values. The agreement of the relative scale of the present work with those of the three previous workers is, therefore, another indication of the internal consistency of the results of the present work.

A perplexing point is the disagreement of the Griffiths and Osherovich results with those of this experiment. Among the results of other workers presented in Table I, only those of Griffiths and Osherovich were obtained by direct lifetime measurements. It is seen, however, that the values of these two workers are much larger than the corresponding values of the present work. Upon examination of the descriptions of their experiments, a point of variance with the present experiment is noted: The minimum lifetimes measured with both the Griffiths and Osherovich experimental setups were much greater than the minimum lifetime measured here (Griffith, 39 nsec, and Osherovich,[21] 33 nsec, as compared with the present work, 5 nsec). Thus, it seems possible that the disagreement in the direct lifetime measurements could arise from the inability of the Griffiths and Osherovich experimental setups to measure short enough lifetimes. However, since relevant experimental details of the two previous experiments are unpublished, the discrepancies remain basically unresolved.

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Note added in proof. Since this manuscript was submitted, delayed-coincidence determinations by Bennett *et al.*[22] of the lifetimes of several $2p$ (Paschen notation) levels in Ne I have come to the attention of the author. The mean lives in nanoseconds given by these workers are as follows: $2p_1$, 15 ± 1; $2p_2$, 19 ± 2; $2p_4$, 19 ± 1; and $2p_6$, 29 ± 12.

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Stochastic Acceleration*

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Charged particles moving under the influence of randomly time-varying electromagnetic fields may be expected to experience a net acceleration. This process is analyzed for the special case of nonrelativistic motion in a static uniform magnetic field and time-varying electric field. Acceleration parallel and transverse to the magnetic field are considered separately. In the weak-field approximation, the motion may be described by a Fokker-Planck equation. The coefficients of this equation are expressible in terms of the correlation functions for the electric fields. In certain cases, the coefficients may be expressed in terms of the energy spectrum of the field. The Fokker-Planck equation derived for motion along the magnetic field is closely related to an equation of the quasilinear theory of plasma instability. One may also show that the equation is closely related to the phenomenon of Landau damping. Longitudinal acceleration is effected by waves with phase velocities slightly greater than the particle velocity. A similar statement is true for transverse acceleration, except that the "resonant" waves are in addition shifted by the particle gyrofrequency. In the absence of any other effects (such as "loading" of the accelerating field by the accelerated particles), the transverse energy distribution tends to a Maxwellian form with a temperature which increases linearly with time. The same is true for longitudinal acceleration if the spectrum of the electric field is flat over the range of phase velocities of interest. The equations related to transverse acceleration show that the high-energy electrons observed in the transition region outside the earth's magnetosphere may have been accelerated by quite weak random or quasirandom electric fields.

I. INTRODUCTION

THERE is good reason to try to understand the mechanisms by which charged particles may be accelerated in random electromagnetic fields. Laboratory experiments have shown that electrons may be accelerated to high energy by the electric field generated by a beam-plasma instability.[1,2] Spacecraft experiments have shown that high-energy electrons are produced in the transition region between the magnetosphere and the bow shock.[3,4] It has been proposed by Bernstein,

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Fredericks, and Scarf[5] that this region is unstable to the growth of ion acoustic waves and that the oscillatory electric fields so generated are responsible for the observed high-energy electrons.[6] The acceleration of charged particles in solar flares[7,8] and other violent astronomical events gives further weight to the problem of particle acceleration.

Stix[9] and Fredericks, Scarf, and Bernstein[10] have considered cyclotron acceleration of electrons in "quasi-random" electric fields—the field being assumed to be coherent for a given length of time and then to suffer a random change in phase. This article will treat what is basically the same problem—namely, the acceleration of charged particles in a magnetic field by random electric fields—except that we shall regard the fluctuations in electric field as a stationary random process to be described by appropriate correlation functions. We shall find that, if we make a "weak-field" approximation, it is possible to describe the acceleration process in terms of the second-order correlation functions only. Hence it is often possible to relate the process to the energy spectrum of the field.

