Regge Poles in πN Scattering

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(Received 20 August 1965)

Exploiting the unitarity and analyticity properties of relativistic partial-wave amplitudes for pion-nucleon scattering, the partial-wave amplitude has been analytically continued in the complex J plane. The Froissart-Gribov (F-G) representation in π-N scattering is holomorphic in the domain $\mathrm{Re}J>\alpha+\frac{1}{2}$ provided the invariant amplitudes and their absorptive parts in the t and u channels are bounded by t^α, u^α for fixed s and large t,u. Using the N/D method of analytic continuation it has been shown that it is possible to continue the F-G representation in the interesting region $\mathrm{Re}J\leq\alpha+\frac{1}{2}$ and it is meromorphic there. This has been done under some reasonable assumptions on the asymptotic growth of the spectral function. It is further shown that the N/D method of analytic continuation does not work beyond the line $J=-\frac{1}{2}$ and at $J=0$ there is an accumulation of an infinite number of poles at threshold.

INTRODUCTION

In a previous paper[1] we investigated the problem of analytic continuation of relativistic partial-wave amplitude in the complex angular-momentum plane for identical pseudoscalar particles. The purpose of the present paper is to extend this formalism to the problem of pion-nucleon scattering. The notation of complex angular momenta is introduced by obtaining a Froissart-Gribov (F-G) representation for π-N scattering, which is a straightforward generalization of the F-G representation in π-π scattering. Apart from the dynamical singularities, the F-G representation in π-N scattering contains singularities of $\sqrt{s}$ type which occur in the problem because of spin. Most of the earlier work[2] on π-N scattering has been carried out in the W plane ($W^2=s$) in order to avoid the kinematical singularities of the partial-wave amplitude which appear in the s plane. The above argument for discarding the s plane is not relevant in our approach,[1] since in any case in our problem of analytic continuation we have to define a new *auxiliary amplitude*[1] which contains extra kinematical singularities. However, in order to preserve continuity and connection with earlier work we have in the following considered W as our variable instead of s.

Exploiting unitarity and analyticity properties of the F-G representation we have continued the partial-wave amplitude in the interesting region of the J plane where Regge poles can occur. This has been done under some reasonable assumptions on the asymptotic growth of the spectral functions. We have further shown that the domain of meromorphy of the partial-wave amplitude, using the N/D method of analytic continuation, cannot be extended beyond $J=-\frac{1}{2}$, and at $J=0$ there is an accumulation of poles at the threshold.

[1] Haridas Banerjee and G. C. Joshi, Phys. Rev. **137**, B1576 (1965).
[2] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960); S. C. Frautschi and J. D. Walecka, *ibid.* **120**, 1486 (1960).

I. THE AUXILIARY AMPLITUDE

In π-N scattering, the partial-wave amplitude is given by[3]

$$a_{J\mp1/2}=\frac{E+M}{16\pi W}[A_{J\mp1/2}+(W-M)B_{J\mp1/2}]+\frac{E-M}{16\pi W}[-A_{J\pm1/2}+(W+M)B_{J\pm1/2}],\quad(1)$$

where

$$\{B_J,A_J\}=\int_{-1}^{+1}d(\cos\theta)\{B(s,t,u),A(s,t,u)\}P_J(\cos\theta).\quad(2)$$

Our task now is to express A_J, B_J in terms of an F-G representation.[3] This is done by assuming the following Mandelstam representation for $A(s,t,u)$ and $B(s,t,u)$:

$$A(s,t,u)=\frac{1}{\pi}\int_4^\infty\frac{A_t(s,t')}{t'-t}\left(\frac{t}{t'}\right)^N dt'+\frac{1}{\pi}\int_{(M+1)^2}^\infty\frac{A_u(s,u')}{u'-u}\left(\frac{u}{u'}\right)^N du'+\sum_{n=1}^N t^{n-1}L_n(s)+\sum_{n=1}^N u^{n-1}M_n(s),\quad(3)$$

$$B(s,t,u)=R_s/M^2-s+R_u/M^2-u+\frac{1}{\pi}\int_4^\infty\frac{B_t(s,t')}{t'-t}\left(\frac{t}{t'}\right)^N dt'+\frac{1}{\pi}\int_{(M+1)^2}^\infty\frac{B_u(s,u')}{u'-u}\left(\frac{u}{u'}\right)^N du'+\sum_{n=1}^N t^{n-1}\bar{L}_n(s)+\sum_{n=1}^N u^{n-1}\bar{M}_n(s),\quad(4)$$

where $A_t(s,t)$, $B_t(s,t)$ are the absorptive parts in the t channels and $A_u(s,u)$, $B_u(s,u)$ are the absorptive parts in the u channels. In view of the conditions (3) and (4) we demand that there exist an $\alpha<N$ such that

$$|A_t(s,t),B_t(s,t)|/t^{\alpha+\epsilon}\rightarrow0\quad\text{as}\quad t\rightarrow\infty,\quad(5)$$

and

$$|A_t(s,t),B_t(s,t)|/t^{\alpha-\epsilon}\rightarrow\infty\quad\text{as}\quad t\rightarrow\infty,\quad(6)$$

[3] V. Singh, Phys. Rev. **129**, 1889 (1963).

with similar relations for $A_u(s,u)$ and $B_u(s,u)$. Substituting Eqs. (3) and (4) in (2) we obtain for $l>N$ that

$$A_{J\mp1/2}=\frac{1}{\pi q^2}\left\{\int_4^\infty dt\, Q_{J\mp1/2}\left(1+\frac{t}{2q^2}\right)A_t(s,t)+(-1)^{J\mp1/2}\int_{(M+1)^2}^\infty du\, Q_{J\mp1/2}\left(1+\frac{u-(M^2-1)^2/s}{2q^2}\right)A_u(s,u)\right\}, \quad (7)$$

and

$$B_{J\mp1/2}=\frac{1}{\pi q^2}\left\{(-1)^{J\mp1/2}Q_{J\mp1/2}\left(1+\frac{M^2-(M^2-1)^2/s}{2q^2}\right)+\int_4^\infty dt\, Q_{J\mp1/2}\left(1+\frac{t}{2q^2}\right)B_t(s,t)\right.$$
$$\left.+(-1)^{J\mp1/2}\int_{(M+1)^2}^\infty du\, Q_{J\mp1/2}\left(1+\frac{u-(M^2-1)^2/s}{2q^2}\right)B_u(s,u)\right\}. \quad (8)$$

