposite sign the particle migrates to, but not further than, a Debye distance from the central axis, a result which is independent of the magnetic field or the collision frequency. An expression for the three-dimensional mean-square displacement indicates that the particle diffuses transversely, exponentially faster than it does longitudinally.

It is interesting to note that this second problem in which a charged particle migrates into a charged cylindrical medium does not permit the magnetic-diffusion result of Chapman and Cowling to be recaptured in a uniform continuous manner, as was possible in the first example discussed above. The reason is that the presence of the electric term completely alters the form of the "unperturbed" force equation [see, for instance Eq. (27)]. An important consequence is that a Chapman-Cowling procedure about zero electric field would give the Chapman-Cowling magnetic diffusion coefficient as the lowest order estimate, which would be an erroneous result. The underlying difficulty is the long-range nature of the Coulomb force.[17]

Included in the latter part of the paper is a brief discussion of the application of these results to the theory of the structure of sheaths. The familiar[18] first-order estimates show that the sheath maximum lies at a Debye distance from the disturbing wall. This result is corrected to account for the field generated by the wall charge. The augmented Debye distance squared appears as $\hat{d}^2=(2\pi\epsilon_0 kT+q\rho_L)/q^2n$ where ρ_L is the charge density of an effective line and n is the number density of the medium.

Finally the conditions of the first example are relaxed to give the motion of a neutral particle in a neutral fluid. Although the asymptotic formula agrees with that of previous investigators,[19,20] there is disagreement for earlier times. In the included analysis the mean-square (three-dimensional) displacement $\langle r_3{}^2\rangle$ is given by

$$\text{(a)}\quad \langle r_3{}^2\rangle=(2C^2/\nu^2)\left\{\frac{(y-2)}{2}+\frac{1}{y}(1-e^{-y})\right\},$$

while in the work referred to one finds

$$\text{(b)}\quad \langle r_3{}^2\rangle=(2C^2/\nu^2)(y-1+e^{-y}).$$

The difference between the two formulations is most likely due to the fact that in the included analysis $\langle r^2\rangle$ is a time average quantity while the classical results pertain to an ensemble average quantity.

In this vein a discussion on "ergodic differential equations" is included. An ergodic differential equation is one which has an ergodic solution, i.e., a solution whose time and ensemble averages are the same. It is shown that some of the equations which arise in the included area of investigation are not ergodic, whence the ensemble average becomes distinct. Furthermore, trivial examples serve to illustrate that a characteristic difference in form between the time and ensemble average is similar to the difference in form (viz., e^{-t} versus e^{-t}/t) which arises between the two averages (a) and (b) presented above.

II. ANALYSIS

The motion of an isolated "test" charge in a viscous medium in which electric and magnetic fields are embedded is given by

$$m\ddot{\mathbf{r}}=q\dot{\mathbf{r}}\times\mathbf{B}+q\mathbf{E}-\nu m\dot{\mathbf{r}}+m\mathbf{\Gamma}. \qquad (1)$$

In this equation m and q are the mass and charge of the Brownian particle, respectively, and $\mathbf{E}$ and $\mathbf{B}$ are the imposed fields. The magnetic field is steady and in the z direction, while $\mathbf{E}$ is radial. The interaction between the particle and the medium is given in terms of the collision frequency ν and the stochastically fluctuating field $\mathbf{\Gamma}$, characteristic to Brownian motion. The field $\mathbf{\Gamma}$ incorporates short-range discontinuous collisions, while the coefficient ν gives rise to a smooth dissipative effect. Both of these phenomenological parameters are, to lowest order, independent of the applied fields, (i.e., they do not vanish if the fields are turned off).

[16] See Eq. (26) and Ref. 24.
[17] See R. L. Liboff, Phys. Fluids 2, 40 (1959) for extended references on this topic.
[18] See L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), and J. L. Delcroix, *Introduction to the Theory of Ionized Gases* (Interscience Publishers, Inc., New York, 1960).
[19] L. S. Ornstein, Versl. Acad. Amst. **26**, 1005 (1917).
[20] R. Fürth, Z. Physik. **2**, 244 (1920).

