SU_4 Mass Formula and Particle Classification Schemes*

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The mathematical derivation of the general mass formula of SU_4 is given. The derivation is preceded by a survey of the mathematical aspects of the group SU_4 and its representations. The possible elementary-particle symmetry schemes based on the Lie algebra of SU$_4$, most of which have already been proposed, are classified and systematically discussed. The extension of SU_4 to SU_8 by inclusion of ordinary spin is discussed and shown to rule out two of the three classes of models. Finally, a simple SU_8 model is sketched.

I. INTRODUCTION

UNLIKE the search for the SU_3 symmetry scheme[1] which was motivated by the desire to unify two well-established quantities, isospin and strangeness, or hypercharge, investigation of consequences of a possible even higher unitary symmetry is purely speculative. It is as if one would have looked for SU_3 at the time when only nucleons and pions were known, i.e., when all known baryons and all known mesons had the same hypercharge (or strangeness; the distinction here is meaningless because of the superselection rule due to the conservation of baryon number). It must be added that only one (or maybe two) observed facts, which we discuss below, suggest that SU_3 is not yet the end of the story.

Nevertheless, since we do not understand either isospin, or hypercharge, or SU_3, or what it means that a symmetry is sometimes broken, it seems perfectly legitimate to speculate further.

The increase in rank from 1 to 2 in generalizing SU_2 to SU_3 suggests the direction in which one might proceed, namely the rank-3 groups. Of the three compact simple groups of rank 3, $SU_4 \cong A_3 \cong D_3$ recommends itself. Firstly, it is the one with the smallest dimension; it has only 15 group parameters as compared to 21 for B_3 and C_3, and therefore it is the most cautious speculation. Secondly, nature's predilection for the unitary groups displayed now twice may be of deeper significance and it is therefore surely worthwhile to catch this hint and to stay in line (in the line of the groups SU_n, that is). Thirdly, the reduction of SU_4 with respect to SU_3 is certainly more natural than the same reduction of $B_3=O_7$ and $C_3=Sp_6$. Since it is particularly striking that SU_3 seems to be broken in the simplest way possible from the mathematical point of view, this too might be taken as a clue. That we do not consider semisimple groups, i.e., direct products, is clear; the second factor not connected to SU_3 would have to be chosen entirely *ad hoc*.

In the following we present (in Sec. III) the derivation of the general mass formula[2] of SU_4 after first reviewing the mathematical aspects of the group SU_4 and its representations (Sec. II). The various possible elementary-particle symmetry schemes based on the Lie algebra of SU_4, most of which have already been proposed,[3] are classified and systematically discussed in Sec. IV.

Finally, in Sec. V, we briefly discuss the inclusion of ordinary spin, which leads us from $SU_4 \otimes SU_2$ to SU_8 and we sketch a simple model.

II. MATHEMATICAL PRELIMINARIES

A. SU_4 and Its Irreducible Representations

The elements of the group[4] SU_4 depend upon 15 parameters. Its rank is 3, that is, 3 members of the Lie algebra can be diagonalized simultaneously or what is the same, maximal toroids (the manifold generated by 3 members of the Lie algebra and linear combinations) are of dimension 3. Physically this means that there is another quantum number besides I_3 and Y which will be conserved to the extent that the SU_4 symmetry is valid. The irreducible representations are labelled accordingly by three non-negative integers[5] $(\Lambda_1,\Lambda_2,\Lambda_3)$, $\Lambda_i=0, 1, 2, \cdots$. The dimension of the representation

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[1] M. Gell-Mann, California Institute of Technology Synchrotron Laboratory Report No. CTSL–20 (unpublished); Phys. Rev. **125**, 1067 (1962); Y. Ne'eman, Nucl. Phys. **26**, 222 (1961); D. Speiser and J. Tarski, Institute for Advanced Study, Princeton, 1961 (unpublished), and J. Math. Phys. **4**, 588 (1963).

[2] R. J. Oakes and D. Speiser, Phys. Rev. Letters **13**, 579 (1964).

[3] P. Tarjanne and V. Teplitz, Phys. Rev. Letters **11**, 447 (1963); Y. Hara, Phys. Rev. **134**, B701 (1964); B. J. Bjorken and S. L. Glashow, Phys. Letters **11**, 255 (1964); D. Amati, H. Bacry, J. Nuyts, and J. Prentki, Nuovo Cimento **34**, 1732 (1964); I. S. Gerstein and M. L. Whippman, Phys. Rev. **136**, B829 (1964); P. Tarjanne, *ibid.* **136**, B1532 (1964); M. Hama, Progr. Theoret. Phys. (Kyoto) **32**, 411 (1964); R. Baier and P. Urban (to be published).

[4] J.-P. Antoine and D. Speiser, J. Math. Phys. **5**, 1226, 1560 (1964).

[5] Here we follow the usual convention, but remark that it is not the most useful for all purposes; in Ref. 4, for example, strictly positive integers $p_i=\Lambda_i+1$ are used.

$(\Lambda_1,\Lambda_2,\Lambda_3)$ is then[4,6]

$$D(\Lambda)=\tfrac{1}{12}(\Lambda_1+1)(\Lambda_2+1)(\Lambda_3+1)(\Lambda_1+\Lambda_2+2)(\Lambda_2+\Lambda_3+2)\times(\Lambda_1+\Lambda_2+\Lambda_3+3). \quad (1)$$

The unit representation is (0,0,0), and the 3 fundamental representations from which all other ones can be built up are (1,0,0), (0,1,0), and (0,0,1) as indicated in Table I. The adjoint (regular) representation is (1,0,1) and is of dimension 15. The characters of these representations have been constructed by Wigner in his celebrated paper.[7] Wigner however was interested in the group as viewed as O_6 rather than as SU_4 and his presentation of the weight diagrams therefore shows the reduction with respect to $O_4(\cong D_2)$ rather than with respect to SU_3. However, it is easy to obtain the desired reduction starting from Wigner's work.

The representations of SU_4 fall into one of 4 classes according to whether[4,7a]

$$C=\Lambda_1+2\Lambda_2+3\Lambda_3 \quad (2)$$

has the value 0, 1, 2, or 3 (mod 4). These four classes form the cyclic group of order 4 with respect to formation of the direct product. Class 0 ($C=0$) is the neutral element, etc. The complex conjugate of each representation belonging to class 1 will belong to class 3. These representations are simply connected and have four elements in the center. Representations of class 2 faithfully represent the group of rotations in a 6-dimensional Euclidean space $O_6\simeq SU_4/Z_2$.[8] They are doubly connected and have two elements in the center. Finally, the representations of class 0 have only the identity in the center and are quadruply connected. They faithfully represent the adjoint group, SU_4/Z_4. The representation $(\Lambda_1,\Lambda_2,\Lambda_3)$ is the complex conjugate of the representation $(\Lambda_3,\Lambda_2,\Lambda_1)$. The representations $(\Lambda_1,\Lambda_2,\Lambda_1)$ are real and belong to either class 0 or class 2. There are no symplectic representations. The representations are conveniently listed in a 3-dimensional lattice (Cartan-Stiefel diagram[4]) which is shown in Fig. 1.

TABLE I. Fundamental representations of SU_4.

Representation	Dimensions	Weight diagram	Roots
(1,0,0)	4	tetrahedron	complex
(0,1,0)	6	octahedron	real
(0,0,1)	4	tetrahedron	complex

[6] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, Inc., New York, 1950).

[7] E. P. Wigner, Phys. Rev. **51**, 106 (1937).

[7a] This fact has been known for a long time; it was already implicit in the work of Weyl and Stiefel. The quantity C (see also Appendix A) has been called "plurality type" by C. R. Hagen and A. J. Macfarlane, J. Math. Phys. **5**, 1335 (1964), where earlier references can be found. However, we feel it more appropriate to speak of "classes" of representations (in the sense of equivalence classes or cosets).

[8] We denote by Z_n the discrete group composed of the n roots $1^{1/n}$.

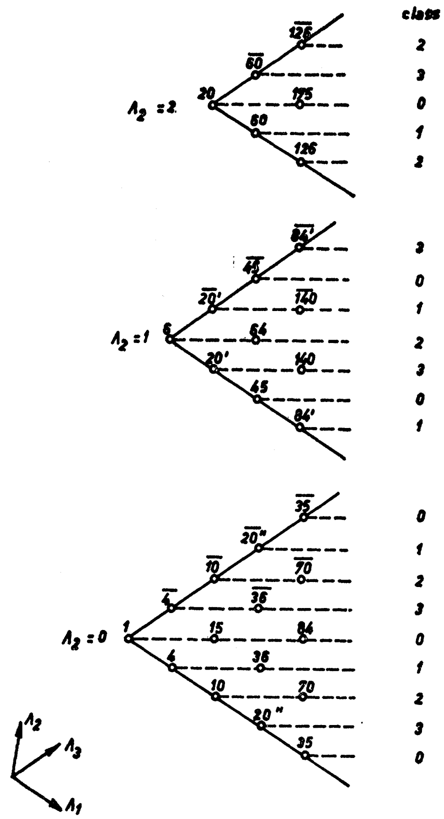

FIG. 1. SU_4 Cartan-Stiefel diagram.

