N/D Calculation with Inelastic Unitarity of πN Scattering*

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We calculated the real part δ of the πN phase shifts for partial waves $J\leq\frac{3}{2}$ using the N/D equations with inelastic unitarity. The generalized potential is determined by considering single-exchange diagrams for the nucleon, the N^* (1238 MeV) and the ρ (760 MeV). The inelastic factor η is taken from the recent, extensive complex phase-shift analyses. A straight cutoff W_c on the dispersion integrals in the energy plane W is used to eliminate the high-energy divergences associated with the exchange of particles with spin ≥ 1. Full numerical solutions of the integral equation for the N function are obtained by the matrix-inversion technique. The single cutoff W_c is separately adjusted for each J to give the best fit to the two coupled waves $l=J-\frac{1}{2}$ and $J+\frac{1}{2}$. For comparison, we also calculate the δ's using elastic unitarity, i.e., $\eta(W)\equiv 1$. By including inelastic effects, we obtain better agreement with the phase-shift analyses, except for the S_{31} and P_{13} partial waves. In particular, our calculation of the D_{13} phase shift agrees with the phase-shift analyses for $\delta_{D_{13}}$ up to $E_L\approx 450$ MeV (whereas the solution for $\eta\equiv 1$ gives a δ which is much too small). The P_{11} partial wave is of special importance since (in addition to the nucleon pole) it contains a possible resonance at $E_L\sim 570$ MeV which is very inelastic. We did two different calculations of the $I=\frac{1}{2}$, $J=\frac{1}{2}$ partial wave: (i) W_c was adjusted to yield the nucleon pole as a bound state. The residue (related to $g_{\bar{N}N\pi}{}^2$) is approximately twice what it should be. Both the S_{11} and P_{11} phase shifts are in violent disagreement with the phase-shift analyses. (The calculations with inelastic effects gave only a slight improvement over the $\eta\equiv 1$ calculations.) (ii) The nucleon pole was included in the direct channel at the correct position with the correct residue and W_c was adjusted so that no zero appeared in the D function. We then obtained quantitative fits to the low-energy S_{11} and P_{11} phase shifts.

I. INTRODUCTION

RECENTLY, extensive energy-dependent complex phase analyses of the experimental data on pion-nucleon scattering have been performed by Roper[1] and by Auvil, Donnachie, Lea, and Lovelace.[2] The analyses done for incident-pion laboratory kinetic energies, E_L, up to 700 MeV have shown many interesting features. In particular, the S matrix element $S\equiv\eta e^{i\delta}$ for the P_{11} partial wave has δ going through $\pi/2$ at $E_L\sim 575$ MeV and the inelastic factor η becoming very small. Significant inelasticity (i.e., $\eta\ll 1$) is found in several partial waves.

Many attempts have been made to calculate πN scattering theoretically by solving partial-wave dispersion relations using the N/D method.[3,4,5] Inelastic effects have generally been ignored in these calculations. The effects of higher mass inelastic channels can be very important even though one is interested in an energy region in which only the elastic channel being considered (πN channel) is open since the solution of the integral equation involves knowledge of the functions over all physical energies.

In this paper we calculate the real part of the πN phase shifts δ for partial waves $J\leq\frac{3}{2}$ by solving the single-channel N/D equations with inelastic unitarity derived by Frye and Warnock.[6] The input inelastic factors η are taken from the phase-shift analyses.[1,2] We use the generalized potential found by considering single-particle exchange diagrams for the nucleon, the N^*(1238 MeV), and the ρ(760 MeV). The high-energy divergences associated with the exchange of particles with spin ≥ 1 are eliminated (following Ball and Wong[3]) by using a straight cutoff W_c on the dispersion integrals. We also did calculations using a smooth cutoff of the type employed by Abers and Zemach[4] with the result that the low-energy phase shifts were essentially the same for the two types of cutoffs. Full numerical solutions for the N function are obtained by the matrix-inversion technique.

We present, in Sec. II, the N/D equations with inelastic unitarity and the generalized potential that we use in our calculations. The results of these calculations are described in Sec. III.