In order to simplify the present calculations, we consider the acceleration of particles to nonrelativistic energies. We assume, furthermore, that the magnetic field is uniform, and that the electric field may be regarded as steady and homogeneous in the statistical sense. A further approximation of the present treatment is that the gyroradius is assumed to be small compared with the transverse scale for variations of electric field.

If the above approximations are accepted, the transverse motion will have no effect on the longitudinal motion so that one may consider longitudinal acceleration (parallel to the magnetic field) without reference to the transverse acceleration. We shall also consider transverse acceleration independently of longitudinal acceleration, but this requires separate justification. One reason for making such a separation is that longitudinal acceleration is effected by low-frequency electric fields whereas the transverse acceleration is effected by electric fields with frequencies comparable to the gyrofrequency.

II. ACCELERATION PARALLEL TO MAGNETIC FIELD

We wish to consider the behavior of a distribution of charged particles in a uniform magnetic field, moving under the influence of random electric fields. We ignore the effect of collisions. If the collective behavior of the particles may be neglected (that is, if we neglect the reaction of the motion of the charged particles on the electric and magnetic fields), we may discuss the behavior of the distribution by considering a "test" particle. Since the distribution is homogeneous and since we are now neglecting motion transverse to the magnetic field, the particles may be appropriately described by the velocity distribution function $f(v,t)$ where v is the component of velocity parallel to the magnetic field. If it can be shown that the change in the velocity Δv in an interval of time Δt is such that $\langle\Delta v\rangle$ and $\langle(\Delta v)^2\rangle$ contain contributions linear in Δt, whereas all higher products have expectation values which vary as a higher power of Δt, the time development of the distribution function may be described by the Fokker-Planck equation[11,12]

$$\frac{\partial f}{\partial t}=-\frac{\partial}{\partial v}\left(\left\langle\frac{\Delta v}{\Delta t}\right\rangle f\right)+\frac{1}{2}\frac{\partial^2}{\partial v^2}\left(\left\langle\frac{(\Delta v)^2}{\Delta t}\right\rangle f\right). \quad (2.1)$$

We shall find that such a description is indeed possible in the "weak-field" limit, in which it is possible to assign a time interval which is long compared with the coherence time of the electric field but short compared with the time scale for change of the distribution function.

If the z coordinate is chosen to be parallel to the magnetic field, the equation of motion may be written as

$$\frac{d^2z}{dt^2}=\frac{dv}{dt}=\frac{q}{m}E_z(z,t). \quad (2.2)$$

We regard the electric field as being weak and accordingly carry out a perturbation analysis in powers of the strength of the electric field. Thus we write the trajectory equation as

$$z=z_0+v_0t+Z^{\rm I}(t)+Z^{\rm II}(t)+\cdots, \quad (2.3)$$

where z_0, v_0 are the position and velocity of the particle at time $t=0$; $Z^{\rm I}(t)$ is the perturbation of the orbit which is linear in the electric field, etc. Thus we assign the initial conditions

$$Z^{\rm I}(0)=Z^{\rm II}(0)=\cdots=\dot{Z}^{\rm I}(0)=\dot{Z}^{\rm II}(0)=\cdots=0. \quad (2.4)$$

On substituting (2.3) into (2.2), we obtain a sequence of equations, of which the first two are as follows:

$$\ddot{Z}^{\rm I}=\frac{q}{m}E_z(z_0+v_0t,\, t), \quad (2.5)$$

$$\ddot{Z}^{\rm II}=\frac{q}{m}Z^{\rm I}E_{z,z}(z_0+v_0t,\, t). \quad (2.6)$$

A suffix following a comma indicates a partial derivative.