In view of the above classification of the representations one must distinguish between three different symmetry groups: SU_4/Z_4, SU_4/Z_2, and SU_4.

(i) SU_4/Z_4. Only representations of class 0 are used.
(ii) SU_4/Z_2. Representations of class 0 and class 2 occur.
(iii) SU_4. All representations are used.[9]

This distinction is the analog of the well-known distinction between SU_3/Z_3 (eightfold way) and SU_3 (Sakata model); however, since 4 is not a prime number, more than two possibilities occur. Nevertheless, it is clear that the algebra above does not completely determine the model. The various possibilities will be discussed below.

B. Decomposition of Representations of SU_4 into Representations of SU_3

If we want to take into account the effects due to interactions which do not respect the full SU_4 symmetry, but preserve only SU_3, we must decompose the representations of SU_4 into representations of SU_3. This decomposition is the exact analog of the (Y,I) content of a SU_3 multiplet, and it can be obtained using the

[9] In this case it is not the same whether the baryons transform as a representation of class 1 or class 3.

well-known branching rule of Weyl.[6] Here we want to translate this rule and to express the decomposition of the character into our geometrical language. It then says: in the Cartan-Stiefel diagram (and in the weight diagram as well) one axis, which we call X, is distinguished. The SU_4 weight diagrams then are decomposed ("sliced") into sheets perpendicular to this axis X, and the weights on every sheet form one or several weight diagrams, i.e., representations of SU_3. Clearly X represents the new quantum number occurring in the SU_4 theory, and plays there the same role as the hypercharge does in SU_3.

We consider, as usual,[4] a 4-dimensional Euclidean space referred to an orthonormal basis e_i $(i=1\cdots 4)$ with coordinates f_i. The nonzero roots of SU_4 are the vectors $e_{ij}\equiv e_i-e_j(1\leq i,\, j\leq 4)$, contained in the hyperplanes $\sum_i f_i=0$. We take as roots of the SU_3 subgroup the vectors $e_{ij}(1\leq i,\, j\leq 3)$ so that the X axis will be in the direction of the vector $(e_{14}+e_{24}+e_{34})$; in order to fix the scale, we decide to take for the X coordinate the value

$$X=\tfrac{1}{4}\sum_{i=1}^{3}(f_i-f_4)=-f_4.$$

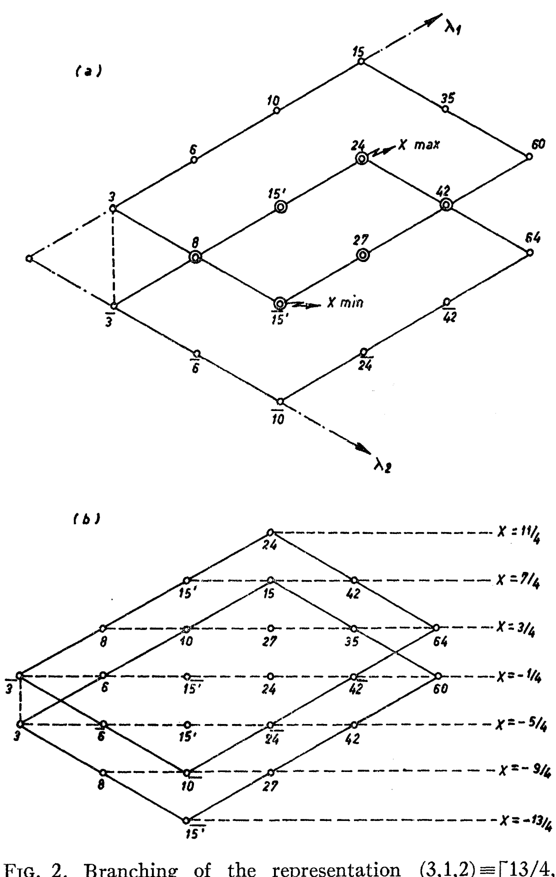

FIG. 2. Branching of the representation $(3,1,2)\equiv[13/4, \frac{1}{4}, -\frac{3}{4}, -11/4]$ of SU_4, of dimension 540: (a) the parallelograms in the SU_3 Cartan-Stiefel diagram; (b) the "interlaced" parallelograms.

We can now state the desired branching rule, together with the value of X for the different SU_3 multiplets.

Theorem. The SU_4 representation $(\Lambda_1,\Lambda_2,\Lambda_3)$ decomposes into the SU_3 representations (λ_1,λ_2) in the following way. Starting from every point on the line $\lambda_1+\lambda_2=\Lambda_2$ in the SU_3 Cartan-Stiefel diagram, draw on the lattice a parallelogram with edges of length Λ_1 and Λ_3 along the axes λ_1 and λ_2, respectively. All representations belonging to any lattice point of one of those (Λ_2+1) parallelograms are contained in $(\Lambda_1,\Lambda_2,\Lambda_3)$. It is then clear that the number of SU_3 representations contained in $(\Lambda_1,\Lambda_2,\Lambda_3)$ is $(\Lambda_1+1)(\Lambda_2+1)(\Lambda_3+1)$.

Furthermore, if the SU_3 representation (λ_1,λ_2) lies in the parallelogram starting at the point (λ_1',λ_2') the value of its X coordinate is given by

$$X(\lambda_1,\lambda_2)=(\lambda_1-\lambda_2)-\tfrac{3}{2}(\lambda_1'-\lambda_2')-\tfrac{3}{4}(\Lambda_1-\Lambda_3). \quad (3)$$

The proof of this theorem is elementary, but involves some tedious geometrical considerations; it is therefore postponed to Appendix A.

For the applications, we want to obtain the complete decomposition together with the spatial position of the SU_3 multiplets in the weight diagram. This is possible at once, if we perform the graphical construction suggested by the theorem (see Appendix A); we merely reproduce the (Λ_2+1) parallelograms *in the reverse order and "interlaced"* in the following way: first the upper one, starting at $(\Lambda_2,0)$; then the second one, starting at $(\Lambda_2-1,\, 1)$, is to be drawn one step *above*, so that the lattice line containing this point $(\Lambda_2-1,\, 1)$ falls onto the second line of the first parallelogram; then the third parallelogram comes two steps above the first one, and so on. The resulting figure contains all the SU_3 multiplets, each on its proper plane $X=$constant. This construction can be seen better on an example. Figure 2 shows the complete decomposition of the representation (3,1,2), of dimension 540.

According to this rule, the adjoint representation of SU_4, the **15**, will reduce into the SU_3 representations $\mathbf{3}+\mathbf{1}+\mathbf{8}+\bar{\mathbf{3}}$, as shown in Fig. 3. The decompositions of the low-dimensional representations are listed in Tables II–IV. The decomposition of the representations which are conjugate to those listed in the tables are simply obtained by changing the sign of X and conjugating

TABLE II. Branching of the class 0 $(\Lambda_1+2\Lambda_2+3\Lambda_3\equiv 0)$ representations of SU_4 into representations of SU_3.

Representation $(\Lambda_1,\Lambda_2,\Lambda_3)$	Dimension $D(\Lambda)$	SU_3 decomposition $X=2$	$X=1$	$X=0$	$X=-1$	$X=-2$	$X=-3$
(0,0,0)	1			1			
(1,0,1)	15		3	1, 8	$\bar{3}$		
(0,2,0)	20		$\bar{6}$	8	6		
(4,0,0)	35		15	10	6	3	1
(2,1,0)	45		15′	8, 10	$\bar{3}$, 6	3	
(2,0,2)	84	6	3, 15′	1, 8, 27	$\bar{3}$, $\overline{15}'$	6	

TABLE III. Branching of the class 1 ($\Lambda_1+2\Lambda_2+3\Lambda_3\equiv1$) representations of SU_4 into representations of SU_3.

Representation $(\Lambda_1,\Lambda_2,\Lambda_3)$	Dimension $D(\Lambda)$	SU_3 decomposition $X=9/4$	$X=5/4$	$X=1/4$	$X=-3/4$	$X=-7/4$	$X=-11/4$	$X=-15/4$
(1,0,0)	4			3	1			
(0,1,1)	$\overline{20}'$		$\overline{3}$	3, $\overline{6}$	8			
(0,0,3)	$\overline{20}''$	1	$\overline{3}$	$\overline{6}$	$\overline{10}$			
(2,0,1)	36		6	3, 15′	1, 8	$\overline{3}$		
(5,0,0)	56		21	15	10	6	3	1
(1,2,0)	60		15′	$\overline{6}$, 15′	8, 10	6		
(3,1,0)	84′		24	15, 15′	8, 10	$\overline{3}$, 6	3	

the SU_3 multiplets, and therefore are not listed separately. Also, the class 3 representations are just those conjugate to the class 1 representation, as noted above, and are not repeated in the tables.

Finally a word must be said about the unique labeling of states within a particular SU_4 multiplet. Because of the appearance of multiple weights, one requires the Casimir operators of subgroups as in the case of SU_3, in addition to I_3, Y, and X in order to uniquely specify the states. We choose $\mathbf{I}^2$, i.e., the Casimir-Racah operator C_2 of the isospin subgroup, and the two Casimir-Racah operators C_2 and C_3 of the SU_3 subgroup containing I_3 and Y. One can verify that all the six operators $X, Y, I_3, \mathbf{I}^2, C_2, C_3$ commute with each other. Furthermore, from the decomposition explained above one sees that they are a complete set of commuting observables in the space of the internal degrees of freedom.