Our object at this point is to obtain an equation for $r^2=x^2+y^2$. To this end we first note the relation.

$$(r^2)''=2\mathbf{r}\cdot\ddot{\mathbf{r}}+2\dot{r}^2. \tag{2}$$

Primes and dots are used interchangeably to denote time derivatives. Forming the scalar product of Eq. (1) with $\mathbf{r}$ and using Eq. (2) to eliminate $\mathbf{r}\cdot\ddot{\mathbf{r}}$, there results

$$(r^2)''-2\dot{r}^2+\nu(r^2)'-2(q/m)\mathbf{r}\cdot\mathbf{E}=2\Omega\mathbf{b}\cdot\mathbf{r}\times\dot{\mathbf{r}}+2\mathbf{r}\cdot\mathbf{\Gamma}. \tag{3}$$

The Larmor frequency Ω is given by

$$\Omega=qB/m, \tag{4}$$

while $\mathbf{b}$ is a unit vector in the direction of $\mathbf{B}$. Solving Eq. (1) for $\dot{\mathbf{r}}$ and substituting the resulting form into the right side of Eq. (3) gives the desired result,

$$\begin{aligned}\tfrac{1}{2}[(r^2)''+\alpha(r^2)']-(q/m)\mathbf{r}\cdot\mathbf{E}\\ =\dot{r}^2+(\Omega/\nu)\mathbf{b}\cdot[(q/m)\mathbf{r}\times\mathbf{E}+\mathbf{r}\times\mathbf{\Gamma}\\ -(1/m)\mathbf{r}\times\mathbf{F}]+\mathbf{r}\cdot\mathbf{\Gamma},\end{aligned} \tag{5}$$

$$\nu\alpha\equiv\nu^2+\Omega^2. \tag{6}$$

The force field $\mathbf{F}$ is the instantaneous force on the particle, i.e.,

$$\mathbf{F}=m\ddot{\mathbf{r}}. \tag{7}$$

Since the electric field is radial,

$$\mathbf{E}\cdot\mathbf{r}=rE;\quad \mathbf{r}\times\mathbf{E}=0. \tag{8}$$

Equation (5), still an exact relation, then reduces to the desired form,

$$\begin{aligned}\tfrac{1}{2}[(r^2)''+\alpha(r^2)']-(q/m)rE=\dot{r}^2+\mathbf{r}\cdot\mathbf{\Gamma}\\ +(\Omega/\nu)\mathbf{b}\cdot[\mathbf{r}\times\mathbf{\Gamma}-(1/m)\mathbf{r}\times\mathbf{F}].\end{aligned} \tag{9}$$

Our primary interest at this point involves the mean square displacement,

$$\delta\equiv\langle r^2\rangle\equiv\frac{1}{t}\int_0^t r^2(\tau)d\tau, \tag{10}$$

in the limit $\nu t\gg 1$, i.e., after many "collisions" have taken place. The inverse of the latter equation is

$$(t\delta)'=r^2. \tag{11}$$

Note that in going to a representation of the average variable δ, as given by Eq. (11) an element of precision is lost. Although r^2 is uniquely determined by δ, knowing r^2 determines δ only to within an additive term (K/t) where K is any constant.

Inserting the relation (11) into Eq. (9) yields,

$$\begin{aligned}(t\delta)'''+\alpha(t\delta)''-2(q/m)Er=2(t\langle\dot{r}^2\rangle)'\\ +2(\Omega/\nu)[(t\langle\mathbf{b}\cdot(\mathbf{r}\times\mathbf{\Gamma})\rangle)'-(1/m)(t\langle\mathbf{b}\cdot(\mathbf{r}\times\mathbf{F})\rangle)]\\ +2(t\langle\mathbf{\Gamma}\cdot\mathbf{r}\rangle)'.\end{aligned} \tag{12}$$

This equation, together with the relation (10), is still an exact form. Owing to the vector products, however, it is not self-contained.

To remedy the situation, consider first the form $\mathbf{r}\times\mathbf{\Gamma}$. This vector is the torque of $\mathbf{\Gamma}$ about the origin. If $\mathbf{\Gamma}$ is a random force field, the average of this torque must vanish. If $\langle\mathbf{b}\cdot(\mathbf{r}\times\mathbf{\Gamma})\rangle$ were positive, say, then the force would exert a preferred rotation about the origin. But this is contradictory to our assumption that $\mathbf{\Gamma}$ is random. Therefore, this average quantity vanishes.

The $\langle\mathbf{\Gamma}\cdot\mathbf{r}\rangle$ term may be neglected, in the random phase approximation. Namely, consider that $\Gamma_r\sim\exp i\omega_1 t$ and $r\sim\exp i\omega_2 t$. Then the average inner product will contain terms like $\int_0^t\cos(\omega_1-\omega_2)t$. If the phases ω_1 and ω_2 are random, in a sufficiently long time interval, the cos term is positive as often as it is negative and the integral tends to zero.