In order to get rid of the factors of the form $(-1)^{J\mp1/2}$ in Eqs. (7) and (8), we define odd and even amplitudes denoted by $a_{J\mp1/2}{}^{e,o}(W)$. For example, $a_{J-1/2}{}^e$ contains J values $J=\frac{1}{2}, \frac{5}{2}, \frac{9}{2},\cdots$ and $a_{J-1/2}{}^o(W)$ contains $J=\frac{3}{2}, \frac{7}{2}, \frac{11}{2},\cdots$. The amplitudes $a_{J\mp1/2}{}^{e,o}$ can now be defined for complex values of J as they satisfy Carlson's theorem, but they will have a different continuation in the complex J plane. In our analysis we shall consider the amplitude $a_{J-1/2}{}^e(W)$; however, a similar analysis can be carried out for other amplitudes.

In view of conditions (5) and (6), representation (1) for $a^{e,o}(J\mp\frac{1}{2}, W)$ defines a holomorphic function of J in the region $\mathrm{Re}J>\alpha+\frac{1}{2}$. Our problem now is to analytically continue $a^{e,o}(J\mp\frac{1}{2}, W)$ in the interesting region $\mathrm{Re}J\leq\alpha+\frac{1}{2}$.

The first step in doing this is the introduction of an appropriate auxiliary function. Our choice of the auxiliary amplitude in the π-π scattering was crucially dependent on the l-asymptotic behavior of $a_+(l,s)$ [see Sec. II of Ref. 1]. Let us apply the same arguments here. Considering the contribution of the nucleon pole term in Eq. (8) we find

$$Q_{J-1/2}\left(1+\frac{M^2-(M^2-1)^2/s}{2q^2}\right)\sim\left(\frac{[s-(M^2+2)]^{1/2}+[M^2-(M^2-1)^2/s]^{1/2}}{[s-(M^2+2)]^{1/2}-[M^2-(M^2-1)^2/s]^{1/2}}\right)^{-(J-1/2)}\frac{1}{J^{1/2}},\quad \text{when}\quad |J|\to\infty, \quad (9)$$

which indicates the need of a factor of the form

$$\left(\frac{[s-(M^2+2)]^{1/2}+[M^2-(M^2-1)^2/s]^{1/2}}{[s-(M^2+2)]^{1/2}-[M^2-(M^2-1)^2/s]^{1/2}}\right)^{J-1/2},$$

in the representation of $a^{e,o}(J\mp\frac{1}{2}, W)$. It can also be shown that with this choice of the auxiliary function all the F-G representations occurring in Eqs. (7) and (8) are bounded in J. We would like to point out at this stage that the choice of the auxiliary amplitude corresponding to the second and third terms of the right-hand side of Eq. (8) introduces extraneous poles in the definition of $a^{e,o}(J\mp\frac{1}{2}, W)$. Therefore, in view of Eq. (9) we define

$$h^{e,o}(J\mp\tfrac{1}{2}, W)=\left(\frac{[s-(M^2+2)]^{1/2}+[M^2-(M^2-1)^2/s]^{1/2}}{[s-(M^2+2)]^{1/2}-[M^2-(M^2-1)^2/s]^{1/2}}\right)^{J-1/2}a^{e,o}(J\pm\tfrac{1}{2}, W)\frac{16\pi W}{E\pm1} \quad (10)$$

$$=\frac{\lambda(J,W)}{(4q^2)^{J-1/2}}a^{e,o}(J\mp\tfrac{1}{2}, W)\frac{16\pi W}{E+1}, \quad (11)$$

where

$$\lambda(J,W)\equiv([s-(M^2+2)]^{1/2}+[M^2-(M^2-1)^2/s]^{1/2})^{2(J-1/2)}. \quad (12)$$

$h^{e,o}(J\mp\frac{1}{2}, W)$ as defined in Eq. (10) satisfies all the requirements of an auxiliary function [See Sec. II and Ref. 1], i.e.,

(i) $h^{e,o}(J\mp\frac{1}{2}, W)$ does not contain any extra singularities in the W plane and there is a finite gap between the cuts arising out of the crossed-channel singularities and the unitarity cuts. This will be accomplished in the following section by appropriately choosing the cuts of the auxiliary function.

(ii) The s-asymptotic behavior of $h^{e,o}(J\mp\frac{1}{2}, W)$ is J-independent.

(iii) $h^{e,o}(J\mp\frac{1}{2}, W)$ is a bounded function of J.

(iv) $h^{e,o}(J\mp\frac{1}{2}, W)$ does not introduce any extraneous poles in $a^{e,o}(J\mp\frac{1}{2}, W)$. This is because

$$[s-(M^2+2)]^{1/2}+[M^2-(M^2-1)^2/s] \quad (13)$$

never vanishes on the physical sheet.

In this connection let us discuss a more interesting problem, which is a peculiar feature of unequal-mass kinematics, i.e.,

$$q^2\to\infty,\quad \text{when}\quad W\to0. \quad (14)$$

Such a behavior of q^2 introduces $(J-\frac{1}{2})$-order poles in

$a^{e,o}(J\mp\frac{1}{2}, W)$ if we define

$$H=\frac{a^{e,o}(J\mp\frac{1}{2}, W)}{(4q^2)^{J-1/2}}. \tag{15}$$

In our definition of the auxiliary amplitude, Eq. (10), the above mentioned difficulty is absent. This is because

$$\lambda(J,W)/(4q^2)^{J-1/2}\to \text{constant}, \quad \text{when} \quad W\to 0. \tag{16}$$

The problem of auxiliary amplitudes has recently been considered by Martin and Uretsky,[4] in connection with the one-nucleon-exchange problem in pion-nucleon scattering. They defined an auxiliary amplitude:

$$h(J\mp\tfrac{1}{2}, W)=(W^2/4q^2)^{J-1/2}a(J\mp\tfrac{1}{2}, W). \tag{17}$$

The above amplitude, however, is not convenient for our analysis. This is because Eq. (17) does not satisfy requirement (iv) as mentioned for the auxiliary amplitude.