We calculated the phase shifts δ using both inelastic and elastic unitarity and compared the results to the phase-shift analysis of Roper.[1] Because of spin-effect complications, the calculations involve the simultaneous solution of the N/D equations in the energy plane W for the two partial waves with the same values of I and J, i.e., $l=J-\frac{1}{2}$ and $J+\frac{1}{2}$. We have one adjustable parameter, the cutoff W_c, which is used to give the best fit to the two partial waves.[7] For example, in calculating the P_{33} and D_{33} partial waves, we adjust W_c to produce

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[1] L. Roper, Phys. Rev. Letters **12**, 340 (1964); L. Roper and R. Wright, Phys. Rev. **138**, B921 (1965).
[2] P. Auvil, A. Donnachie, A. Lea, and C. Lovelace, Phys. Letters **12**, 76 (1964).
[3] J. Ball and D. Wong, Phys. Rev. **133**, B179 (1964); **138**, AB4(E) (1965).
[4] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963).
[5] W. Frazer and J. Fulco, Phys. Rev. **119**, 1420 (1960); S. Frautschi and J. Walecka, *ibid.* **120**, 1486 (1960).
[6] G. Frye and R. Warnock, Phys. Rev. **130**, 478 (1963).
[7] Among the coupling constants, only the interactions of the ρ with the nucleon are not well known and we allow these to vary somewhat (keeping, of course, one set of values to determine the potential for all the partial waves).

the 3-3 resonance at the observed energy. By including inelastic effects, we obtain better agreement with Roper's analysis than for the elastic-unitarity calculations with the exception of the S_{31} and P_{13} partial waves. It seems reasonable to expect better agreement here also when it becomes possible to treat the higher energy inelastic effects realistically.[8] Inelastic effects are particularly important in computing the D_{13} partial wave. We are able to fit Roper's $\delta(E_L)$ up to 450 MeV by including them in the calculation whereas the solution for $\eta\equiv 1$ is not close at all. By using the above approach we were able to obtain reasonably good agreement with Roper's phases for all of the partial waves except the S_{11} and P_{11} which we now discuss in detail.

The approach used by Ball and Wong[3] was to compute the nucleon as a bound state of πN system. W_c can then be adjusted to produce the nucleon at the observed energy. Following this procedure we found that it was impossible to fit the low-energy phase shifts $\delta_{S_{11}}$ and $\delta_{P_{11}}$ or even the scattering lengths,[9,10]

$$\begin{aligned} a_1 &= \quad 0.171\pm 0.005\,, \\ a_{11} &= -0.101\pm 0.007\,, \end{aligned} \tag{1}$$

of the S_{11} and P_{11} partial waves, respectively. The value which we obtain for a_1 even has the wrong sign. The calculated value of $g_{\bar{N}N\pi}{}^2$ from the residue of the nucleon pole is too large by a factor[11] of 2. Including inelastic effects leads to only a slight improvement over the pure elastic calculation.[12,13]

We felt that our failure to obtain a good fit to Roper's values for $\delta_{S_{11}}$ and $\delta_{P_{11}}$ is closely related to the fact our calculated residue for the nucleon pole was too large. There is no reason to expect a quantitative fit to these low-energy phase shifts if our solution does not fit the nearest singularity in the πN system. Because of these considerations, we decided to include the nucleon pole in the direct channel in the generalized potential. This procedure forces the nucleon pole to appear at the correct position with the correct residue $(-\frac{2}{3}g_{\bar{N}N\pi}{}^2/4\pi)$ in the solution provided that we now adjust W_c so that no zero appears in the D function below the elastic threshold. We find that it is now possible to adjust the cutoff to obtain the experimental values of the scattering lengths (1) and a good fit to the low-energy S_{11} and P_{11} phase shifts found by Roper.[1]

[8] We let $\eta\to 1$ at the cutoff W_c.

[9] J. Hamilton and W. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

[10] We use units $\hbar=c=m_\pi=1$.

[11] It is interesting that the relativistic calculation yields an N^*_{33} which is too broad and a nucleon with too large a residue whereas the static Chew-Low calculation yields the correct value for the N^*_{33} width as well as the correct residue for the nucleon pole in the static reciprocal bootstrap [see G. Chew, Phys. Rev. Letters **9**, 233 (1962) and F. Low, *ibid.* **9**, 279 (1962)].

[12] Note that the single-channel N/D calculation with inelastic unitarity is not always equivalent to the multichannel ND^{-1} calculation [see M. Bander, P. Coulter, and G. Shaw, Phys. Rev. Letters **14**, 207 (1965)].

[13] The results here were not sensitive to the values of the ρ couplings.