We assume that the "expectation value" or "average

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value" of the electric field vanishes,

$$\langle \mathbf{E} \rangle = 0. \tag{2.7}$$

Our assumption that the electric field is steady and homogeneous in the statistical sense implies that the second-order correlation functions may be expressed as

$$R_{\alpha\beta}(\zeta,\tau) = \langle E_\alpha(z,t) E_\beta(z+\zeta,\, t+\tau) \rangle, \tag{2.8}$$

where α, $\beta = x$, y, z. The following notation will be used for the Fourier transforms of the correlation functions:

$$R_{\alpha\beta}(\zeta,\tau) = \int\int dk\, d\omega\, e^{i(k\zeta-\omega\tau)} S_{\alpha\beta}(k,\omega), \tag{2.9}$$

$$S_{\alpha\beta}(k,\omega) = (2\pi)^{-2} \int\int d\zeta\, d\tau\, e^{-i(k\zeta-\omega\tau)} R_{\alpha\beta}(\zeta,\tau). \tag{2.10}$$

Unless otherwise specified, limits of integration will be $-\infty$ to ∞.

If the Fourier transform of $E_z(z,t)$ is written as $\tilde{E}_z(k,\omega)$, an equivalent statement of (2.8) is found to be

$$\langle \tilde{E}_\alpha(k,\omega) \tilde{E}_\beta(k',\omega') \rangle = S_{\alpha\beta}(k,\omega) \delta(k+k') \delta(\omega+\omega'). \tag{2.11}$$

If we now write $W_{\text{II}}{}^T$ for the total energy density of the longitudinal electric field, so that

$$W_{\text{II}}{}^T = \frac{1}{8\pi} \langle E_z{}^2 \rangle, \tag{2.12}$$

we find that this quantity may be expressed as

$$W_{\text{II}}{}^T = \int\int dk\, d\omega\, W_{\text{II}}(k,\omega), \tag{2.13}$$

where

$$W_{\text{II}}(k,\omega) = \frac{1}{8\pi} S_{zz}(k,\omega). \tag{2.14}$$

We now proceed to calculate the coefficients in the Fokker-Planck equation (2.1). We wish to find the contribution to these coefficients which is of lowest order in the electric field, that is, quadratic. For this reason, the second coefficient may be calculated from Z^{I}. Integration of (2.5) gives

$$\dot{Z}^{\text{I}}(t) = \frac{q}{m} \int_0^t dt' E_z(z+vt',\, t'). \tag{2.15}$$

(We now drop the suffix on z and v.) Hence

$$\langle (\dot{Z}^{\text{I}}(t))^2 \rangle = \left(\frac{q}{m}\right)^2 \int_0^t dt' \int_0^t dt'' R_{zz}(v(t'-t''),\, t'-t''). \tag{2.16}$$

If we now assume that t is long compared with the coherence time defined by the function R_{zz}, (2.16) may be approximated by

$$\langle (\dot{Z}^{\text{I}}(t))^2 \rangle = t \left(\frac{q}{m}\right)^2 \int d\tau' R_{zz}(v\tau,\tau). \tag{2.17}$$

Hence we find that the second Fokker-Planck coefficient is expressible as

$$\left\langle \frac{(\Delta v)^2}{\Delta t} \right\rangle = 2\pi \left(\frac{q}{m}\right)^2 \int dk S_{zz}(k,vk). \tag{2.18}$$

In order to calculate the first Fokker-Planck coefficient, it is necessary to evaluate $\dot{Z}^{\text{II}}$. On integrating (2.6), we see that

$$\dot{Z}^{\text{II}}(t) = \frac{q}{m} \int_0^t dt' Z^{\text{I}}(t') E_{z,z}(z+vt',\, t'). \tag{2.19}$$

We find, from (2.5) and the boundary conditions (2.4), that

$$Z^{\text{I}}(t) = \frac{q}{m} \int_0^t dt'(t-t') E_z(z+vt',\, t'). \tag{2.20}$$

We find from (2.19) and (2.20) that

$$\langle \dot{Z}^{\text{II}}(t) \rangle = \frac{q^2}{m^2} \int_0^t dt' \int_0^{t'} dt''(t'-t'') R_{zz,\zeta}(v(t'-t''),\, t'-t''). \tag{2.21}$$