II. ANALYTICITY OF $h^{e,o}(J\mp\frac{1}{2}, W)$ IN THE W PLANE

The analyticity properties of $h^{e,o}(J\mp\frac{1}{2}, W)$ as defined in Eq. (10) essentially depend upon the F-G representations for $A_{J\mp1/2}$ and $B_{J\mp1/2}$. The cuts in the F-G representation arise from the known properties of the $Q_{J\mp1/2}$ functions and the spectral functions $A_t(s,t)$, $A_u(s,t)$ and $B_t(s,t)$, $B_u(s,u)$.

Let us consider the F-G representations term by term. The first term on the right-hand side of Eq. (8), which comes from the u-channel nucleon pole, gives rise to two short cuts, i.e.,

$$\begin{gathered}-(M^2+2)^{1/2}\le W\le -(M^2-1)/M, \\ (M^2-1)/M\le W\le (M^2+2)^{1/2}.\end{gathered} \tag{18}$$

The second term of Eq. (8), which comes from the t channel, i.e., $\pi\pi\to N\bar{N}$, gives rise to a circular cut and a cut along the imaginary axis, i.e.,

$$\begin{gathered}|W|=(M^2-1)^{1/2}, \\ -i\infty\le W\le +i\infty.\end{gathered} \tag{19}$$

The contribution of the crossed πN channel or the u channel is given by the third term in (8), which gives rise to the following cuts:

$$\begin{gathered}-(M-1)\le W\le (M-1), \\ -i\infty\le W\le +i\infty.\end{gathered} \tag{20}$$

Apart from these singularities we also have an extra cut of one function, namely, $Q_{J-1/2}(z)$, which comes for the noninteger values of $J-\frac{1}{2}$. Such a branch cut of (8) when mapped in the W plane fills the gap between the right-hand and the left-hand unitarity cuts coming from $A_u(s,u)$ and $A_t(s,t)$. This kinematical branch cut can be easily removed provided we choose the cuts for $(4q^2)^{J-1/2}$ as follows:

$$\begin{gathered}-M-1\le W\le M+1, \\ i\infty\le W\le 0.\end{gathered} \tag{21}$$

Further, if we choose the cuts for

$$([s-(M^2+2)]^{1/2}+[M^2-(M^2-1)^2/s]^{1/2})^{2(J-1/2)}$$

as

$$\begin{gathered}-(M^2+2)^{1/2}\le W\le -(M^2-1)/M, \\ (M^2-1)/M\le W\le (M^2+2)^{1/2}, \\ i\infty\le W\le 0,\end{gathered} \tag{22}$$

we find that there is a finite gap between the cuts on the real axis.

III. STUDY OF THE LEFT-HAND FUNCTION

The partial-wave amplitude as defined in Eq. (1) contains two types of amplitudes, namely, $a(J-\frac{1}{2}, W)$ and $a(J+\frac{1}{2}, W)$ representing transitions from $J-\frac{1}{2}$ to $J-\frac{1}{2}$ and from $J+\frac{1}{2}$ to $J+\frac{1}{2}$ states, respectively. For the purpose of unique interpolation we further subdivide each of these amplitudes into odd and even amplitudes, represented by $a^{e,o}(J-\frac{1}{2}, W)$ and $a^{e,o}(J+\frac{1}{2}, W)$.

Let us now consider the amplitude $a^e(J-\frac{1}{2}, W)$, which is a holomorphic function of J in the region $\mathrm{Re}J>\alpha+\frac{1}{2}$. Our subsequent discussion in this paper is devoted to the analytic continuation of $a^e(J-\frac{1}{2}, W)$ to region $\mathrm{Re}J\le\alpha+\frac{1}{2}$. For the other amplitudes, however, a similar analysis can be carried out.

It follows from Eq. (11) that

$$h^e(J-\tfrac{1}{2}, W)=\frac{\lambda(J,W)}{(4q^2)^{J-1/2}}a^e(J-\tfrac{1}{2}, W)\frac{16\pi W}{E+1}. \tag{23}$$

If $2i\alpha_{1,2}(J-\frac{1}{2}, W)$, $2i\alpha_3(J-\frac{1}{2}, W)$, and $2i\alpha_4(J-\frac{1}{2}, W)$, $2i\alpha_5(J-\frac{1}{2}, W)$ are the discontinuities of $h^e(J-\frac{1}{2}, W)$ across the short nucleon cuts

$$[-(M^2+2)^{1/2}\le W\le -(M^2-1)/M]$$

and $[(M^2-1)/M\le W\le (M^2+2)^{1/2}]$, the crossed-nucleon cut $[-(M-1)\le W\le (M-1)]$, the circular π-π cut $[|W|=(M^2-1)^{1/2}]$, and the imaginary cut

$$[-i\infty\le W\le i\infty],$$

respectively, then the left-hand function is given by

$$F^e(J-\tfrac{1}{2}, W)=\frac{1}{\pi}\int_{-(M^2+2)^{1/2}}^{-(M^2-1)/M} dW'\frac{\alpha_1(J-\frac{1}{2}, W')}{W'-W}+\frac{1}{\pi}\int_{(M^2-1)/M}^{(M^2+2)^{1/2}} dW'\frac{\alpha_2(J-\frac{1}{2}, W')}{W'-W}$$
$$+\frac{1}{\pi}\int_{-(M-1)}^{M-1} dW'\frac{\alpha_3(J-\frac{1}{2}, W')}{W'-W}+\frac{\gamma}{\pi}\int_0^{2\pi} id\phi\frac{\alpha_4(J-\frac{1}{2}, \phi)}{We^{-i\phi}-\gamma}+\frac{1}{\pi}\int_{-\infty}^{\infty} dy\frac{\alpha_5(J-\frac{1}{2}, y)}{y+iW}, \tag{24}$$