We show in the Appendix that if the D function vanishes at some energy s_0 below the elastic threshold for a particular generalized potential, then adding a term to the generalized potential of the form $g/(s-s_0)$, with g arbitrary, does not change the solution of the N/D equation. Thus if a dynamical pole appears at $s=s_0$, then neither the solutions or the residue of this pole can be changed by adding a pole term at $s=s_0$ to the generalized potential. In our first approach, in which we adjust W_c so that the nucleon pole appears as a bound state, i.e., as a zero in the D function, we were unable to obtain the correct value for $g_{\bar{N}N\pi}{}^2$ which undoubtedly means that our approximation to the generalized potential (and to η) is not good enough. The success of our second approach (in which we force the nucleon to have the correct position and residue) in obtaining good fits to $\delta_{S_{11}}$ and $\delta_{P_{11}}$ cannot be regarded as evidence for the elementarity of the nucleon. We conclude only that the nucleon pole must have the correct position and residue in the solution in order to have the calculated S_{11} and P_{11} phase shifts agree with experiment.

Our results may be summarized as follows: By including inelastic effects and using the generalized potential of Ball and Wong[3] we are able to obtain reasonably good agreement with the low-energy phase-shift analysis of Roper[1] except for the S_{11} and P_{11} partial waves. By forcing the nucleon pole in our solution of the $J=\frac{1}{2}$, $I=\frac{1}{2}$ partial wave to have the correct position *and* residue we are also able to fit these phase shifts. We are unable to make a definite statement about the P_{11} resonance found by Roper since our results in *this* energy region are sensitive to the ρN coupling (see Fig. 7 and the accompanying discussion in Sec. III). Our results do indicate that $\delta_{P_{11}}$ becomes large at relatively low energy.[14]

II. FORMULATION OF THE PROBLEM

A. Choice of Amplitude and the N/D Equations

The nucleon spin introduces a factor of $s^{1/2}$ in the πN partial wave amplitudes. Singularities of this type are avoided by working in the total-energy (W) plane where $s=W^2$. The amplitude we consider may be written as (omitting isospin indices)[10]

$$h_J(W)\equiv(\eta_J(W)e^{2i\delta_J(W)}-1)/2i\rho_J(W)\,, \tag{2}$$

where δ_J is the real part of the phase shift, η_J is the inelastic factor, and ρ_J is a kinematical factor which we define by

$$\rho_J(W)=(E+m)(k^2/s)^J\,, \tag{3}$$

with m the nucleon mass, k the (center-of-mass) momentum and $E\,[=(s+m^2-1)/2W]$ the nucleon

[14] R. Dalitz and R. Moorhouse, Phys. Letters **14**, 159 (1965) discuss the possibility that the P_{11} enhancement may not be a resonant state.

energy. The quantities η_J and δ_J are defined by

$$\begin{aligned}\eta_J(W)&=\eta_{l+}(W) & W&>W_E\,,\\ &=\eta_{(l+1)-}(W)\,, & W&<-W_E\,,\end{aligned}\tag{4}$$

$$\begin{aligned}\delta_J(W)&=\delta_{l+}(W) & W&>W_E\,,\\ &=\delta_{(l+1)-}(W)\,, & W&<-W_E\,,\end{aligned}\tag{5}$$

where $W_E(=m+1)$ is the elastic threshold and $l\pm$ means the state such that $J=l\pm\frac{1}{2}$.

We use the N/D equations with inelastic unitarity derived by Frye and Warnock.[6] If we make one subtraction in the D function at $W=0$,[15] then we have

$$\begin{aligned}\frac{2\eta_J(W)}{1+\eta_J(W)}\mathrm{Re}N_J(W)&=\bar{B}_J(W)+\frac{1}{\pi}\left\{\int_{-\infty}^{-W_E}+\int_{W_E}^{\infty}\right\}\frac{2\rho_J(W')\mathrm{Re}N_J(W')}{1+\eta_J(W')}\\&\quad\times\left[\bar{B}_J(W')-\frac{W}{W'}\bar{B}_J(W)\right]\frac{dW'}{W'-W}\,,\end{aligned}\tag{6}$$

$$\mathrm{Re}D_J(W)=1-\frac{W}{\pi}P\left\{\int_{-\infty}^{-W_E}+\int_{W_E}^{\infty}\right\}\times\frac{2\rho_J(W')\mathrm{Re}N_J(W')}{[1+\eta_J(W')]W'}\frac{dW'}{(W'-W)}\,,\tag{7}$$

$$\bar{B}_J(W)=h_J{}^L(W)+\frac{1}{\pi}P\left\{\int_{-\infty}^{-W_I}+\int_{W_I}^{\infty}\right\}\times\frac{1-\eta_J(W')}{2\rho_J(W')}\frac{dW'}{(W'-W)}\,,\tag{8}$$