The remaining cross term is the $\mathbf{b}$ component of the torque of the total force $\mathbf{F}$ about the origin. The fact that the $\mathbf{B}$ field causes a preferred sense of rotation of the test particle between collisions does not necessarily imply that $\langle\mathbf{b}\cdot(\mathbf{r}\times\mathbf{F})\rangle$ is finite. Indeed, even in the limit of no collisions, unless the origin is an interior point of the particle orbit, $\langle(\mathbf{r}\times\mathbf{F})\cdot\mathbf{b}\rangle=0$. If $\Omega\gg\nu$ and the radius R of the domain of motion is large compared to a Larmor radius R_Ω, then the ratio of orbits which enclose the origin to those which do not is $(R_\Omega/R)^2$. (Assume a uniform density, then all orbits in a circle or radius 2 R_Ω about the origin enclose the origin. Particles exterior to this central circle do not.) For sufficiently large magnetic field this ratio becomes vanishingly small. In the more consistent limit, $\nu\gg\Omega$, so that the particle never completes a Larmor orbit. The force field $\mathbf{F}$ is randomized by collisions and $\langle\mathbf{b}\cdot(\mathbf{r}\times\mathbf{F})\rangle$ vanishes. Inasmuch as this is the domain of interest for the ensuing analysis, we will set $\langle\mathbf{b}\cdot(\mathbf{r}\times\mathbf{F})\rangle$ equal to zero.

Note that all of the terms neglected are averages of two distinct randomly fluctuating variables, and therefore can be consistently neglected within the random phase approximation. Such a formal procedure necessitates that the interval over which average quantities are defined be sufficiently long. It is this formal procedure that implies that the defining equations, Eq. (10), includes the limit, $t\rightarrow\infty$.

Our final assumption regards the mean-square speed $\langle\dot{r}^2\rangle$ (after a sufficiently long time interval). Were there no electric field imposed in the problem, then it would be consistent to speak of the test particle and medium being in equilibrium at the temperature T. In this case $\langle\dot{r}^2\rangle=2kT/m=C^2$, the two-dimensional mean thermal speed. With an electric field present this is not generally the case, inasmuch as the electric field will clearly add to the energy of the particle. However, if collisions are frequent enough, the electric field never gets the chance to contribute to the energy of the particle. This will be the case if the work done on the particle by the electric field E between collisions is small compared to the thermal energy kT. If λ is the mean free path, then this condition is satisfied if $kT\gg\lambda Eq$. With this constant in mind, we set $\langle \dot{r}^2\rangle = C^2$ to obtain[21] our starting equation [from Eq. (12)].

$$(t\delta)'''+\alpha(t\delta)''-2(q/m)Er=2C^2. \tag{13}$$

This latter assumption regarding the mean thermal speed may formally be viewed as a first approximation in an iteration scheme. The next-order equations would then include a corrected value of $\langle r^2\rangle$ obtained from the first iterate. Inasmuch as the included analysis obtains only lowest order results, they are applicable only to a medium of sufficiently high temperature.

In passing we note that an alternative method of solution is to solve for the ensemble average of r^2. Ensemble averages, we recall, are employed to obtain expressions for the physically relevant time average. However, it is quite clear that an ordinary differential equation whose homogeneous solution grows in time is not ergodic.[22] By way of example consider the trivial case,

$$Y'-\gamma Y=0.$$

If $\bar{Y}$ is the ensemble average, and $\langle Y\rangle$ the time average, there results,

$$\bar{Y}=Ae^{\gamma t}\neq\langle Y\rangle = A(e^{\gamma t}/\gamma t),$$

and the differential equation is not ergodic. If on the other hand the inhomogeneous solution is constant, and dominates the long time behavior, the equation is ergodic. Consider, for example,

$$Y'+\gamma T=\eta.$$

Then

$$\bar{Y}=\eta/\gamma=\langle Y\rangle.$$

and the equation is ergodic. Since the equations we will encounter are of both types, and the formulation of the time average is not difficult, ensemble averages will be ignored. Equation (13) will serve as our starting equation in the subsequent analysis. The recipe is to solve for δ and then pass to the limit $\nu t\gg 1$.