This may be rearranged as

$$\langle \dot{Z}^{\text{II}}(t) \rangle = \frac{q^2}{m^2} \int_0^t dt' \int_0^{t'} d\tau\, \tau R_{zz,\zeta}(v\tau,\tau). \tag{2.22}$$

If one now assumes that t is long compared with the coherence time of the function $R_{zz}(\zeta,\tau)$, (2.22) may be approximated by

$$\langle \dot{Z}^{\text{II}}(t) \rangle = t \frac{q^2}{m^2} \int_0^\infty d\tau\, \tau R_{zz,\zeta}(v\tau,\tau). \tag{2.23}$$

Hence the first Fokker-Planck coefficient may be expressed as

$$\left\langle \frac{\Delta v}{\Delta t} \right\rangle = \frac{1}{2} \frac{q^2}{m^2} \frac{\partial}{\partial v} \int d\tau\, R_{zz}(v\tau,\tau), \tag{2.24}$$

where we have made use of the fact that $R_{zz}(v\tau,\tau)$ is even in τ.

On using (2.9), we see that (2.24) may be rewritten as

$$\left\langle \frac{\Delta v}{\Delta t} \right\rangle = \pi \frac{q^2}{m^2} \frac{\partial}{\partial v} \int dk\, S_{zz}(k,vk). \tag{2.25}$$

We may now make use of (2.18) and (2.25) to obtain

(2.1) in the form

$$\frac{\partial f}{\partial t}=\frac{\partial}{\partial v}\left(D(v)\frac{\partial f}{\partial v}\right), \qquad (2.26)$$

where $D(v)$, the coefficient of "diffusion in velocity space," is given by

$$D=\pi\frac{q^2}{m^2}\int dkS_{zz}(k,vk). \qquad (2.27)$$

This may be re-expressed in terms of the energy spectrum as

$$D=8\pi^2\frac{q^2}{m^2}\int dkW_{\shortparallel}(k,vk). \qquad (2.28)$$

III. CYCLOTRON (TRANSVERSE) ACCELERATION

Particle acceleration may be effected also by electric fields directed transverse to the magnetic field. In this case, one would expect that the most important contribution to acceleration comes from that part of the electric field spectrum which is resonant with the cyclotron motion of the particle. We now consider this acceleration mechanism ("cyclotron" acceleration), by calculation analogous to that of the preceding section.

For present purposes, we concentrate on the equations of motion for the transverse degrees of freedom

$$\begin{aligned}\frac{d^2x}{dt^2}-\Omega\frac{dy}{dt}&=\frac{q}{m}E_x,\\ \frac{d^2y}{dt^2}+\Omega\frac{dx}{dt}&=\frac{q}{m}E_y,\end{aligned} \qquad (3.1)$$

where Ω is the gyrofrequency

$$\Omega=qB/mc. \qquad (3.2)$$

If q is negative, one may preferably reverse the signs attached to Ω in the above equations so that Ω remains positive.

We represent the solution of Eqs. (3.1) by equations

$$\begin{aligned}x&=X+\tfrac{1}{2}(Ae^{-i\Omega t}+A^*e^{i\Omega t}),\\ y&=Y-\tfrac{1}{2}i(Ae^{-i\Omega t}-A^*e^{i\Omega t}),\end{aligned} \qquad (3.3)$$

which are equivalent to

$$\begin{aligned}x&=X+r\cos(\Omega t+\alpha),\\ y&=Y-r\sin(\Omega t+\alpha),\end{aligned} \qquad (3.4)$$

where

$$A=re^{i\alpha}. \qquad (3.5)$$

If the electric field vanishes, X, Y, and A are all constants. If the electric field is nonzero, X, Y, and A are all functions of time, upon which we may impose the conditions

$$\begin{aligned}\dot{X}+\tfrac{1}{2}(\dot{A}e^{-i\Omega t}+\dot{A}^*e^{i\Omega t})&=0,\\ \dot{Y}-\tfrac{1}{2}i(\dot{A}e^{-i\Omega t}-\dot{A}^*e^{i\Omega t})&=0,\end{aligned} \qquad (3.6)$$

so that the velocity components are expressible as

$$\begin{aligned}v_x&=-\tfrac{1}{2}i\Omega(Ae^{-i\Omega t}-A^*e^{i\Omega t}),\\ v_y&=-\tfrac{1}{2}\Omega(Ae^{-i\Omega t}+A^*e^{i\Omega t}).\end{aligned} \qquad (3.7)$$