where W_I is the inelastic threshold and the generalized potential $h_J{}^L(W)$ is that part of $h_J(W)$ which is regular in the physical region $(|W|\geq W_E)$.[16] Note that $\rho_J(W)$ has been defined in (3) so that the proper behavior of the phase shift $(\delta\propto k^{2l+1})$ at the elastic threshold for both parity states (5) is guaranteed. The solution for the amplitude is completed by the relations

$$\mathrm{Im}D_J(W)=-\frac{2\rho_J(W)}{1+\eta_J(W)}\mathrm{Re}N_J(W),\quad |W|>W_E\tag{9}$$

$$\mathrm{Im}N_J(W)=\frac{1-\eta_J(W)}{2\rho_J(W)}\mathrm{Re}D_J(W),\quad |W|>W_E\,,\tag{10}$$

and otherwise

$$\mathrm{Im}N_J(W)=D_J(W)\mathrm{Im}h_J{}^L(W)\,.\tag{11}$$

In terms of the invariant amplitudes A and B, we have

$$\begin{aligned}h_J(W)=\frac{s^{J-1/2}}{16\pi k^{2J-1}}&\Bigg\{[A_{J-1/2}(s)+(W-m)B_{J-1/2}(s)]\\&+\left(\frac{E-m}{k}\right)^2[-A_{J+1/2}(s)+(W+m)B_{J+1/2}(s)]\Bigg\}\,,\end{aligned}\tag{12}$$

where

$$A(s,t)=\tfrac{1}{2}\sum_{l=0}^{\infty}(2l+1)A_l(s)P_l(\cos\theta)\,,\tag{13}$$

$$B(s,t)=\tfrac{1}{2}\sum_{l=0}^{\infty}(2l+1)B_l(s)P_l(\cos\theta)\,,\tag{14}$$

and θ is the center-of-mass scattering angle. As usual we have

$$s=[(k^2+m^2)^{1/2}+(k^2+1)^{1/2}]^2\,,$$
$$t=-2k^2(1-\cos\theta)\,,$$

and

$$u+s+t=2m^2+2\,.$$

B. Generalized Potential

We use the same generalized potential that was found by Ball and Wong[3] by considering the single exchange diagrams[17] for N and N^*(1238 MeV) in the "crossed u channel" and ρ (760 MeV) in the t channel. Then for the two isotopic spin amplitudes we have[3,16]

$$\begin{aligned}{}^{(1/2,3/2)}A^L(s,t)&=(2,1)(6\pi)(\gamma_2/m_\rho{}^2-t)(2s+t-2m^2-2)\\&\quad+(\tfrac{4}{3},\tfrac{1}{3})(8\pi\gamma_{33}/(m_{33}{}^2-u))\{(m_{33}-m)(E_{33}+m)^2\\&\quad+\tfrac{3}{2}(m_{33}+m)[m^2+1-\tfrac{1}{2}m_{33}{}^2\\&\qquad-s+(m^2-1)^2/2m_{33}{}^2]\}\,,\end{aligned}\tag{15}$$

and

$$\begin{aligned}{}^{(1/2,3/2)}B^L(s,t)&=(1,-2)\frac{g_{\bar{N}N\pi}{}^2}{m^2-u}\\&\quad+(2,-1)(-12\pi)\frac{\gamma_1+2m\gamma_2}{m_\rho{}^2-t}\\&\quad+(-\tfrac{4}{3},-\tfrac{1}{3})\frac{8\pi\gamma_{33}}{m_{33}{}^2-u}\Bigg\{-(E_{33}+m)^2\\&\quad+\tfrac{3}{2}\left[m^2+1-\tfrac{1}{2}m_{33}{}^2-s+\frac{m^2-1}{2m_{33}{}^2}\right]\Bigg\}\,,\end{aligned}\tag{16}$$

where m_ρ is the mass of the ρ(760 MeV), m_{33} is the mass of the N^*(1238 MeV), and E_{33} is the nucleon energy at the 3-3 resonance. In these equations, $g_{\bar{N}N\pi}$ is the renormalized πN coupling constant. The N^* residue γ_{33} may be obtained from the experimental width of the N^* (in the narrow width approximation). The residues γ_1 and γ_2 for the electric and magnetic couplings of the

[15] The amplitude N/D is of course independent of the subtraction point. This is one of the checks that we make for our numerical program.

[16] The superscript L signifies that these functions are regular in the physical s region.

[17] Abers and Zemach, Ref. 4, use a somewhat different N^* term.