[4] A. W. Martin and J. L. Uretsky, Phys. Rev. **135**, B803 (1964).

where (in the last two integrals)

$$W'=(M^2-1)^{1/2}e^{i\phi}=\gamma e^{i\phi}$$

and

$$W'=iy.$$

Following the procedure of Ref. 1 (see Sec. II), we can easily calculate

$$\sum_{k=1}^{5}\alpha_k(J-\tfrac{1}{2},W).$$

And after some lengthy but straightforward calculations, we finally obtain

$$F^e(J-\tfrac{1}{2},W)=\sum_{k=1}^{8}F_k{}^e(J-\tfrac{1}{2},W),\tag{25}$$

where

$$F_1{}^e(J-\tfrac{1}{2},W)=\frac{4}{\pi}\int_4^\infty dt\int_{(M+1)^2}^\infty ds'\frac{\rho_{12}(s',t)}{s'-s}[\chi(s)Q_{J-1/2}(z(s,t))-\chi(s')Q_{J-1/2}(z(s',t))]+4\int_4^\infty dt\,Q_{J-1/2}(z(s,t))\phi_2(t,\Sigma-s-t)\chi(s),\tag{26}$$

$$F_2{}^e(J-\tfrac{1}{2},W)=\frac{4}{\pi}\int_{(M+1)^2}^\infty du\int_{(M+1)^2}^\infty ds'\frac{\rho_{13}(s',u)}{s'-s}[\chi(s)Q_{J-1/2}(z'(s,u))-\chi(s')Q_{J-1/2}(z'(s',u))]+4\int_{(M+1)^2}^\infty Q_{J-1/2}(z'(s,u))\psi_2(\Sigma-s-u,u)\chi(s)du,\tag{27}$$

$$F_3{}^e(J-\tfrac{1}{2},W)=\frac{4}{\pi}\int_4^\infty dt\int_{(M+1)^2}^\infty ds'\frac{\bar\rho_{12}(s',t)}{s'-s}[\chi'(s)Q_{J-1/2}(z(s,t))-\chi'(s')Q_{J-1/2}(z(s',t))]+4\int_4^\infty dt\,Q_{J-1/2}(z(s,t))\bar\phi_2(t,\Sigma-s-t)\chi'(s),\tag{28}$$

$$F_4{}^e(J-\tfrac{1}{2},W)=\frac{4}{\pi}\int_{(M+1)^2}^\infty du\int_{(M+1)^2}^\infty ds'\frac{\bar\rho_{13}(s',u)}{s'-s}[\chi'(s)Q_{J-1/2}(z'(s,u))-\chi'(s')Q_{J-1/2}(z(s',u))]+4\int_{(M+1)^2}^\infty du\,Q_{J-1/2}(z(s,u))\bar\psi_2(\Sigma-s-u,u)\chi'(s),\tag{29}$$

$$F_5{}^e(J-\tfrac{1}{2},W)=-\frac{4}{\pi}\int_4^\infty dt\int_{(M+1)^2}^\infty ds'\frac{\rho_{12}(s',t)}{s'-s}[\chi''(s)Q_{J+1/2}(z(s,t))-\chi''(s')Q_{J+1/2}(z(s',t))]-4\int_4^\infty dt\,Q_{J+1/2}(z(s,t))\chi''(s)\phi_2(t,\Sigma-s-t),\tag{30}$$

$$F_6{}^e(J-\tfrac{1}{2},W)=\frac{4}{\pi}\int_{(M+1)^2}^\infty du\int_{(M+1)^2}^\infty ds'\frac{\rho_{13}(s',u)}{s'-s}[\chi''(s)Q_{J+1/2}(z'(s,u))-\chi''(s')Q_{J+1/2}(z'(s',u))]+4\int_{(M+1)^2}^\infty du\,Q_{J+1/2}(z'(s,u))\chi''(s)\psi_2(\Sigma-s-u,u),\tag{31}$$

$$F_7{}^e(J-\tfrac{1}{2},W)=\frac{4}{\pi}\int_4^\infty dt\int_{(M+1)^2}^\infty ds'\frac{\bar\rho_{12}(s',t)}{s'-s}[\chi'''(s)Q_{J+1/2}(z(s,t))-\chi'''(s')Q_{J+1/2}(z(s',t))]+4\int_4^\infty dt\,Q_{J+1/2}(z(s,t))\bar\phi_2(t,\Sigma-s-t)\chi''(s),\tag{32}$$

$$F_8{}^e(J-\tfrac{1}{2},W)=-\frac{4}{\pi}\int_{(M+1)^2}^\infty du\int_{(M+1)^2}^\infty ds'\frac{\bar\rho_{13}(s',u)}{s'-s}[\chi'''(s)Q_{+1/2}(z'(s,u))-\chi'''(s')Q_{J+1/2}(z'(s',u))]-4\int_{(M+1)^2}^\infty du\,\chi'''(s)Q_{J+1/2}(z(s,u))\bar\psi_2(\Sigma-s-u,u)+\frac{4}{\pi}\frac{\lambda(J,W)}{(4q^2)^{J+1/2}}Q_{J-1/2}\left(1+\frac{M^2-(M^2-1)^2/s}{2q^2}\right)(W-M)-\frac{16}{\pi}\frac{\lambda(J,W)}{(4q^2)^{J+3/2}}Q_{J+1/2}\left(1+\frac{M^2-(M^2-1)^2/s}{2q^2}\right)(E-M)^2(W+M),\tag{33}$$

where the symbols are defined in Appendices A and B [see Eqs. (A1) to (A9) and (B8) to (B13)]. For a check on the calculation one can see that $F^e(J-\frac{1}{2}, W)$, as defined in (25), has only those cuts which are given by Eqs. (18) to (22) and further that $F^e(J-\frac{1}{2}, W)$ has no right-hand or left-hand physical cut. The discontinuities of $F^e(J-\frac{1}{2}, W)$ across the unphysical cuts are the same as those of the auxiliary partial-wave amplitude $h^e(J-\frac{1}{2}, W)$.