On substituting (3.7) into (3.1) and using (3.6), we find that

$$\dot{X}=(c/B)E_y,\quad \dot{Y}=-(c/B)E_x, \qquad (3.8)$$

and

$$\dot{A}=i(c/B)e^{i\Omega t}(E_x+iE_y). \qquad (3.9)$$

Equations (3.8) lead to coefficients for spatial diffusion. On making the now familiar assumptions, and on noting that the arguments of E_x, E_y are to be taken as $z+vt$, t, since we are ignoring correlation between the longitudinal motion and transverse motion, we obtain the following formulas:

$$\begin{aligned}\langle(\Delta x)^2/\Delta t\rangle&=2\pi(c^2/B^2)\int dkS_{xx}(k,vk),\\ \langle\Delta x\Delta y/\Delta t\rangle&=-2\pi(c^2/B^2)\int dkS_{xy}(k,vk),\\ \langle(\Delta y)^2/\Delta t\rangle&=2\pi(c^2/B^2)\int dkS_{yy}(k,vk).\end{aligned} \qquad (3.10)$$

The same procedure may be applied to Eq. (3.9), and we obtain the Fokker-Planck coefficient governing diffusion in the "plane" defined by the complex variable A:

$$\begin{aligned}\langle|\Delta A|^2/\Delta t\rangle\\ =2\pi(c^2/B^2)\int dk[S_{xx}(k,vk+\Omega)-iS_{yx}(k,vk+\Omega)\\ +iS_{xy}(k,vk+\Omega)+S_{yy}(k,vk+\Omega)].\end{aligned} \qquad (3.11)$$

The above equations simplify in the special case that the electric field is symmetrical (in the statistical sense) about the direction of the magnetic field. The total energy density of the transverse electric field,

$$W_\perp{}^T=\frac{1}{8\pi}\langle E_x{}^2+E_y{}^2\rangle \qquad (3.12)$$

may then be analyzed as

$$W_\perp{}^T=\int\int dk\,d\omega\,W_\perp(k,\omega), \qquad (3.13)$$

where

$$W_\perp(k,\omega)=\frac{1}{4\pi}S_{xx}(k,\omega)=\frac{1}{4\pi}S_{yy}(k,\omega). \qquad (3.14)$$

If we make the further assumption that right-hand and left-hand circular polarization has equal statistical weight, S_{xy} and S_{yx} vanish, so that Eqs. (3.10) and (3.11) become

$$\langle(\Delta x)^2/\Delta t\rangle=\langle(\Delta y)^2/\Delta t\rangle=8\pi^2(c^2/B^2)\int dkW_\perp(k,vk)\,,$$
$$\langle\Delta x\Delta y/\Delta t\rangle=0\,, \tag{3.15}$$

$$\langle|\Delta A|^2/\Delta t\rangle=16\pi^2(c^2/B^2)\int dkW_\perp(k,\,vk+\Omega)\,. \tag{3.16}$$

It may be more convenient to express the transverse acceleration in terms of the transverse energy instead of the complex altitude A. These quantities are related by

$$w=\tfrac{1}{2}m\Omega^2|A|^2. \tag{3.17}$$

Hence we find that

$$\langle\Delta w/\Delta t\rangle=\Gamma(v)\,,\quad \langle(\Delta w)^2/\Delta t\rangle=2w\Gamma(v)\,, \tag{3.18}$$

where

$$\Gamma(v)=8\pi^2\frac{q^2}{m}\int dkW_\perp(k,\,vk+\Omega)\,. \tag{3.19}$$

We see from (3.18) that the Fokker-Planck equation for the distribution function $F(w,v,t)$ of the transverse energy w becomes

$$\frac{\partial F}{\partial t}=\Gamma\frac{\partial}{\partial w}\left(w\frac{\partial F}{\partial w}\right). \tag{3.20}$$