Case 1. Motion About a Charged Wire

In the first example we consider the Brownian motion of charged particles about a straight uniformly charged wire which is parallel to a constant uniform magnetic field. If ρ_L is the charge density of the line (Coulombs/m) then the related electric field is

$$\mathbf{E}[\rho_L/2\pi\epsilon_0 r^2]\mathbf{r}, \tag{14}$$

$$V^2\equiv q\rho_L/2\pi\epsilon_0 m, \tag{15}$$

where V is a characteristic velocity.

In terms of these parameters Eq. (13) appears as

$$(t\delta)'''+\alpha(t\delta)''=2(V^2+C^2)\equiv 2\beta^2. \tag{16}$$

The solution is

$$t\delta=A+Fe^{-\alpha t}+(\beta^2/\alpha^3)[(\alpha t)^2+D(\alpha t)], \tag{17}$$

where A, F, and D are constants to be determined.

The initial conditions are given by the constraint that the particle move uniformly in the early times, so that about $t=0$, $\langle r^2\rangle\sim t^2$, or that $(t\delta)\sim t^3$, which is equivalent to three pieces of initial data. This constraint, together with the exponential form of the general solution serves to determine the coefficient which multiplies the t^3 term.

Expanding Eq. (17) and then setting the first three coefficients equal to zero gives

$$A+F=0, \tag{18a}$$

$$(\beta/\alpha)^2D-\alpha F=0, \tag{18b}$$

$$\alpha^3F+2\beta^2=0, \tag{18c}$$

from which there results

$$(\alpha^2/2\beta^2)\delta=(1/y)(1-e^{-y})+(y-2)/2. \tag{19}$$

In this formula y is the nondimensional time αt.

In the limit as $y\gg 2$

$$\delta\sim\beta^2 t/\alpha \tag{20}$$

which gives the desired result,

$$\langle r^2\rangle\sim(V^2+C^2)t/\nu(1+(\Omega/\nu)^2). \tag{21a}$$

The sole effect of the electric field is to augment the thermal diffusing influence by an additive term V^2. The variable $\langle r^2\rangle$ in Eq. (21)-a represents the mean square displacement of a particle constrained to move in plane (hitherto called the "transverse" displacement). It can be made to represent the projected two-dimensional displacement, for an unconstrained particle, by merely changing C^2 to $2C^2/3$. Furthermore, the z component of Eq. (1) can easily be shown to yield $\langle z^2\rangle \sim C^2t/3\nu$. It follows that if C^2 in Eq. (21-a) is changed to $2C^2/3$ and then the resulting formula is added to $\langle z^2\rangle$, one obtains an expression[23] for $\langle r_3^2\rangle=\langle x^2+y^2+z^2\rangle$, with the **B** field in the z direction and the **E** field in the (x,y) plane. There results

$$\langle r_3^2\rangle\sim(C^2t/3\nu)\left\{1+\frac{1}{(1+(\Omega/\nu)^2)}[2+(3V^2/C^2)]\right\}. \tag{21b}$$

[21] In setting the two-dimensional entity $\langle\dot{r}^2\rangle$ equal to C^2 we are tacitly assuming that the Brownian particle is constrained to move in two dimensions. For two-dimensional motion in three-dimensional space, on the other hand, $\langle\dot{r}^2\rangle=2C^2/3$. For the first case $C^2=2kT/m$, while for the second $C^2=3kT/m$. In either case $\langle\dot{r}^2\rangle=\langle\dot{x}^2+\dot{y}^2\rangle=2kT/m$. Also, for the first case $\langle x^2\rangle=\langle r^2\rangle/2$, while for the second, $\langle x^2\rangle=\langle r_3^2\rangle/3$. The three-dimensional radius vector is r_3.

[22] An "ergodic differential equation" is one which has an ergodic solution.

[23] See S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943) [first article in Ref. 1(a)] for discussions of three-dimensional Brownian motion. It is also an excellent review article on Brownian motion.

If q and ρ_L are oppositely charged then V^2 is a negative number, $-v^2$, and Eq. (21a) reads

$$\langle r^2\rangle \sim (C^2-v^2)t/\nu(1+(\Omega/\nu)^2). \tag{22}$$

From this equation it follows that the diffusion is still "outward," granted that

$$(kT/m) > q\rho_L/2\pi\epsilon_0 m. \tag{23}$$

This criterion is independent of the strength of the magnetic field.