C. Kronecker Products of Irreducible Representations of SU_4

The Kronecker product of two irreducible representations of SU_4 can be reduced to a sum of irreducible representations using the standard techniques.[6,10] The product of an arbitrary representation $(\Lambda_1,\Lambda_2,\Lambda_3)$ with another representation can be obtained directly or deduced from products of $(\Lambda_1,\Lambda_2,\Lambda_3)$ with the three fundamental representations (1,0,0), (0,1,0), and (0,0,1). These latter products, which can be easily obtained geometrically,[10] are the following:

$$(\Lambda_1,\Lambda_2,\Lambda_3)\otimes(1,0,0) = (\Lambda_1+1,\Lambda_2,\Lambda_3)\oplus(\Lambda_1-1,\Lambda_2+1,\Lambda_3) \oplus(\Lambda_1,\Lambda_2-1,\Lambda_3+1)\oplus(\Lambda_1,\Lambda_2,\Lambda_3-1) \quad (4a)$$

$$(\Lambda_1,\Lambda_2,\Lambda_3)\otimes(0,1,0) = (\Lambda_1,\Lambda_2+1,\Lambda_3)\oplus(\Lambda_1,\Lambda_2-1,\Lambda_3)\oplus(\Lambda_1+1,\Lambda_2,\Lambda_3-1) \oplus(\Lambda_1-1,\Lambda_2,\Lambda_3+1)\oplus(\Lambda_1+1,\Lambda_2-1,\Lambda_3+1) \oplus(\Lambda_1-1,\Lambda_2+1,\Lambda_3-1) \quad (4b)$$

$$(\Lambda_1,\Lambda_2,\Lambda_3)\otimes(0,0,1) = (\Lambda_1,\Lambda_2,\Lambda_3+1)\oplus(\Lambda_1,\Lambda_2+1,\Lambda_3-1) \oplus(\Lambda_1+1,\Lambda_2-1,\Lambda_3)\oplus(\Lambda_1-1,\Lambda_2,\Lambda_3). \quad (4c)$$

In Eqs. (4) it is to be understood that terms on the right-hand side for which any of the integers are negative are to be omitted.

For convenience we have listed the reductions of the Kronecker products of several of the low-dimensional representations in Table V.

TABLE IV. Branching of the class 2 ($\Lambda_1+2\Lambda_2+3\Lambda_3\equiv2$) representations of SU_4 into representations of SU_3.

Representation $(\Lambda_1,\Lambda_2,\Lambda_2)$	Dimension $D(\Lambda)$	SU_3 decomposition $X=\frac{3}{2}$	$X=\frac{1}{2}$	$X=-\frac{1}{2}$	$X=-\frac{3}{2}$	$X=-\frac{5}{2}$
(0,1,0)	6		$\overline{3}$	3		
(2,0,0)	10		6	3	1	
(0,3,0)	50	$\overline{10}$	$\overline{15}'$	15′	10	
(1,1,1)	64	8	$\overline{3}$, 6, $\overline{15}'$	3, $\overline{6}$, $\overline{15}'$	8	
(3,0,1)	70	10	6, 24	3, 15′	1, 8	$\overline{3}$

III. SU_4 MASS FORMULA

It is an empirical fact (that is not understood) that the mass splittings within an SU_3 multiplet to a good approximation can be represented as matrix elements of an operator which transforms like $Y=0$, SU_2 singlet component of the 8-dimensional representation of SU_3,

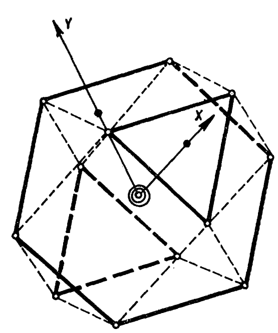

FIG. 3. Branching of the adjoint representation **15** of SU_4 into the SU_3 representations $3\oplus1\oplus8\oplus\overline{3}$.

[10] D. Speiser, in *Group Theoretical Concepts and Methods in Elementary Particle Physics* (Istanbul Summer School, 1962), edited by F. Gürsey (Gordon and Breach Science Publishers, Inc., New York, 1964); Helv. Phys. Acta **38**, 73 (1965); J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963).

TABLE V. Reductions of the Kronecker products of some low-dimensional representations of SU_4.

$4\otimes\bar{4}=1\oplus 15$
$4\otimes 4=6\oplus 10$
$4\otimes 6=\bar{4}\oplus 20'$
$4\otimes 10=20'\oplus 20''$
$4\otimes 4\otimes 4=\bar{4}\oplus 20'\oplus 20'\oplus 20''$
$4\otimes 4\otimes 4\otimes 4=1\oplus 15\oplus 15\oplus 15\oplus 20\oplus 20\oplus 35\oplus 45\oplus 45\oplus 45$
$4\otimes 15=4\oplus\overline{20'}\oplus 36$
$15\otimes 15=1\oplus 15\oplus 15\oplus 20\oplus 45\oplus\overline{45}\oplus 84$
$15\otimes 20=15\oplus 20\oplus 45\oplus\overline{45}\oplus 175$
$15\otimes 20'=\bar{4}\oplus 20'\oplus 20'\oplus 20''\oplus\overline{36}\oplus\overline{60}\oplus 140$
$15\otimes 35=35\oplus 45\oplus 189\oplus 256$
$15\otimes 64=6\oplus 10\oplus\overline{10}\oplus 50\oplus 64\oplus 64\oplus 64\oplus 70\oplus\overline{70}\oplus 126\oplus\overline{126}\oplus 256$
$15\otimes 70=10\oplus 64\oplus 70\oplus 70\oplus 126\oplus 140\oplus 270\oplus 300$
$20\otimes 20=1\oplus 15\oplus 20\oplus 84\oplus 105\oplus 175$

i.e., the adjoint (regular) representation of SU_3. It is then a natural first guess to assume that the SU_4 symmetry will be broken in an analogous manner. That is, that the SU_4 mass splittings transform like the $X=0$, SU_3 singlet component of the 15-dimensional (adjoint) representation of SU_4. Under this assumption it is a straightforward mathematical problem to obtain the SU_4 mass formula which is the analog of Okubo's SU_3 mass formula.[11] (Of course, the physical basis of such a mass formula remains as obscure as ever.) More generally, one might entertain the weaker assumption that the SU_4 mass splittings transform like the $X=0$, SU_3 singlet component of other representations of SU_4. Components with this covariance property occur only in the representations $(\Lambda,0,\Lambda)$ which are all real and belong to class 0. They include the **1**, **15**, **84**, etc. There is no reason to exclude these possibilities *a priori*. We shall consider here only the "obvious" generalization of the SU_3 case and presume that the mass splittings within the SU_4 multiplets can be represented by the matrix elements of an operator which transforms like a component of the 15-dimensional (adjoint) representation of SU_4. To include the mass splittings within the SU_3 multiplets as well as between different SU_3 multiplets we must consider all the $X=Y=0$, SU_2 singlet components of the **15**. There are two such components and assuming the symmetry-breaking interactions conserve X, Y, and **I** it is sufficient to consider only these two operators.

The components of the **15** are represented by a tensor $\psi_\mu{}^\nu$ having one upper and one lower index. These indices range from 1 to 4 and their contraction vanishes. In other words, the **15** is represented by a traceless 4×4 matrix. The diagonal elements all have $X=Y=I_3=0$ and two of them, $\psi_3{}^3$ and $\psi_4{}^4$, are SU_2 (isospin) singlets while the third independent element $(\psi_2{}^2-\psi_1{}^1)$ transforms like the $I_3=0$ component of an isospin vector. Consequently, the mass splittings must transform like a superposition of the matrix elements of $\psi_3{}^3$ and $\psi_4{}^4$ if the electromagnetic splittings are neglected. The electromagnetic splittings, which can be taken to transform like $\psi_1{}^1$, can easily be included simply by applying the results obtained below for $\psi_3{}^3$ and $\psi_4{}^4$ to the component $\psi_1{}^1$ as will be discussed later.