IV. DISCUSSION

The principal result of Sec. II is Eq. (2.26) which shows that the effect of a random electric field on a distribution of charged particles may be represented by a diffusion equation. This result is closely related to one of the equations of the quasilinear theory of instabilities in plasmas.[13–15] In order to compare the formulas, we may assume that the electric field is composed of waves satisfying a dispersion relation

$$\omega=\Omega(k) \tag{4.1}$$

so that the energy density is expressible as

$$W_{\parallel}(k,\omega)=U_{\parallel}(k)\delta(\omega-\Omega(k))\,. \tag{4.2}$$

Equation (2.28) then becomes

$$D=8\pi^2\frac{q^2}{m^2}\int dk\frac{U_{\parallel}(k)}{|v-d\Omega(k)/dk|}\,. \tag{4.3}$$

This equation is identical with that occurring in the quasilinear theories, except that the group velocity occurs in Eq. (4.3). However, in the theories referred to, assumptions are made which imply that the group velocity is small.

One of the interesting aspects of the comparison between our derivation of (2.26) and the corresponding formula of quasilinear theory is that a certain step in the latter derivation [e.g., Eq. (17) of Ref. 15] hinges upon the assumption that the waves of interest are weakly unstable. No such assumption is made in the present derivation.

It is perhaps worth pointing out that Eq. (2.25) is related to the physical mechanism of Landau damping.[16] It shows that, if the phase velocities of waves are grouped about a particular value, particles going slightly faster than the waves will be decelerated. Hence, the faster particles must give up energy to the waves whereas the slower particles gain energy from the waves. This is the mechanism which was proposed by Dawson[17] as a physical explanation of Landau damping.

One may in fact relate the equations of Sec. II to the formula for Landau damping of plasma oscillations. If one considers a beam of electrons described by a distribution function $f(v,t)$, the rate of gain of energy by the beam is given by

$$\partial W_B/\partial t=\int dv\frac{\partial f}{\partial t}\tfrac{1}{2}mv^2 \tag{4.4}$$

which, by (2.26) and (2.28), may be rewritten as

$$\partial W_B/\partial t=-8\pi^2\frac{q^2}{m}\int dv\,fv\int dkW_{\parallel}(k,vk)\,. \tag{4.5}$$

This may be re-expressed as

$$\partial W_B/\partial t=-8\pi^2\frac{q^2}{m}\int\int dkd\omega\,f'(\omega/k)\frac{\omega}{k|k|}W_{\parallel}(k,\omega)\,. \tag{4.6}$$

The energy acquired by the "beam" (namely, the group of electrons with velocities close to the phase velocity of the wave) must be balanced by a loss of energy from the plasma oscillations. The energy of the plasma oscillations is divided equally between electric field and kinetic energy of the plasma electrons. From this we infer that

$$\partial W_B/\partial t=-2(\partial W_{\parallel}{}^T/\partial t)\,. \tag{4.7}$$

Since our equations hold for any distribution of wave energy among wave number and frequency, consistent only with the dispersion relation for the wave species, we may infer that the superposition principle (equivalent to the procedure of linearization based on the

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"small-amplitude" assumption) leads to

$$(\partial/\partial t)W_{11}(k,\omega)=2\gamma(k,\omega)W_{11}(k,\omega)\,, \tag{4.8}$$

where

$$\gamma=2\pi^2\frac{q^2}{m}\frac{\omega}{k|k|}f'(\omega/k)\,. \tag{4.9}$$

This is the familiar formula for the coefficient of Landau damping of plasma oscillations.

Our method of deriving the Landau-damping coefficient for plasma oscillations suggests that formulas derived for stochastic acceleration may be used to estimate Landau damping of complex wave species.