Let us now examine the J-asymptotic behavior of the representation (25). This is fairly simple because of the presence of $\gamma_{J\mp1/2}(z)$ functions with arguments greater than unity. Using a similar procedure as for Ref. 1, we obtain

$$F^e(J-\tfrac{1}{2}, W)\sim O(c/|J|^{1/2}), \quad |J|\to\infty. \tag{34}$$

The above asymptotic behavior of $F^e(J-\frac{1}{2}, W)$ will remain true throughout the domain of validity of the representation (25).

IV. THE N/D METHOD AND THE DOMAIN OF MEROMORPHY

We now apply the usual N/D formalism to $h^e(J-\frac{1}{2}, W)$ such that

$$h^e(J-\tfrac{1}{2}, W)=N(J-\tfrac{1}{2}, W)/D(J-\tfrac{1}{2}, W) \tag{35}$$

where $D(J-\frac{1}{2}, W)$ contains the unitarity cuts $[-\infty\leq W\leq -(M+1)$ and $(M+1)\leq W\leq\infty]$, and the rest of the cuts, as defined by Eqs. (18) to (22), are contained in $N(J-\frac{1}{2}, W)$. The unitarity relation for $h^e(J-\frac{1}{2}, W)$ for $\mathrm{Re}J>\alpha+\frac{1}{2}$, $W>M+1$ is given by

$$h^e(J-\tfrac{1}{2}, W+i\epsilon)-h^e(J-\tfrac{1}{2}, W-i\epsilon)=2i\phi(J,W)h^e(J-\tfrac{1}{2}, W+i\epsilon)h^e(J-\tfrac{1}{2}, W-i\epsilon), \tag{36}$$

where

$$\phi(J,W)=q\frac{(4q^2)^{J-1/2}}{\lambda(J,W)}R(J,W)\frac{E+M}{16\pi W}. \tag{37}$$

In the elastic-unitarity approximation, however, $R(J,W)=1$. The functions $N(J-\frac{1}{2}, W)$ and $D(J-\frac{1}{2}, W)$ satisfy the following integral equations:

$$D(J-\tfrac{1}{2}, W)=1-\frac{W-W_0}{\pi}\int_{-\infty}^{-M-1}dW'\frac{\phi(J,W')N(J-\frac{1}{2}, W')}{(W'-W_0)(W'-W)}-\frac{W-W_0}{\pi}\int_{M+1}^{\infty}dW'\frac{\phi(J,W')N(J-\frac{1}{2}, W')}{(W'-W_0)(W'-W)} \tag{38}$$

and

$$N(J-\tfrac{1}{2}, W)=F^e(J-\tfrac{1}{2}, W)$$
$$+\frac{1}{\pi}\int_{-\infty}^{-M-1}\frac{F^e(J-\frac{1}{2}, W')-[(W-W_0)/(W'-W_0)]F^e(J-\frac{1}{2}, W)}{W'-W}N(J-\tfrac{1}{2}, W')\phi(J,W')dW'$$
$$+\frac{1}{\pi}\int_{M+1}^{\infty}\frac{F^e(J-\frac{1}{2}, W')-[(W-W_0)/(W'-W_0)]F^e(J-\frac{1}{2}, W)}{W'-W}N(J-\tfrac{1}{2}, W')\phi(J,W')dW'. \tag{39}$$

The functions appearing in the above equations can easily be obtained, for negative values of W, by applying the well-known MacDowell symmetry[5] to Eq. (1), i.e.,

$$a^{e,o}(J-\tfrac{1}{2}, -W)=-a^{e,o}(J+\tfrac{1}{2}, W), \tag{40}$$

which, in view of Carlson's theorem, can be continued to the noninteger values of $(J-\frac{1}{2})$. The method for obtaining $F^e(J-\frac{1}{2}, -W)$ is thus the same as for $F^e(J-\frac{1}{2}, W)$.

In the problem of π-π scattering,[1] in order to prove the existence and the boundedness (in s) of the left-hand function, we assumed certain bounds for the amplitude and its absorptive part. Similarly, in the present investigation we assume the following conditions:

$$|\phi_{1,2}|, \quad |\psi_{1,2}|, \quad |\mu_{1,2}|\sim t^{1-\gamma}/s, \quad \text{for} \quad s,t>R \tag{41}$$

(where R is any arbitrary large positive number) and

$$|\bar{\phi}_{1,2}|, \quad |\bar{\psi}_{1,2}|, \quad |\bar{\mu}_{1,2}|\sim t^{1/2-\gamma}/s, \quad \text{for} \quad s,t>R, \tag{42}$$

where $\gamma>0$ and the other notations are defined in Appendix A and B [see Eqs. (A1) to (A9)]. These assumptions are clearly more restrictive than our assumptions in Ref. 1. We have made them in order to simplify the problem and to avoid the complications due to subtractions in the dispersion relations for $A_t(s,t)$ and $A_u(s,u)$. Our ansatz regarding the asymptotic behavior, as in (41) and (22), is consistent with the observation made by other authors[6] that the $B(J-\frac{1}{2}, s)$ amplitude should vanish $\sqrt{s}$ times faster than the amplitude $A(J-\frac{1}{2}, s)$. This can easily be seen, if we substitute Eqs. (41) and (42) into Eqs. (7) and (8), respectively. We would like to point out at this stage

[5] S. W. MacDowell, Phys. Rev. **116**, 774 (1959).

[6] V. Singh and B. M. Udgaonkar, Phys. Rev. **123**, 1487 (1961).

that restrictions (41) and (42) on the asymptotic growth of the spectral functions were first used by Mandelstam[7] in connection with the problem of π-π scattering.