Let us first consider the general form of any tensor $T_\mu{}^\nu$. We can express the matrix elements of $T_\mu{}^\nu$ in terms of the generators of SU_4 which we denote by $F_\mu{}^\nu$. These satisfy the following:

$$[F_\alpha{}^\beta,F_\mu{}^\nu]=F_\alpha{}^\nu\delta_\mu{}^\beta-F_\mu{}^\beta\delta_\alpha{}^\nu\,, \tag{5a}$$

$$F_\mu{}^{\nu\dagger}=F_\nu{}^\mu\,, \tag{5b}$$

$$F_\mu{}^\mu=0\,. \tag{5c}$$

The most general form of $T_\mu{}^\nu$ can be shown by the same argument as Okubo's[11] to be

$$\begin{aligned}T_\mu{}^\nu=a\delta_\mu{}^\nu+bF_\mu{}^\nu+c(F_\mu{}^\alpha F_\alpha{}^\nu+F_\alpha{}^\nu F_\mu{}^\alpha-\tfrac{1}{2}\delta_\mu{}^\nu F_\alpha{}^\beta F_\beta{}^\alpha)\\+d(F_\alpha{}^\beta F_\mu{}^\alpha F_\beta{}^\nu+F_\beta{}^\nu F_\alpha{}^\beta F_\mu{}^\alpha+F_\mu{}^\alpha F_\beta{}^\nu F_\alpha{}^\beta\\-\tfrac{3}{4}\delta_\mu{}^\nu F_\alpha{}^\gamma F_\beta{}^\alpha F_\gamma{}^\beta+F_\alpha{}^\beta F_\beta{}^\nu F_\mu{}^\alpha+F_\beta{}^\nu F_\mu{}^\alpha F_\alpha{}^\beta\\+F_\mu{}^\alpha F_\alpha{}^\beta F_\beta{}^\nu-\tfrac{3}{4}\delta_\mu{}^\nu F_\alpha{}^\gamma F_\gamma{}^\beta F_\beta{}^\alpha)\,,\end{aligned} \tag{6}$$

where it is to be understood that Eq. (6) stands between states of a given irreducible representation of SU_4. The symmetric combinations of generators have been chosen merely to simplify the calculation. To write Eq. (6) in terms of the more familiar quantum numbers Q, I_3, and X we make the identifications of these quantum numbers with the eigenvalues f_α of the diagonal components $F_\alpha{}^\alpha(\alpha=1,2,3,\text{ or }4)$ as follows (these are precisely the f_α used in Sec. II B):

$$I_3=\tfrac{1}{2}(f_1-f_2)\,, \tag{7a}$$

$$Y=-f_3\,, \tag{7b}$$

$$X=-f_4\,. \tag{7c}$$

Here the hypercharge $Y=-f_3$ is *not* quite the same as the usual hypercharge, insofar as the direction of Y does *not* lie in the plane $X=0$. Actually, $\tilde{Y}=Y+\frac{1}{3}X$ is parallel to the plane $X=0$ and therefore orthogonal to the direction of X. It will also prove quite useful to introduce the Casimir operators C_1, C_2, and C_3 of the U_3 subgroup whose multiplets lie in planes orthogonal to the direction of the new quantum number X. These are conveniently taken to be

$$C_1=F_i{}^i\,, \tag{8a}$$

$$C_2=F_i{}^jF_j{}^i\,, \tag{8b}$$

$$C_3=F_i{}^kF_j{}^iF_k{}^j+F_i{}^kF_k{}^jF_j{}^i\,, \tag{8c}$$

[11] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962); M. Gell-Mann, Ref. 1. This mass sum rule is perfectly general: in the reduction $SU_{l+1}\to SU_l$, the adjoint representation of SU_{l+1} is decomposed as $\text{Adj}^{(l+1)}\to\text{Adj}^{(l)}\oplus 1^{(l)}\oplus D^{(l)}\oplus\bar{D}^{(l)}$, where $D^{(l)}$ is the elementary representation $l\equiv(1,0\cdots 0)$ of SU_l, $\bar{D}^{(l)}$ its conjugate. Then, if the symmetry breaking term belongs to $\text{Adj}^{(l+1)}$, we have the mass sum rule:

$$lM_{D^{(l)}}+lM_{\bar{D}^{(l)}}=(l+1)M_{1^{(l)}}+(l-1)M_{\text{adj}^{(l)}}.$$

where i, j, and k are summed from 1 to 3. More explicitly, for a U_3 submultiplet[12] $(\frac{1}{3}(2\lambda_1+\lambda_2+X), \frac{1}{3}(\lambda_2-\lambda_1+X), -\frac{1}{3}(\lambda_1+2\lambda_2-X))$ which corresponds to the SU_3 multiplet (λ_1,λ_2) that lies in the plane $f_4=-X$, one finds

$$C_1=+X\,, \tag{9a}$$

$$C_2=\tfrac{1}{3}C_1^2+\tfrac{2}{3}(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2+3\lambda_1+3\lambda_2)\,, \tag{9b}$$

$$C_3=(2/9)(\lambda_1-\lambda_2)(2\lambda_1+\lambda_2+3)(\lambda_1+2\lambda_2+3) -2C_1C_2+(4/9)C_1^3\,. \tag{9c}$$

It is now a straightforward but tedious matter to show that

$$\{F_4{}^\alpha,F_\alpha{}^4\}-\tfrac{1}{2}F_\beta{}^\alpha F_\alpha{}^\beta=C_1^2-C_2+\tfrac{1}{2}\hat{C}_2 \tag{10}$$

and

$$F_\alpha{}^\beta\{F_4{}^\alpha,F_\beta{}^4\}+F_4{}^\alpha\{F_\beta{}^4,F_\alpha{}^\beta\} +F_\beta{}^4\{F_4{}^\beta,F_\beta{}^\alpha\}-\tfrac{3}{4}F_\alpha{}^\gamma\{F_\beta{}^\alpha,F_\gamma{}^\beta\} =-C_1^3+3C_1C_2-C_3-3C_1\hat{C}_2+2C_1+\tfrac{1}{4}\hat{C}_3\,. \tag{11}$$

Here $\hat{C}_n$ are the Casimir operators of U_4 which are defined by simply extending the summations from 1 to 4 in Eqs. (8). Since the $\hat{C}_n$ are just constants for any SU_4 multiplet they do not contribute to the mass splittings and can be dropped. The most general form of the matrix element of $T_4{}^4$ in a given irreducible representation of SU_4 is then of the form

$$T_4{}^4=a+bX+c(C_2-X^2)+d(C_3-3XC_2+X^3)\,, \tag{12}$$

where C_2 and C_3 refer to the U_3 multiplets orthogonal to the direction of X. The parameters a, b, c, and d are undetermined constants which depend on the particular SU_4 multiplet.

Not all of the terms in Eq. (12) are independent for certain representations of SU_4. For any representation of SU_4 the number of independent terms in the matrix element of an operator that transforms like a component of the **15** is just the number of times the **15** appears in the product of the representation with its complete conjugate. This number is given by the following *theorem:*

The Kronecker product $(1,0,1)\otimes(\Lambda_1,\Lambda_2,\Lambda_3)$ contains the representation $(\Lambda_1,\Lambda_2,\Lambda_3)$ precisely as many times as the number of integers Λ_1, Λ_2, and Λ_3 which are nonzero. This can be proved by computing explicitly the direct product with help of formulas (4), or by the geometrical method as indicated in Appendix B.

To obtain the general SU_4 mass formula[13] we simply superpose the splittings which transform like $T_3{}^3$ and $T_4{}^4$.[14] The latter are given explicitly by Eq. (12) from which the analogous formula for $T_3{}^3$ can be obtained by simply permuting the variables. That is, one merely interchanges X and Y and reinterprets the Casimir operators C_2 and C_3 as referring to U_3 multiplets which lie in planes orthogonal to the direction of Y rather than X. The general mass formula for an arbitrary representation of SU_4 is then[2]

$$\begin{aligned}M=M_0+M_1{}^{(x)}X+M_2{}^{(x)}(C_2{}^{(x)}-X^2)\\ +M_3{}^{(x)}(C_3{}^{(x)}-3XC_2{}^{(x)}+X^3)\\ +M_1{}^{(y)}Y+M_2{}^{(y)}(C_2{}^{(y)}-Y^2)\\ +M_3{}^{(y)}(C_3{}^{(y)}-3YC_2{}^{(y)}+Y^3)\,,\end{aligned} \tag{13}$$

where the superscripts on the Casimir operators indicate the appropriate U_3 subgroup. Actually, Eq. (13) also holds if the U_3 Casimir operators C_n are replaced, respectively, by the simpler SU_3 Casimir operators C_n'. For the SU_3 representations (λ_1,λ_2) the operators C_n' have the values

$$C_1'=0\,, \tag{14a}$$

$$C_2'=\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2+3\lambda_1+3\lambda_2\,, \tag{14b}$$

and

$$C_3'=(\lambda_1-\lambda_2)(2\lambda_1+\lambda_2+3)(\lambda_1+2\lambda_1+3)\,. \tag{14c}$$

The mass formula can alternatively be derived from the preceding theorem in a quite simple way.[2] First we write down the following ansatz which contains all terms up to the third order:

$$M=a+bX+c(C_2'+\alpha X^2)+d(C_3'+\beta XC_2'+\gamma X^3)\,. \tag{15}$$

Since the preceding theorem shows that the representation $(\Lambda,0,0)$ will be split into equally spaced representations of SU_3, we can insert the values for C_2', C_3', and $X=\lambda_1-\frac{3}{4}\Lambda$ into this formula and require that all nonlinear terms in λ_1 vanish. Then the constants α, β, and γ are determined to be those of Eq. (12).