One may obtain a simple solution of the Eq. (2.26) in the special case that the spectrum of the electric field fluctuations is such that the diffusion coefficient $D(v)$ is constant over the range of particle velocities of interest. (That is, the wave spectrum is, in a sense, that of "white noise.") A particular solution of (2.26) is then

$$f(v,t)=(m/2\pi\mathbf{k}T)^{1/2}e^{-(mv^2/2\mathbf{k}T)}\,, \tag{4.10}$$

where

$$T=(mD/\mathbf{k})t\,. \tag{4.11}$$

That is, the distribution is Maxwellian with a temperature which increases linearly with time. One would expect that a distribution which is initially sharply peaked about zero energy will tend asymptotically to the form (4.10).

In the course of Sec. III, we have derived formulas for spatial diffusion coefficients. These could be incorporated into a Fokker-Planck equation which allows for spatial variations. If, for instance, only the effects represented by Eqs. (3.10) were to be considered, the appropriate equation would be

$$\frac{\partial f}{\partial t}=-\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\left\langle\frac{(\Delta x)^2}{\Delta t}\right\rangle f\right)-\frac{\partial^2}{\partial x\partial y}\left(\left\langle\frac{\Delta x\Delta y}{\Delta t}\right\rangle f\right)-\frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\left\langle\frac{(\Delta y)^2}{\Delta t}\right\rangle f\right). \tag{4.12}$$

These equations represent a slight generalization of formulas derived by Spitzer[18] for the same basic problem.

We now consider Eq. (3.20), representing our principal result for transverse stochastic acceleration. We find that this equation leads asymptotically to a Maxwellian distribution

$$F(w,t)=(1/\mathbf{k}T)e^{-w/\mathbf{k}T}\,, \tag{4.13}$$

where

$$T=(\Gamma/\mathbf{k})t\,, \tag{4.14}$$

since Γ does not depend on w.

As an example of the application of this theory, we may inquire into the possibility that high-energy electrons, observed in the "bow-shock" region between the magnetosphere and the undisturbed solar wind, have been accelerated by the cyclotron stochastic mechanism discussed in Sec. III. We suppose that an electron is accelerated from an initial energy of 1 eV (corresponding to a supposed solar-wind temperature of about 10 000°) to a final energy of 30 keV. If this acceleration takes place while an electron is traversing a region of thickness approximately R_E, the radius of the earth, the time available for this acceleration is approximately 10 sec, assuming that the longitudinal energy is not greatly affected by the transverse acceleration process. Hence, using (4.14), we find that $\Gamma=4\times10^{-9}$ erg sec^{-1}. Hence we find, from (3.19), that the electric field energy per unit (radian) bandwidth is 2×10^{-19} erg cm^{-3} sec. If the electric field energy is in effect distributed over a band of width comparable with the gyrofrequency of an electron, the total energy density will be $W_\perp{}^T=2\times10^{-16}$ erg cm^{-3}, since the gyrofrequency $\Omega=10^3$ sec^{-1} for a typical magnetic field strength of 5γ, i.e., 5×10^{-5} G. This energy density corresponds to an electric field of about 6×10^{-8} esu or about 20 μ V/cm. Hence we see that the observed acceleration can be achieved by quite weak electric fields.

In pursuing the problem of stochastic acceleration, it will be most important to consider the "loading" of the driving electric field by the currents set up by the accelerated particles. In order to close this problem, it would be necessary also to investigate the mechanism for generation of the electric fields. This may be due to instabilities, as suggested by Scarf, Bernstein, and Fredericks[6] and by Stix.[9] Hence an extension of the theory given in this article will be closely related to the quasilinear theory of plasma instabilities.

Nevertheless, it seems to be worthwhile to pursue problems such as that of stochastic acceleration without reference to a particular mechanism for generation of the fluctuating electric fields. Since acceleration, when it occurs, may accelerate particles to relativistic energies, there is need to extend the present treatment to the relativistic regime. This will be undertaken in a later article.

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