The above results indicate that the mean-square displacement $\langle r^2\rangle_E$, without the magnetic field is reduced by the factor

$$\langle r^2\rangle_{EB}/\langle r^2\rangle_E = [1+(\Omega/\nu)^2]^{-1} \tag{24}$$

in the presence of the magnetic field, for both subcases of like and oppositely charged wire and test particle.

In the limit of vanishing electric and magnetic fields Eq. (21b) gives the mean-square displacement of a neutral particle diffusing through a neutral medium

$$\langle r_3^2\rangle \sim (C^2/\nu)t. \tag{25a}$$

Comparing this with the Einstein result,

$$\langle r_3^2\rangle \sim (2C^2/\nu)t, \tag{25b}$$

we see that the two formulas differ by a factor of 2. This is best explained by the fact that the Einstein average is an ensemble average while in the included analysis, a time average is calculated. Finally, we note that a generalization of Eq. (25a) to early time behavior is obtained by setting $\alpha^2=\nu^2$ and $\beta^2=2C^2/3$ in Eq. (19) and adding the result to $\langle z^2\rangle$. This latter variable is also obtained from Eq. (19) by setting $\alpha^2=\nu^2$ and $\beta^2=C^2/3$. Adding the two results gives Eq. (a) which is discussed in the Introduction.

Case 2. Motion in a Uniformly Charged Cylindrical Medium

In this second example the cylindrical geometry is uniformly filled with a viscous medium of constant charge density. A steady magnetic field oriented parallel to the axis of symmetry permeates the medium. If n is the number density of the medium then the charge density is nq; where for convenience, the unit of charge of the medium q is the same as that of the test particle. The electric field is

$$\mathbf{E}=\mathbf{r}(qn/2\epsilon_0)\equiv \mathbf{r}(m/2q)\omega_p^2 \tag{26}$$

which also serves to define the plasma frequency[24] ω_p.

If this latter form is substituted into Eq. (13), there results

$$(t\delta)'''+\alpha(t\delta)''-\omega_p^2(t\delta)'=2C^2. \tag{27}$$

[24] Note that ω_p includes the mass of the Brownian particle—not of the medium.

The solution is

$$(\omega_p^2/C^2)(t\delta)=A(e^{D_+t}-1)+F(e^{D_-t}-1)-t, \tag{28}$$

where the exponents are the roots

$$2D_\pm=-\alpha\pm[\alpha^2+(2\omega_p)^2]^{1/2}. \tag{29}$$

The form of Eq. (28) is chosen to insure that the particle behaves kinematically about $t=0$ viz $\delta\sim t^2$. This serves to determine the constants A and F,

$$A=D_-/(D_+[D_--D_+]), \tag{30a}$$

$$F=D_+/(D_-[D_+-D_-]). \tag{30b}$$

In the diffusion problem at hand αt, containing the term νt, is a large number. The two relevant expansions of $D_\pm$ are

$$D_+\sim +\omega_p^2/\alpha, \tag{31a}$$

$$D_-\sim -\alpha. \tag{31b}$$

These in turn supply the proper limiting forms of the coefficients A and F,

$$F\sim -\omega_p^2/\alpha^3, \tag{32a}$$

$$A\sim \alpha/\omega_p^2. \tag{32b}$$

The corresponding form of Eq. (28) is

$$\omega_p^2\delta/C^2 \sim t^{-1}\{-(\omega_p^2/\alpha^3)[e^{-\alpha t}-1] + (\alpha/\omega_p^2)[\exp(\omega_p^2 t/\alpha)-1]-t\} \tag{33a}$$

which finally appears as

$$\delta=\langle r^2\rangle \sim [C^2\alpha/t\omega_p^4]\exp\{\omega_p^2 t/\alpha\}. \tag{33b}$$

An expression for $\langle r_3^2\rangle$ is obtained in the same manner as described after Eq. (21a). There results

$$\langle r_3^2\rangle \sim (2C^2t/3\nu)\{1+(\nu\alpha/t^2\omega_p^4)\exp(\omega_p^2 t/\alpha)\}. \tag{33c}$$

The particle diffuses transversely, exponentially faster than it does longitudinally.

Furthermore an important implication of Eq. (33b) is that one may not approach the case of zero-charge density uniformly. The long-range nature of the Coulomb interaction imposes an "on-off" effect, so that results which appear in its absence are vastly removed from those which appear in the presence of the interaction, for arbitrarily small charge density. A Chapman-Cowling iteration about zero-charge density would fail, yielding as it would the incorrect lowest order estimate, viz. Eq. (21) with $V^2=0$.