In applying the SU_4 mass formula care must be taken since eigenstates of $C_2{}^{(x)}$ and $C_3{}^{(x)}$ are not necessarily eigenstates of $C_2{}^{(y)}$ and $C_3{}^{(y)}$. For example, in the center $(X=Y=I_3=0)$ of the **15** let $\psi_1{}^{(x)}$ and $\psi_8{}^{(x)}$ be, respectively, the SU_3 singlet and octet components with respect to the SU_3 subgroup whose multiplets are orthogonal to the direction of X. Then the linear combination

$$\begin{aligned}\psi_1{}^{(y)}=\tfrac{1}{3}\psi_1{}^{(x)}-\tfrac{2}{3}\sqrt{2}\psi_8{}^{(x)}\,,\\ \psi_8{}^{(y)}=\tfrac{1}{3}\psi_8{}^{(x)}+\tfrac{2}{3}\sqrt{2}\psi_1{}^{(x)}\,,\end{aligned} \tag{16}$$

are, respectively, the SU_3 singlet and octet components with respect to the SU_3 subgroup whose multiplets are orthogonal to the direction of Y. This gives singlet-octet mixing in a definite manner, as well as a transition moment, and shows that the physical particles, i.e., the eigenstates of the mass operator, are not pure eigen-

[12] U_3 multiplets are designated by their highest weight vector (f_1,f_2,f_3) where $\lambda_1=f_1-f_2\geqslant 0$, $\lambda_2=f_2-f_3\geqslant 0$.

[13] We neglect here cross terms of the form XY, etc. With these terms added the mass formula would contain too many parameters, and would be useless (except in very peculiar cases). The same thing occurs in the SU_3 theory with simultaneous Y and Q breaking. See, for example, M. Gourdin, Ergeb. Exakt. Naturw. **36**, 1 (1964).

[14] It would seem more physical to break the symmetry along X and $\tilde{Y}=Y+\frac{1}{3}X$, which is the "true" hypercharge. But this would give a much less symmetrical formula and in fact amounts to the same.

states of either of the Casimir operators. The degree of mixing is determined by diagonalizing the mass matrix and is dependent upon the phenomenological parameters appearing in the mass formula [Eq. (13)].

Before proceeding to discuss some specific models based on SU_4 symmetry, we conclude this section with some general remarks about the SU_4 mass formula.

(i) For the tetrahedral and octahedral representations of SU_4, which are denoted by $(\Lambda,0,0)$ or $(0,0,\Lambda)$, and $(0,\Lambda,0)$, respectively, the mass formula has only linear terms, i.e., $M=M_0+M_xX+M_yY$, as a consequence of the above theorem. [In fact, it is easy to see that

$$(\Lambda,0,0)\otimes(0,0,\Lambda)=(\Lambda,0,\Lambda)\oplus(\Lambda-1,\,0,\,\Lambda-1)\oplus\cdots\oplus(0,0,0).] \quad (17)$$

These are then completely analogous to the equally spaced triangular representations of SU_3. (They are the only representations of SU_4 which have exactly one SU_3 multiplet on each X plane; this is obvious from the above branching rule.)

(ii) Since the TCP theorem excludes terms which are different for particle and antiparticle there can be no linear terms in the mass formula for bosons. Therefore the masses of bosons belonging to an SU_4 representation of the form $(0,\Lambda,0)$ would not be split except in higher order.

(iii) If we consider only splittings along one direction, say X, then the orthogonal SU_3 multiplets are split apart but there are no splittings with these SU_3 submultiplets. For the **15** which contains the SU_3 submultiplets **3** with $X=+1$, **1** and **8** with $X=0$, and $\bar{\mathbf{3}}$ with $X=-1$ we have the mass sum rule (where the symbols can denote the masses or the squared masses, depending on the model).

$$3M_3+3M_{\bar{3}}=4M_1+2M_8 \quad (18)$$

which is the direct generalization of the Gell-Mann–Okubo octet mass formula of SU_3.[11] It can happen that no sum rule exists, for example in the case of the **20′** (1,1,0) (see Table III) where the four masses are linear combinations of four independent parameters. (Only three of them are independent in the case of the **15**.)

(iv) The electromagnetic mass splittings can easily be taken into account in models in which either the electromagnetic current or the electromagnetic mass splitting has the transformation properties of the tensor component T_1^1. The form of the matrix elements of T_1^1 is obtained from Eq. (12) by replacing X by $-Q$ and using the Casimir operators of the SU_3 subgroup orthogonal to Q. In the case of assuming the electromagnetic current transforms like T_1^1 all of the higher order terms in Q, C_2-Q^2, and $C_3+3QC_2-Q^3$ must be adjoined to the mass formula, as well, since the mass splittings are of second order in the current. Note however that this breaking term is the most natural one from the mathematical point of view, but by no means the only one possible; this will be discussed elsewhere.

IV. POSSIBLE SU_4 SYMMETRY SCHEMES

Let us now systematically discuss the classification of the known elementary particles in the various symmetry schemes that can be based on the SU_4 algebra. We restrict ourselves to the following particles whose quantum numbers are established[15]:

$\frac{1}{2}^+$ baryons:	$N(939)$, $\Lambda(1115)$, $\Sigma(1195)$, $\Xi(1317)$;
$\frac{3}{2}^+$ baryon resonances:	$N_{3/2}^*(1238)$, $Y_1^*(1385)$, $\Xi_{1/2}^*(1530)$, $\Omega_0(1675)$;
0^- mesons:	$\pi(137)$, $\eta(548)$, $K(496)$;
1^- mesons:	$\rho(750)$, $\omega(783)$, $\varphi(1020)$, $K^*(891)$.

As remarked above, there are three distinct symmetry groups to be considered, namely, SU_4/Z_4, SU_4/Z_2, and SU_4, which are distinguished by the classes of representation they use. Most of the possible models which we now discuss have already been proposed.[3]

Case (i): SU_4/Z_4

Symmetry schemes based on the group SU_4/Z_4 use only the class 0 representations and are the most straightforward extension of the eightfold way of SU_3. The bosons must belong to one of the real representations, and as can be seen from Table II the simplest choices are the **15** and the **20** for both pseudoscalar and vector mesons. We eliminate the **20**, since, as noted above, there is no mass splitting among bosons in the **20** except in higher order. (Note also that if the vector mesons are gauge particles they necesssarily belong to the **15**.) In this scheme the 1^- mesons can decay strongly into two 0^- mesons since the **15** is contained in the product **15**⊗**15** as indicated in Table V.

Table II shows the smallest representations for the $\frac{1}{2}^+$ baryons are **15** and **20**, while the $\frac{3}{2}^+$ baryon resonances could belong to the **35**. However, these assignments are untenable if the mesons are to be in the **15** since neither **15**⊗**15** nor **15**⊗**20** contains the **35** as can be seen from Table V. However, both **15**⊗**15** and **15**⊗**20** do contain the **45**, making it the most reasonable choice for the $\frac{3}{2}^+$ resonances. We note that the most straightforward generalization of the eightfold way would be the following multiplet assignments:

0^- mesons:	**15**
1^- mesons:	**15**
$\frac{1}{2}^+$ baryons:	**15**
$\frac{3}{2}^+$ resonances:	**45**

With these assignments several questions naturally arise:

[15] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **36**, 977 (1964).

(a) Where is the ninth $X=0$ pseudoscalar meson? Presumably the 960-MeV $\pi\pi\eta$ resonance[15] is to be identified with this state.

(b) Where is the octet of $X=0$ $\frac{3}{2}^+$ resonances in the **45**? The only candidates[15] seem to be the $Y_1^*(1660)$ and the $\Xi_{1/2}^*(1810)$.

(c) Where is the ninth $\frac{1}{2}^+$ baryon in the **15**? The only candidate,[15] $Y_0^*(1405)$, seems to have the wrong parity.

(d) Why have none of the $|X|=1$ states been found? Such particles should be produced in pairs in $\bar{p}p$ annihilation, for example. A rough application of the mass sum rule (18) (with squared masses) in the case of the pseudoscalar mesons, taking for M_1 960 MeV and for M_8 a mean mass of the SU_3 octet gives $M_3=M_{\bar{3}}\simeq 850$ MeV. Such a low value of the mass would perhaps have already permitted the detection of a pair of those particles. At any rate this is a critical point of the SU_4 theory and it gives a possibility of experimental tests.

Case (ii): SU_4/Z_2

As noted above the class 2 representations do not contain self-conjugate particles so the bosons must still be assigned to a class 0 representation, the simplest one being the **15**. The $\frac{1}{2}^+$ baryons can go in either the **64**, the **70**, or the $\overline{\mathbf{70}}$ while the $\frac{3}{2}^+$ resonances can go in the **50** or the **70**. If the $\frac{1}{2}^+$ baryons are assigned to the **64**, then the $\frac{3}{2}^+$ resonances cannot belong to the **70** or $\overline{\mathbf{70}}$ but can belong to the **50** in which case $X=-\frac{3}{2}$ for both the baryons and their resonances.

Evidently, any of these schemes involves unattractively large multiplets.