It is now possible to show (see Appendices A and B), that if conditions (41) and (42) are satisfied, there exists a left-hand function $F^e(J-\frac{1}{2}, W)$, such that

(i) $F^e(J-\frac{1}{2}, W)$ is holomorphic for

$$\mathrm{Re}J>\max(1-\gamma, -1)+\tfrac{1}{2}; \quad (43)$$

(ii) $F^e(J-\frac{1}{2}, W)$ asymptotically vanishes as

$$|F^e(J-\tfrac{1}{2}, W)|\sim cs^{-\delta}, \quad s\to\infty \quad (44)$$

where

$$\delta=\min(1,\gamma).$$

The above properties of $F^e(J-\frac{1}{2}, W)$ are sufficient to guarantee that the integral Eq. (39) is nonsingular. Equation (39) will be nonsingular provided

$$\int_{-\infty}^{-M-1} |F^e(J-\tfrac{1}{2}, W)|^2\, dW<\infty\,, \quad (45)$$

$$\int_{M+1}^{\infty} |F^e(J-\tfrac{1}{2}, W)|^2\, dW<\infty\,, \quad (46)$$

$$\int_{M+1}^{\infty}\int_{M+1}^{\infty} |K(W,W')|^2\, dW\, dW'<\infty\,, \quad (47)$$

$$\int_{-\infty}^{-M-1}\int_{-\infty}^{-M-1} |K(W,W')|^2\, dW\, dW'<\infty\,, \quad (48)$$

where

$$K(W,W')=\frac{F^e(J-\frac{1}{2}, W')-[(W-W_0)/(W'-W_0)]F^e(J-\frac{1}{2}, W)}{W'-W}\phi(J,W'). \quad (49)$$

For $\gamma>\frac{1}{2}$ the conditions (45) and (46) are clearly satisfied; thus, in order to show that Eq. (39) is nonsingular we have only to show that the integrals I and I' defined by Eqs. (47) and (48), respectively, should converge. Substituting $W'=W$ in I and I', we obtain

$$I\approx\int_{M+1}^{\infty} dW\, W^{1-4\gamma}\int_{(M+1)/W}^{\infty} \left|\frac{\lambda^{-2\gamma}-\lambda^{-1}}{\lambda-1}\lambda\right|^2 d\lambda<\infty\,,$$

and

$$I'\approx\int_{-\infty}^{-M-1} dW\, W^{1-4\gamma}\int_{-\infty}^{-M-1} \left|\frac{\lambda^{-2\gamma}-\lambda^{-1}}{\lambda-1}\lambda\right|^2 d\lambda<\infty\,,$$

provided $\gamma>\frac{1}{2}$. We have thus shown that $N(J-\frac{1}{2}, W)$ is a meromorphic function of J in the domain

$$\mathrm{Re}J>\max(-1, 1-\gamma')+\tfrac{1}{2}\,,$$

where $\gamma'=\max(\frac{1}{2},\gamma)$. It follows that $h^e(J-\frac{1}{2}, W)$, and therefore $a^e(J-\frac{1}{2}, W)$, is a meromorphic function of J in the domain $\mathrm{Re}J>\max(-1, 1-\gamma')+\frac{1}{2}$.

V. EXTENSION OF THE DOMAIN OF MEROMORPHY AND ACCUMULATION OF POLES AT $J=0$

Let us now come to the question of how far to the left we can extend the domain of meromorphy. So far we have shown that the partial-wave amplitude $a^e(J-\frac{1}{2}, W)$ is meromorphic in the domain $\mathrm{Re}J>\frac{3}{2}-\gamma'$, with $\gamma'=\max(\frac{1}{2},\gamma)$. However, at $J=-\frac{1}{2}$, we find that: (i) The representation for $F^e(J-\frac{1}{2}, W)$ develops fixed poles owing to the presence of $Q_{J-1/2}$ functions [see Eq. (25)]. (This feature has also been observed by several other authors.) (ii) The kernel of the integral Eq. (39) is no longer square integrable.

We therefore find that our method does not work beyond the line $\mathrm{Re}J=-\frac{1}{2}$. Consistency demands that $\gamma<2$, which means the amplitudes and their absorptive parts should not vanish faster than t^{-1}/s, $t^{-3/2}/s$ in t [see Eqs. (41) and (42)]. In the case of π-π scattering a similar situation has been discussed in Ref. 1.

In the π-π scattering,[1] apart from the singularities at negative integral values of l, we also considered the accumulation of poles at $l=-\frac{1}{2}$ at threshold. Let us investigate this feature in the present problem. The behavior of $D(J-\frac{1}{2}, W)$ [Eq. (38)] at $W^2=(M+1)^2$ and at $J=0$ is given by

$$D(J-\tfrac{1}{2}, W)=1-[W^2-(M+1)^2]^J\psi(J)\,,$$

where $\psi(J)$ is at most mermorphic at $J=0$. If $\psi(J)$ is nonzero and bounded in the neighborhood of $J=0$, then the zeros of $D(J-\frac{1}{2}, W)$ will be given by

$$J=\frac{a}{\ln[W^2-(M+1)^2]}\pm\frac{2im\pi}{\ln[W^2-(M+1)^2]}\,,$$

where $a=-\ln\psi(0)$ and $m=1, 2, 3\cdots$. The above equation clearly shows the accumulation of poles of $a^e(J-\frac{1}{2}, W)$ in the neighborhood of $J=0$ and at threshold.

In conclusion, we have shown that from the Mandelstam representation with a finite number of subtractions for the total scattering amplitude, in π-N scattering, there exists a domain of holomorphy $\mathrm{Re}J>\alpha+\frac{1}{2}$ of the Froissart-Gribov representation defined by Eq. (1). Under some reasonable assumptions on the asymptotic growth of the spectral functions Eqs. (41) and (42) it is possible to analytically continue the F-G representation with the help of the N/D method of analytic continuation. And it has been shown that the F-G representation is meromorphic in the domain $\mathrm{Re}J>\frac{3}{2}-\gamma'$ with

[7] S. Mandelstam, Ann. Phys. (N. Y.) 21, 302 (1963).