Returning to the main stream of the analysis it is again instructive to note that the two-dimensional mean square displacement is diminished by the factor

$$\langle r^2\rangle_{EB}/\langle r^2\rangle_E = (1+(\Omega/\nu)^2)\exp\{-\omega_p^2\Omega^2 t/\nu(\nu^2+\Omega^2)\} \tag{34}$$

in the presence of the magnetic field.

In the event that medium and test particle are oppositely charged, ω_p^2 is a negative number and the asymptotic form of Eq. (33) is

$$\langle r^2\rangle \sim C^2/|\omega_p^2| \equiv d^2. \tag{35}$$

After many collisions the particle migrates to, but not further than a Debye distance d from the axis of symmetry. The result being obviously an equation of energies is independent of the magnetic field.

III. SOME COMMENTS RELATING TO THE SHEATH PROBLEM

Standard qualitative descriptions of the formation of a sheath about a conducting surface in a plasma are as follows. The electrons being of higher thermal velocity than the ions strike the conducting surface more frefiuently. The fraction which attach charge the surface negatively. The field so generated causes electrons to be reflected into a positive residue all at temperature T, until some steady-state sheath structure emerges.

Clearly the Brownian motion of a negative particle moving away from a negative source into a positively charged viscous medium is very relevant to the model just described. Comparison of Eqs. (16) and (27) indicate that the desired result is obtained by merely changing C^2 and ω_p^2 in Eq. (27) to read C^2+V^2 and $-\omega_p^2$, respectively. One then obtains

$$\langle r^2\rangle \sim (C^2+V^2)/\omega_p^2 = \tilde{d}^2 \tag{36}$$

in place of Eq. (35). The augmented Debye distance $\tilde{d}$ is

$$\tilde{d}^2 = (2\pi\epsilon_0 kT + q\rho_L)/q^2 n. \tag{37}$$

As a low-order estimate this result appears to be more accurate than previous results[18] which do not account in any way for the field generated by the conductor.

Concluding we note also that a magnetic field normal to the direction of diffusion would not in any way influence the result as given by the equation above. This, of course, might be easily substantiated by noting the difference in the position of the sheath maximum with and without such a magnetic field.

IV. DOMAIN OF VALIDITY OF INCLUDED ANALYSIS

Because of the invariance of $r(t)$ under the transformation $\delta \rightarrow \delta' = \delta + (A/t)$, there is an inherent uncertainty in the included analyses of the order of A/t. The arbitrary constant A is related to the natural periods in the problem, which for a neutral environment reduces to the collision period. After a moderate number of collisions the results presented above become more certain.

Granted that the extended Langevin equation, Eq. (1), is a valid description and that the uncertainty spoken of above is removed, there remains an approximation inserted early in the analysis which merits justification. This is that $\langle \dot{r}^2\rangle$ is the square of the constant thermal speed C. For the problems treated above the electric field does work on the particles between collisions and their mean square velocity increases. If, however, this work $qE\lambda$ (λ = mean free path) is small compared to the thermal energy kT the time between collisions is not long enough for the electric field to impart a significant amount of energy. Under such circumstances setting $\langle \dot{r}^2\rangle = C^2$ is a first-order solution in which the energy imparted by the electric field between collisions is a small perturbation.

The domain of validity becomes $qE\lambda \ll kT$. For the first problem where the radial electric field is due to a uniformly charged straight wire, the criterion of validity becomes

$$r \gg r_{\text{cr}} = q\rho_L\lambda/2\pi\epsilon_0 kT. \tag{38}$$

For arbitrary temperature and charge density the formalism remains valid for r large compared to r_{cr}. To insure that this is satisfied in an actual experiment one need only design the radius of the center wire to be of the order of r_{cr}.

In the second problem where the electric field is that due to a uniformly charged medium, the condition $qE\lambda \ll kT$ becomes,

$$r \ll r_{\text{cr}} = kT 2\epsilon_0/q^2 n\lambda \tag{39}$$

so that for sufficiently large temperature the analysis becomes valid for this problem also.

ACKNOWLEDGMENTS

The author is grateful to Professor P. C. T. deBoer and Professor F. Jelinek for rendering assistance in the preparation of this paper. This research was supported in part by a grant to the College of Engineering of Cornell University by the Ford Foundation, and in part by the Mechanics Branch of the Office of Naval Research under contract Nonr401(25) with the Graduate School of Aerospace Engineering, Cornell University.