Case (iii): SU_4

In schemes based on the full group SU_4 the mesons still can be assigned to the **15**. The $\frac{1}{2}^+$ baryons can go in the **20′**, $\overline{\mathbf{20}}'$, **36**, $\overline{\mathbf{36}}$, **60**, $\overline{\mathbf{60}}$, etc.$\cdots$, while the $\frac{3}{2}^+$ resonances can belong to the **20″**, **56**, **60**, **84′**, etc$\cdots$. Choosing the simplest assignments **20′** or $\overline{\mathbf{20}}'$ for the $\frac{1}{2}^+$ baryons leads to **20″** or **60**, respectively for the $\frac{3}{2}^+$ resonances. Let us consider the former choice in detail, since it is the most attractive one, for several reasons. First, it gives the most economical generalization of SU_3, namely:

0^- mesons:	**15**
1^- mesons:	**15**
$\frac{1}{2}^+$ baryons:	**20′**
$\frac{3}{2}^+$ resonances:	**20″**.

With these assignments, the only new particle whose production is not inhibited by X conservation (implying pair production) is the ninth pseudoscalar meson, which has probably been found [remark (a) above].

With regard to the masses, the only trouble comes from the mesons, where the $|X|=1$ triplets in the **15** could have a low mass [around 850 MeV, as indicated in Case (i)]. For the $\frac{1}{2}^+$ baryons in the **20′**, there is no mass sum rule, while for the $\frac{3}{2}^+$ resonances in the **20″** we have an equal spacing rule (see Sec. III); as in both cases, the "ordinary" particles have $X=\frac{3}{4}$, which is the highest value in the supermultiplet, *all* the other particles could be very heavy, and thus have escaped detection, without contradicting a first-order mass formula.

Another aspect of the models based on the full SU_4 is that these are the only ones which can include the quark model[16] of SU_3 (whatever the charge of the quarks). The quarks being by definition the fundamental objects, it is natural to consider four of them, with spin-$\frac{1}{2}$, belonging to the fundamental representation **4** (of class 1). As a consequence the mesons must be contained in

$$\mathbf{4}\otimes\bar{\mathbf{4}}=\mathbf{1}\oplus\mathbf{15}\ \text{(class 0)},$$

while the baryons and the resonances go into

$$\mathbf{4}\otimes\mathbf{4}\otimes\mathbf{4}=\bar{\mathbf{4}}\oplus\mathbf{20}'\oplus\mathbf{20}'\oplus\mathbf{20}''\ \text{(class 3)}.$$

Clearly this implies the use of the full SU_4 (for the physical particles!) and gives precisely the above assignment. It should be noted that it is not possible to build an SU_4/Z_4 model with quarks, because a bound state of four spin-$\frac{1}{2}$ quarks can only be a boson.

The same situation occurs—independently of the quark hypothesis—when one tries to generalize the current SU_6 model[17] by combining SU_4 with ordinary spin in a SU_8 scheme: only the models starting from the full SU_4 can contain both bosons and fermions, the other ones contain only bosons. This is discussed in the following section.

Note added in proof. We emphasize the importance of the new quantum number which distinguishes bosons of class 0 and bosons of class 2. This new quantum number is *real* and is an eigenvalue of a Hermitian operator, therefore it is an observable. Its value is $+1$ for bosons of class 0, -1 for bosons of class 2. The latter therefore can be produced only *in pairs*.

V. INCLUSION OF ORDINARY SPIN: EMBEDDING OF SU_4 INTO SU_8

It has been proposed recently,[17] with some success, to enlarge the eightfold-way scheme by combining it with the ordinary spin: the direct product $SU_3\otimes SU_2$ is then embedded into a group SU_6. (We consider here the static case only.) This extension has met so far neither full experimental confirmation nor theoretical justification, but at any rate it must be taken at least as a strong indication. Now, SU_4 seems to be a reasonable symmetry group for elementary particle physics. There-

[16] M. Gell-Mann, Phys. Letters **8**, 214 (1964); G. Zweig (unpublished).

[17] F. Gürsey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964); A. Pais, *ibid.* **13**, 175 (1964); B. Sakita, Phys. Rev. **136**, B1756 (1964); F. Gürsey, A. Pais, and L. A. Radicati, Phys. Rev. Letters **13**, 299 (1964).

fore, it is very natural to generalize still further the idea of spin-unitary spin independence and to embed the direct product $SU_4 \otimes SU_2$ into SU_8. It has been shown by two of us[18] that this procedure has two attractive virtues. First it is possible to find a very economical choice of representations which *preserves all the results of* SU_6 and thus automatically gives several experimental verifications; second, this procedure provides an *a priori discrimination between the three types of* SU_4 *schemes:* the embedding is physically admissible in the case of the full SU_4, but not in the case of SU_4/Z_2 or SU_4/Z_4, because the resulting representations would contain only bosons.

Let us begin with the second point; it is merely an application to the case $SU_{2n} \to SU_n \otimes SU_2$ of the branching rules for simple Lie groups given by Whippman.[19] Let us label an irreducible representation $\{m\}_k$ of SU_k by a Weyl partition $\{m_1, m_2, \cdots m_k\}$ consisting of k non-negative, nonincreasing integers. We define then the quantity [compare Eq. (2)]

$$C(m)=\sum_{i=1}^{k} m_i. \tag{19}$$

Note that the value of $C(m)$ is not unique, as the Weyl partition is not: all the partitions $\{m_1+a, m_2+a, \cdots m_k+a\}$, where a is any positive integer, define the same representation of SU_k, so that all the possible values of $C(m)$ differ by an integer multiple of k. Clearly the class of the representation $\{m\}_k$ is given by[4]

$$C=\lambda_1+2\lambda_2+\cdots+(k-1)\lambda_{k-1}\equiv C(m) \bmod k \tag{20}$$
$$C=0, 1, 2 \cdots k-1,$$

where $\lambda_i = m_i - m_{i+1}$ are the affine coordinates (components of the highest weight) used throughout this paper.

Whippman's rule now says: let $\{m'\}_n$ and $\{m''\}_2$ be irreducible representations of SU_n and SU_2, respectively. Then the direct product $\{m'\}_n \otimes \{m''\}_2$ induces a representation into SU_{2n} provided

$$C(m')=C(m''). \tag{21}$$

This representation can be reduced into irreducible constituents according to

$$(\{m'\}_n \otimes \{m''\}_2)_{2n} = \sum_m \oplus C_{mm'm''} \{m\}_{2n},$$

where, for every m in the decomposition,

$$C(m)=C(m')=C(m'').$$

Furthermore, the restriction of $\{m\}_{2n}$ to the subgroup $SU_n \otimes SU_2$ is given by

$$(\{m\}_{2n})_{n\times 2} \to \sum_{m'm''} \oplus C_{mm'm''} (\{m'\}_n \otimes \{m''\}_2)$$

with the *same* coefficients $C_{mm'm''}$. These are tabulated by Coleman.[20]

Applying this rule to the case $n=3$ ($SU_6 \to SU_3 \otimes SU_2$) and using (20), we find the following reduction scheme:

$$(C)_6 \to (C')_3 \otimes (C'')_2$$

with

$$(C')_3 \equiv (C)_6 \bmod 3, \quad (C')_3 = 0, 1, 2$$
$$(C'')_2 \equiv (C)_6 \bmod 2, \quad (C')_2 = 0, 1$$
$$(0) \to (0)\otimes(0), \quad (1) \to (1)\otimes(1), \quad (2) \to (2)\otimes(0),$$
$$(3) \to (0)\otimes(1), \quad (4) \to (1)\otimes(0), \quad (5) \to (2)\otimes(1).$$

Remembering that class 0 of SU_2 means integer spin, class 1 half-integer spin, we see that class-0 representations of SU_3 (i.e., faithful representations of SU_3/Z_3) can contain both bosons and fermions in a SU_6 scheme. This is precisely the case in the current model.[17] Note that the same is true for any *odd* value of n.

On the other hand, for $n=4$, we get

$$(0) \text{ and } (4) \to (0)\otimes(0),$$
$$(1) \text{ and } (5) \to (1)\otimes(1),$$
$$(2) \text{ and } (6) \to (2)\otimes(0),$$
$$(3) \text{ and } (7) \to (3)\otimes(1).$$

So that a model based on SU_4/Z_4 (class 0 only) or SU_4/Z_2 (class 0 and class 2) can contain only bosons; fermions can be put in representations of class 1 or 3 only. The same property holds for any *even* value of n.

We conclude that if we want to combine a SU_4 internal symmetry group with SU_2 into a SU_8 scheme, we are forced to use the full SU_4 and thus the full SU_8: bosons will be assigned to even classes and fermions to odd classes.

We now proceed to the first point and consider a specific model which preserves the results of SU_6. We start from the simplest assignment in SU_4 given in Sec. IV and obtain the following scheme:

(*a*) *For the mesons.* Both 0^- and 1^- mesons go in the **15** of SU_4, whereas they have been assigned together to the adjoint representation of SU_6, the **35**. These two requirements can be combined if we choose the adjoint representation **63** of SU_8. (No other choice is possible if the vector mesons are viewed as gauge particles.) Indeed the latter representation reduces in the following ways:

$$SU_4 \otimes SU_2: \quad \mathbf{63} \to (\mathbf{15},\mathbf{3}) \oplus (\mathbf{15},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}),$$
$$SU_6: \quad \mathbf{63} \to \mathbf{35} \oplus 2\times\mathbf{6} \oplus 2\times\bar{\mathbf{6}} \oplus 4\times\mathbf{1}.$$

Table II shows that the $X=0$ part of this representation contains a vector nonet, a pseudoscalar nonet and a vector SU_3 singlet. Thus we have three vector mesons with $I=0$, $Y=0$: two SU_3 singlets [one in (**15**,**3**), the other one in (**1**,**3**)] and a member of a SU_3 octet [in (**15**,**3**)]. Two of them can be identified with the $\omega(783)$

[18] J.-P. Antoine and D. Speiser, University of Louvain, 1965 (unpublished).