$\gamma'=\max(\frac{1}{2},\gamma)$. We have further shown that with the help of N/D method the domain of meromorphy cannot be extended beyond the line $J=-\frac{1}{2}$. However, at $J=0$ there is an accumulation of an infinite number of poles at threshold. It is interesting to note that the domain of meromorphy as obtained by the N/D method is larger than the domain obtained by the analytic completion.[8]

The author is grateful to Dr. Haridas Banerjee for interesting discussions and useful suggestions.

APPENDIX A

In this Appendix we shall prove some results which will be used in Appendix B.

The total scattering amplitude $A(s,t)$ is defined by

$$A(s,t)=\mu_1(s,t)+\mu_2(s,\Sigma-s-t),$$
$$B(s,t)=\frac{R_s}{M^2-s}+\frac{R_u}{M^2-u}+\bar{\mu}_1(s,t)+\bar{\mu}_2(s,\Sigma-s-t), \quad \text{(A1)}$$

where

$$\mu_1(s,t)=\frac{1}{\pi}\int_4^\infty \frac{A_t(s,t)}{(t'-t)}\left(\frac{t}{t'}\right)^N dt'+\sum_{n=1}^N t^{n-1}L_n(s), \quad \text{(A2)}$$

and

$$\mu_2(s,t)=\frac{1}{\pi}\int_{(M+1)^2}^\infty \frac{A_u(s,u')}{u'-u}\left(\frac{u}{u'}\right)^N du' +\sum_{n=1}^N u^{n-1}M_n(s). \quad \text{(A3)}$$

The absorptive parts in the t and u channels are defined as

$$A_t(s,t)=\phi_1(s,t)+\phi_2(t,\Sigma-s-t) \quad \text{(A4)}$$

and

$$A_u(s,u)=\psi_2(\Sigma-s-u,u)+\psi_1(s,u), \quad \text{(A5)}$$

where

$$\phi_1(s,t)=\frac{1}{\pi}\int_{(M+1)^2}^\infty \frac{\rho_{12}(s',t)}{s'-s}ds', \quad \text{(A6)}$$

$$\phi_2(t,\Sigma-s-t)=\frac{1}{\pi}\int_{(M+1)^2}^\infty \frac{\rho_{23}(t,u')}{u'-u}du', \quad \text{(A7)}$$

$$\psi_1(s,u)=\frac{1}{\pi}\int_{(M+1)^2}^\infty \frac{\rho_{13}(s',u)}{s'-s}ds', \quad \text{(A8)}$$

$$\psi_2(\Sigma-s-u,u)=\frac{1}{\pi}\int_4^\infty \frac{\rho_{23}(t',u)}{t'-t}dt', \quad \text{(A9)}$$

with similar relations for the other invariant amplitude $B(s,t,u)$ in which all the double spectral functions are denoted by symbols with bars over them.

In view of our ansatz Eqs. (41) and (42) it follows from Eqs. (A6) to (A9) that

$$\begin{aligned}\rho_{12}(s',t)&\sim t^{1-\gamma}/s',\\ \rho_{23}(t,u')&\sim t^{1-\gamma}/u',\\ \rho_{13}(s',u)&\sim u^{1-\gamma}/s',\end{aligned} \quad \text{(A10)}$$

and

$$\begin{aligned}\bar{\rho}_{12}(s',t)&\sim t^{1/2-\gamma}/s',\\ \bar{\rho}_{23}(t,u')&\sim t^{1/2-\gamma}/u',\\ \bar{\rho}_{13}(s',u')&\sim u'^{1/2-\gamma}/s',\end{aligned} \quad \text{(A11)}$$

for large values of the arguments of the respective spectral functions. Since we have assumed that Eqs. (A6) to (A9) exist without subtraction, we have some further conditions. In (A7), for example,

$$\rho_{12}(s',t)\sim(s')^{-\epsilon}f(t) \quad \text{for} \quad s>R, \quad t<R. \quad \text{(A12)}$$

Similar relations exist for other cases. Using Eqs. (A10), (A11), and (A6) to (A9), carrying out a similar analysis to what we did for the π-π scattering [see Ref. 1, Eq. (A20)], we obtain

$$\begin{aligned}&\int_{-\xi_1}^{+\xi_1}\rho_{12}(v+\gamma,t)v^n dv\sim t^{1-\gamma}, \quad \text{for} \quad t>R,\\ &\int_{-\xi_1}^{+\xi_1}\rho_{23}(t,v+\gamma_1)v^n dv\sim t^{1-\gamma}, \quad \text{for} \quad t>R,\\ &\int_{-\xi_1}^{+\xi_1}\rho_{13}(v+\gamma_1,u)v^n dv\sim u^{1-\gamma}, \quad \text{for} \quad u>R,\\ &\int_{-\xi_2}^{+\xi_2}\rho_{23}(v+\gamma_2,u)v^n dv\sim u^{1-\gamma}, \quad \text{for} \quad u>R,\end{aligned} \quad \text{(A13)}$$

where

$$\xi_1=[-(M+1)^2+R]/2, \quad \gamma_1=-[(M+1)^2+R^2]/2,$$
$$\xi_2=\tfrac{1}{2}(-4+R), \quad \gamma_2=-\tfrac{1}{2}(4+R). \quad \text{(A14)}$$

For the spectral functions of $B(s,t)$ similar relations exist. For example,

$$\int_{-\xi_1}^{+\xi_1}\bar{\rho}_{12}(v+\gamma_1,t)v^n dv\sim t^{1/2-\gamma}, \quad t>R. \quad \text{(A15)}$$

For the asymptotic behavior of the absorptive parts in the t and u channels, we follow a similar procedure to that given in Ref. 1 [see Eq. (A23)], and using the restrictions on $\bar{\mu}_{1,2}$ and μ_{12}, it follows that

$$\int_4^R \{A_t(s,t),B_t(s,t)\}dt\sim s^{-1}, \quad \text{(A16)}$$

and

$$\int_{(M+1)^2}^R \{A_u(s,u),B_u(s,u)\}du\sim s^{-1}, \quad \text{(A17)}$$

for large values of s.

[8] G. C. Joshi (unpublished).