[19] M. Whippman, University of Pennsylvania report, 1964 (unpublished).

[20] A. J. Coleman, Uppsala University, 1963 (unpublished).

and the $\varphi(1020)$; for the third one, the only candidate known so far is the (still dubious) $K\bar{K}\pi$ resonance[21] $E(1415)$, for which the indications are $J^{PG}=1^{--}(?)$ and $I=0(?)$. In fact, owing to the successive breakings $SU_4 \rightarrow SU_3 \rightarrow SU_2$, the physical states are mixtures of these three pure SU_4 states, and nothing can be said about the masses, since we have a two-step mixing: first, two SU_3 singlets (which can already mix) and a member of an octet, this step corresponding to the X breaking of SU_4; then the usual octet-singlet mixing, which corresponds to the $\tilde{Y}$ breaking of SU_3. Since we have no idea of the relative strength of these two effects, we cannot make any prediction of the masses. However, these remarks lead us to a further possibility: the existence of three degenerate vector mesons could perhaps explain the curious discrepancy between pseudoscalar and vector mesons in SU_3 (note that SU_6 gives another explanation[17]); for the pseudoscalar mesons, the octet mass formula works quite well, indicating no appreciable mixing of the η with any singlet [as the $X(960)$], whereas it works very badly for the vector mesons, indicating a large φ-ψ mixing. Is it not possible that the presence of a third vector meson, as suggested by the present scheme, could be responsible for that situation? It could indeed introduce a supplementary large mixing effect (of the order of the $SU_4 \rightarrow SU_3$ breaking), whereas the singlet-octet mixing would be smaller in both cases (of the order of the $SU_3 \rightarrow SU_2$ breaking).

(*b*) *For the* $\frac{1}{2}^+$ *baryons and the* $\frac{3}{2}^+$ *resonances.* The simplest SU_4 assignment here is $\mathbf{20'}$ for the $\frac{1}{2}^+$ baryons and $\mathbf{20''}$ for the $\frac{3}{2}^+$ resonances (see Sec. IV). On the other hand, both of them are assigned to the SU_6 $\mathbf{56}$ representation [(3,0,0,0,0) in the λ_i coordinates]. The simplest SU_8 solution is now to put all of them in the $\mathbf{120}$ representation (3,0,0,0,0,0,0), whose reductions are the following:

$$SU_4 \otimes SU_2: \quad \mathbf{120} \rightarrow (\mathbf{20''},4) \oplus (\mathbf{20'},2)\,,$$
$$SU_6: \quad \mathbf{120} \rightarrow \mathbf{56} \oplus 2\times\mathbf{21} \oplus 3\times\mathbf{6} \oplus 4\times\mathbf{1}\,.$$

In this case, no new SU_4 supermultiplet is introduced and the "ordinary" baryons and resonances (which here have $X=\frac{3}{4}$) just saturate the $\mathbf{56}$ representation[18] of SU_6. We have already shown in Sec. IV that this assignment is compatible with a first-order SU_4 mass formula, if we suppose that all the particles with $X<\frac{3}{4}$ are very heavy. Furthermore, one has in SU_8

$$\mathbf{120} \otimes \mathbf{120} = \mathbf{1} \oplus \mathbf{63} \oplus \mathbf{1232} \oplus \mathbf{13104}\,,$$

so that the baryon-baryon-meson coupling is unique, as was the case in SU_6; all the consequences of this fact [F/D ratio (generalized), form factors ratio, etc.) thus remain valid here.

(*c*) *For the negative parity baryonic resonances.* In order to be complete, we consider these resonances too; Pais,[17] and Gyuk and Tuan[22] have proposed to assign them to the representations $\mathbf{20}$ and $\mathbf{70}$ of SU_6. This leads us in SU_8 to the representations $\mathbf{56}$ (0,0,1,0,0,0,0) and $\mathbf{168}$(1,1,0,0,0,0,0), respectively, with the reductions:

$$SU_4 \otimes SU_2: \quad \mathbf{56} \rightarrow (\bar{\mathbf{4}},4) \oplus (\mathbf{20'},2)$$
$$\mathbf{168} \rightarrow (\mathbf{20'},4) \oplus (\mathbf{20''},2) \oplus (\mathbf{20'},2) \oplus (\bar{\mathbf{4}},2)$$
$$SU_6: \quad \mathbf{56} \rightarrow \mathbf{20} \oplus 2\times\mathbf{15} \oplus \mathbf{6}$$
$$\mathbf{168} \rightarrow \mathbf{70} \oplus 2\times(\mathbf{15}+\mathbf{21}) \oplus 4\times\mathbf{6} \oplus 2\times\mathbf{1}\,.$$

This implies several assignments in SU_4, but here also the "ordinary" particles have the maximum value $X=\frac{3}{4}$ and saturate the $\mathbf{20}$ and the $\mathbf{70}$ of SU_6, so that nothing new is introduced by SU_8 as far as these particles are concerned.

Note added in proof. In this model *all* classes of the governing groups (SU_4, SU_8) are used. This of course would also be the case in SU_3 and SU_6 if quarks will be found, but then one has the new multiplicative quantum number ϵ $(=1^{1/3})$ of a very unattractive character (non-Hermitian operator). It would seem likely that this ϵ too might become in turn an additive quantum number which leads finally to a large group:

$$\epsilon \rightarrow X \rightarrow SU_4(\rightarrow SU_8)$$

exactly as we had:

associate production $\rightarrow Y \rightarrow SU_3(\rightarrow SU_6)$,
charge symmetry $\rightarrow (I_3) \rightarrow SU_2(\rightarrow SU_4)$.

VI. SUMMARY

We have presented the most important mathematical aspects of the group SU_4 and derived the mass formula for an arbitrary SU_4 multiplet under the assumption that the mass splitting transforms like a component of the 15-dimensional adjoint representation. A systematic discussion of the various possible symmetry schemes was given and some of their defects were noted. In particular, we have given two reasons why a model based on the full SU_4 should be the most reasonable one, namely the possibility of containing a quark-model of SU_3 and the possibility of including the ordinary spin to give an SU_8 scheme, and we have sketched a simple model. It should be emphasized however that we have considered the group-theoretical point of view only: No dynamical considerations have been included; our purpose was mainly mathematical, since the physics behind the whole SU_4 theory is still too speculative. The basic ingredient, namely the existence of particles that carry the new quantum number X, evidently is the crucial question to be answerd experimentally. As is well known, "to make rabbit stew one must first catch a rabbit."

[21] This resonance is already quoted in Ref. 15, under the name $KK\pi$ (1410); the indicated values of the quantum numbers come from the review talk of D. C. Colley at the Birmingham Conference on Elementary Particles, April 1965.

[22] I. P. Gyuk and S. F. Tuan, Phys. Rev. Letters **14**, 121 (1965).

Note added in proof. After completion of this work we learned that the problem of the extension $SU_4 \rightarrow SU_8$ was studied independently, but with different motivations and aims, by Gruber and Vitale[23] and by Iwao.[24] We thank Professor Vitale and Professor Iwao for having kindly sent to us an unpublished report of their work.

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APPENDIX A

In this Appendix, we want to prove the theorem of Sec. IIB which gives the reduction $SU_4 \rightarrow SU_3$.

The first part of the theorem is merely the transcription of Weyl's branching rule.[6] The latter, adapted to SU_4 (instead of U_4), can be formulated as follows: Let an irreducible representation of SU_4 be labeled by the Cartesian components of its highest weight (x_1,x_2,x_3,x_4), with

$$\sum_{i=1}^{4} x_i = 0$$

and

$$\Lambda_i = x_i - x_{i+1} \ (i=1,\,2,\,3), \text{ positive integers.} \tag{A1}$$

When restricted to SU_3, this representation is decomposed into all the representations of SU_3 whose highest weight (y_1,y_2,y_3) satisfies the relation:

$$x_1 \geqslant y_1 \geqslant x_2 \geqslant y_2 \geqslant x_3 \geqslant y_3 \geqslant x_4. \tag{A2}$$

In this relation, the y_i's may take all values such that

$$\lambda_i = y_i - y_{i+1} \ (i=1,\,2) \text{ is a positive integer.}$$

Then, by definition

$$X(\lambda_1,\lambda_2) = -y_4 = y_1+y_2+y_3. \tag{A3}$$

Now for each fixed value of y_2, say $y_2 = x_3+q$, with $0 \leqslant q \leqslant x_2-x_3 = \Lambda_2$ [(Λ_2+1) values], one has, independently of each other,

$$x_1 \geqslant y_1 \geqslant x_2$$

so that:

$$\Lambda_2 - q \leqslant \lambda_1 = y_1 - y_2 \leqslant \Lambda_1 + \Lambda_2 - q \ [(\Lambda_1+1) \text{ values}],$$

and

$$x_3 \geqslant y_3 \geqslant x_4$$

so that:

$$q \leqslant \lambda_2 = y_2 - y_3 \leqslant \Lambda_3 + q \ [(\Lambda_3+1) \text{ values}].$$

In the Cartan-Stiefel diagram of SU_3, these inequalities define a parallelogram with edges of length Λ_1, Λ_3 parallel to the axes λ_1, λ_2, respectively (see Fig. 4). This proves the first part of the theorem.