APPENDIX B

In this Appendix, we shall prove certain results regarding the existence and the asymptotic behavior of the left-hand function $F^e(J-\frac{1}{2}, W)$. We shall show that

(i) $F^e(J-\frac{1}{2}, W)$ is holomorphic in the region

$$\mathrm{Re}J>\max(-1, 1-\gamma)+\tfrac{1}{2}; \qquad \text{(B1)}$$

(ii) $F^e(J-\frac{1}{2}, W)$ asymptotically vanishes, i.e.,

$$F^e(J-\tfrac{1}{2}, W)\sim s^{-\delta}, \quad s\to\infty \qquad \text{(B2)}$$

where $\delta=\min(1,\gamma)$.

Let us now consider $F_1{}^e$ as defined in Eq. (25) in the form

$$F_1{}^e(J-\tfrac{1}{2}, W)=4\sum_{k=1}^{4} M_k, \qquad \text{(B3)}$$

where

$$M_1=\int_4^\infty dt\, \phi'(s,t)x(s)Q_{J-1/2}(z(s,t))$$

$$-\frac{1}{\pi}\int_R^\infty dt\int_R^\infty ds'\frac{\rho_{12}(s',t)}{s'-s}x(s')Q_{J-1/2}(z(s',t)) \qquad \text{(B4)}$$

$$M_2=\frac{1}{\pi}\int_R^\infty dt\int_{(M+1)^2}^\infty ds'\frac{\rho_{12}(s',t)}{s'-s}$$

$$\times[x(s)Q_{J-1/2}(z(s,t))-x(s')Q_{J-1/2}(z(s',t))], \qquad \text{(B5)}$$

$$M_3=x(s)\int_4^R dt\, A_t(s,t)Q_{J-1/2}(z(s,t))$$

$$-\frac{1}{\pi}\int_4^R dt\int_R^\infty ds'\frac{\rho_{12}(s',t)}{s'-s}x(s')Q_{J-1/2}(z(s',t)), \qquad \text{(B6)}$$

$$M_4=x(s)\int_R^\infty dt\, \phi_2(t, \Sigma-s-t)Q_{J-1/2}(z(s,t)), \qquad \text{(B7)}$$

In the above and in Eqs. (25) to (33), we have used the following notation:

$$x(s)=\lambda(J,S)/[4q^2(s)]^{J-1/2}, \qquad \text{(B8)}$$

$$x'(s)=(\lambda(J,S)/[4q^2(s)]^{J+1/2})(\sqrt{s}-M), \qquad \text{(B9)}$$

$$x''(s)=4\frac{\lambda(J,s)}{[4q^2(s)]^{J+3/2}}[E(s)-M]^2, \qquad \text{(B10)}$$

$$x'''(s)=4\frac{\lambda(J,s)}{[4q^2(s)]^{J+3/2}}[E(s)-M]^2(\sqrt{s}+M), \qquad \text{(B11)}$$

$$Q_J(1+t/2q^2(s))=Q_J(z(s,t)), \qquad \text{(B12)}$$

$$Q_J\left(1+\frac{u-(M^2-1)^2/s}{2q^2(s)}\right)=Q_J(z'(s,u)), \qquad \text{(B13)}$$

where $\lambda(J,s)$ is defined in Eq. (12) and $\phi'(s,t)$ is defined in the same sense as in Eq. (A12) of Ref. 1.

We shall now show that the required properties of the left-hand function, namely, (B1) and (B2), are true for each M_k [$k=1, 2, 3, 4$].

(i) M_1. In view of the Eqs. (A6) and (A10), the first integral on the right-hand side of (B4) exists for $\mathrm{Re}>\frac{3}{2}-\gamma$ and vanishes asymptotically in s, like (B2). For the second integral of (B4), we use (A10) and follow a similar procedure, to that given for Eq. (B7) of Ref. 1. We finally obtain that the second integral of (B4) has the properties as demanded by (B1) and (B2).

(ii) M_2. Let us write (B5) in the following form:

$$M_2=\int_R^\infty \frac{dt}{t^{J+3/2}}\int_{(M+1)^2}^\infty ds'\rho_{12}(s',t)F(s',t,s), \qquad \text{(B14)}$$

where $F(s',t,s)$ is bounded in t and analytic in s' in the interval $(M+1)^2\leq s'\leq R$. Hence following Eq. (B10) of Ref. 1, we obtain

$$F(s',t,s)=\sum_{n=0}^\infty c_n(s,t)v^n, \qquad \text{(B15)}$$

where

$$v=s'-\tfrac{1}{2}[R+(M+1)^2]. \qquad \text{(B16)}$$

Using (B15) and (A13) in (B14) we obtain

$$M_2\approx\int_R^\infty \frac{dt}{t^{J+1/2}}t^{1-\gamma}; \qquad \text{(B17)}$$

clearly, M_2 exists provided $\mathrm{Re}J>\frac{3}{2}-\gamma$. Similarly, as in Eq. (B13) of Ref. 1 it can be shown that

$$M_2\sim\max(s^{-\gamma},s^{-1}) \quad \text{when} \quad x\to\infty.$$

(iii) M_3. The first integral of (B6) exists for all J such that $\mathrm{Re}J>-\frac{1}{2}$. As s tends to infinity it is given by

$$M_3'\sim\frac{\ln s}{s}\int_4^R A_t(s,t)dt,$$

and using (A16) we obtain

$$M_3'\sim O[s^{-2}].$$

In view of condition (A12) the second integral of (B6) satisfies all the requirements of (B1) and (B2).

(iv) M_4. From the boundedness condition (A10) and Eq. (A7) it follows that

$$\phi_2(t, \Sigma-s-t)\leq\text{constant}(t^{1-\gamma}/s)$$

for large t and $s>0$. Therefore, M_4 exists if $\mathrm{Re}J>\frac{3}{2}-\gamma$ and is bounded asymptotically by $s^{-\gamma}$.

We have thus shown that F_1 as defined in Eq. (25) satisfies the conditions (B1) and (B2). A similar proof exists for all F_k's, and the conditions (B1) and (B2) are satisfied.