[23] B. Gruber and B. Vitale, Nuovo Cimeno **37**, 1805 (1965), and to be published.

[24] S. Iwao, Ann. Phys. (N. Y.) (to be published).

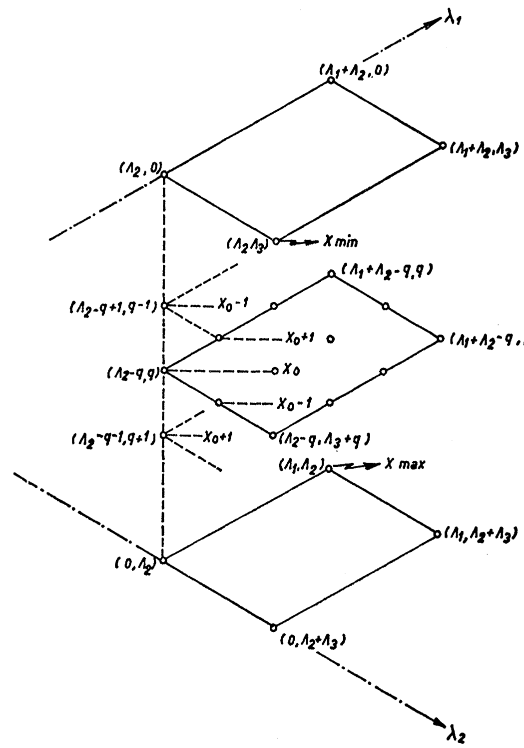

FIG. 4. The general aspect of a decomposition $SU_4 \rightarrow SU_3$. The $(q+1)$th parallelogram starts at the point $(\Lambda_2-q,\,q)$, which has $X = X_0$.

For proving the second part, we use again the coordinates y_i, and we remark that, by (A3):

(i) a step in the λ_1 direction $[y_1,y_2,y_3] \rightarrow [y_1+1,\,y_2,\,y_3]$ *increases* X by 1;

(ii) a step in the λ_2 direction, $[y_1,y_2,y_3] \rightarrow [y_1,\,y_2,\,y_3-1]$ *decreases* X by 1;

(iii) a step horizontally, $[y_1,y_2,y_3] \rightarrow [y_1+1,\,y_2,\,y_3-1]$ *does not change* X;

(iv) a step downwards, $[y_1,y_2,y_3] \rightarrow [y_1,\,y_2+1,\,y_3]$ *increases* X by 1. From this it follows that in each parallelogram the horizontal lines have constant X and are ordered with X *increasing* by unit steps *upwards*, while the left-hand corners of the successive parallelograms are ordered with X *increasing* by unit steps *downwards* (Fig. 4). Consequently, all the representations will be ordered with X increasing by unit steps *upwards* if we draw the parallelograms in the reverse order, in such a way that two points with the same value of X fall on the same horizontal line. This is the "interlacing" described in the text. Furthermore, the extreme values of X are

$$\begin{aligned} X_{\max} &= -y_4 = \tfrac{1}{4}(\Lambda_1+2\Lambda_2+3\Lambda_3), \\ X_{\min} &= -y_1 = -\tfrac{1}{4}(3\Lambda_1+2\Lambda_2+\Lambda_3). \end{aligned} \tag{A4}$$

$X_{\max}$ ($X_{\min}$) corresponds to the weight $[y_1,y_2,y_3,y_4]$

($[y_4,y_3,y_2,y_1]$) which gives the representation (Λ_1,Λ_2) ((Λ_2,Λ_3)), obtained for $q=\Lambda_2$ ($q=0$) (see Fig. 4). These considerations allow us to compute the value of X, for any representation (λ_1,λ_2), in the parallelogram starting at $(\lambda_1',\lambda_2')\equiv(\Lambda_2-q,\, q)$; this gives successively

$$X(\lambda_1,\lambda_2)=X(\Lambda_2-q,\, q)+(\lambda_1-\lambda_1')-(\lambda_2-\lambda_2'),$$
$$X(\Lambda_2-q,\, q)=X(\Lambda_2,0)+q,$$
$$X(\Lambda_2,0)=X(\Lambda_2,\Lambda_3)+\Lambda_3 =-\tfrac{1}{4}(3\Lambda_1+2\Lambda_2+\Lambda_3)+\Lambda_3,$$

and thus

$$X(\lambda_1,\lambda_2)=(\lambda_1-\lambda_2)-\tfrac{3}{2}(\lambda_1'-\lambda_2')-\tfrac{3}{4}(\Lambda_1-\Lambda_3).$$

As an example of the whole construction, Fig. 2 gives the decomposition of (3,1,2).

APPENDIX B

This Appendix contains a geometrical proof of the theorem on Kronecker products stated in Sec. III. Such a proof has been given by Ginibre,[25] for any simple group, but it becomes much simpler if we use the language of Ref. 4.

We shall limit ourself to the case $A_l(\sim SU_{l+1})$, the other cases would be analogous. We have to prove that the Kronecker product

$$D_{\mathrm{adj}}\otimes D=\textstyle\sum_i\oplus D_i \tag{B1}$$

of the adjoint representation D_{adj} of A_l by a given representation $D(\Lambda_1,\Lambda_2,\cdots\Lambda_l)$ contains $(l-m)$ times D itself, where m is the number of vanishing Λ_i's. The decomposition of the direct product (B1) is given as[10]

$$\chi_{\mathrm{adj}}X_D^{(0)}=\textstyle\sum_i\oplus X_i^{(0)}, \tag{B2}$$

where χ_{adj} is the character of D_{adj}, and $X_D^{(0)}$ ($X_i^{(0)}$) is the dominant vector of the characteristic of D (D_i). Now, in Cartesian coordinates [$(l+1)-$dimensional space], we have[4]

$$X_D^{(0)}=[y_1,y_2\cdots y_{l+1}]$$

with

$$\sum_{i=1}^{l+1} y_i=0,$$

$$y_i-y_{i+1}\equiv\Lambda_i+1\geqslant 1,\ \Lambda_i \text{ integer}.$$

$$\chi_{\mathrm{adj}}=\sum_P P[1,0,\cdots 0,\,-1]+l[0,\cdots,0],$$

where $\sum_P$ means the sum on all the permutations of the $(l+1)$ coordinates. Putting these expressions into (B2), we get

$$\chi_{\mathrm{adj}}\cdot X_D^{(0)}=(\textstyle\sum_P P[1,0,\cdots 0,\,-1])[y_1,\cdots,y_{l+1}] +l[y_1,\cdots,y_{l+1}].$$

It remains to evaluate the number of times that $X_D^{(0)}$ occurs in the right-hand side. The second term just gives l times $X_D^{(0)}$. The first term can give another $X_D^{(0)}$ whenever

$$(P[1,0,\cdots,0,\,-1])[y_1,\cdots,y_{l+1}]=P'[y_1,\cdots,y_{l+1}],$$

when P' is some permutation of the coordinates. This term must be added if P' is even and subtracted if P' is odd. We have thus

$$[\cdots\pm1\cdots\mp1\cdots][y_1\cdots y_i\cdots y_k\cdots y_{l+1}] =[y_1\cdots y_i\pm1\cdots y_k\mp1\cdots y_{l+1}]\quad (k>i) =P'[y_1\cdots y_{l+1}].$$

If we take the upper sign, the last equality implies

$$y_i+1>y_{i-1},\quad\text{i.e.,}\quad y_{i-1}-y_i<1$$

or

$$y_{k+1}>y_k-1,\quad\text{i.e.,}\quad y_k-y_{k+1}<1$$

which is impossible in both cases. We take now the lower sign, first with $k>i+1$. We have then

$$y_{i+1}>y_i-1,\quad\text{i.e.,}\quad y_i-y_{i+1}<1$$

or

$$y_k+1>y_{k-1},\quad\text{i.e.,}\quad y_{k-1}-y_k<1$$

which is impossible too. Thus the only possible contribution comes from $k=i+1$:

$$[y_1\cdots y_i-1,\, y_{i+1}+1,\,\cdots y_{l+1}]=P'[y_1\cdots y_{l+1}],$$

which implies

$$y_i-1<y_{i+1}+1,$$

i e.,

$$1\leqslant y_i-y_{i+1}=\Lambda_i+1<2,$$
$$\Lambda_i=0.$$

Since there are m indices i with $\Lambda_i=0$, there will be m terms $X_D^{(0)}$, each of them with a minus sign, as P' is a single transposition. This proves the theorem.

[25] J. Ginibre, J. Math. Phys. 4, 720 (1963).