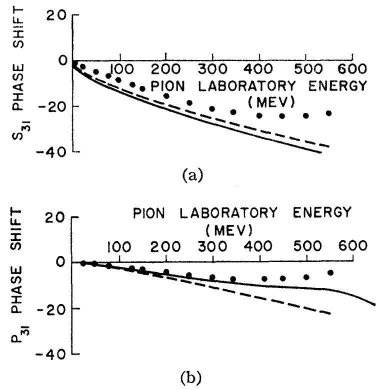

FIG. 1. (a) S_{31} phase shift as a function of the pion laboratory kinetic energy E_L. Here (and in all of the other figures) the solid curve is the solution of the Frye-Warnock N/D equations with inelastic unitary and the dashed curve is the solution for $\eta\equiv 1$. The dots represent the results of the phase shift analysis of Ref. 1. In Figs. 1–6, we used $g_{\bar{N}N\pi}^2/4\pi=14.6$, $\gamma_{33}=0.06$, $\gamma_2=0.27\gamma_1$, and $\gamma_1=-0.84$. The inelastic factor η we use is taken from Ref. 1 in the calculations for Figs. 1, 2, 5, 6, and 7. We use $W_c=19.0$ and $W_c=18.6$ for the elastic and inelastic unitarity calculations, respectively. (b) Same as (a) for the P_{31} partial wave.

ρ to the nucleon may be determined from a fit to the nucleon's isovector electromagnetic form factor.[18,19]

The partial wave projections $h_J{}^L(W)$, our generalized potentials, are obtained from (15) and (16) using (12)–(14).

III. CALCULATIONS AND DISCUSSION

The generalized potential $h_J{}^L(W)$, determined from $A^L(s,t)$ and $B^L(s,t)$ as given by (15) and (16), behaves like $0(W)$ at high energies. This asymptotic behavior makes it impossible to solve the N/D equations. We force the equations to have a unique solution by introducing a straight cutoff on the W integration at $W=W_c$.[20] We obtain full numerical solutions to the N/D equations, employing the matrix-inversion technique to solve the integral for the N function.

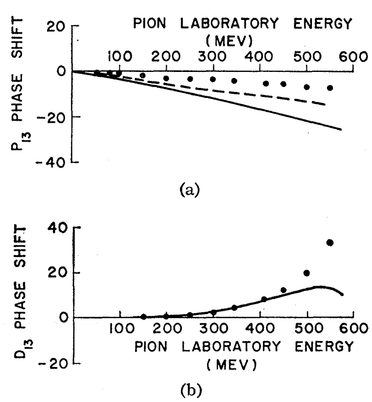

FIG. 2. (a) P_{13} phase shift as a function of pion laboratory kinetic energy. $W_c=18.0$ and 19.0 for the elastic and inelastic unitarity calculations, respectively. (b) Same as (a) for the D_{13} partial wave. The calculated δ using elastic unitarity is too small to be shown on this scale.

The input parameters m_ρ, m_{33}, $g_{\bar{N}N\pi}^2$, and γ_{33} in (15) and (16) for the generalized potential are well determined from experiment: We use $m_\rho=5.4$, $m_{33}=8.8$, $g_{\bar{N}N\pi}^2/4\pi=14.6$, and $\gamma_{33}=0.06$. On the other hand γ_1 and γ_2, proportional to the electric and magnetic couplings of the ρ to the nucleon are not well known. From a fit to the nucleon-isovector electromagnetic form factors, Ball and Wong[18] and Spearman[19] estimate that $\gamma_1\approx-1.0$ and

$$\gamma_2\approx 0.27\,\gamma_1. \tag{17}$$

We will use (17) and treat γ_1 as an adjustable parameter, varying it in the neighborhood of -1.0. For all the calculations shown in Figs. 1–6 we use $\gamma_1=-0.84$.

The inelastic factor η (the remainder of the input for the N/D equations) is taken from the phase-shift analyses of Refs. 1 and 2. This determines η up to $E_L\approx 700$ MeV; we then let η smoothly approach 1 at the cutoff W_c. We numerically solve the N/D equations and adjust W_c for each $J\leq\frac{3}{2}$ to obtain the best fit to the real parts of the phase shifts, δ, for both of the parity states $l=J\pm\frac{1}{2}$ which are coupled in the W plane. The phase shifts for pure elastic scattering are calculated for comparison.

The results of these calculations are shown in Figs. 1–5. The phase shifts $\delta_{l_{2I,2J}}$ calculated using inelastic and elastic unitarity ($\eta(W)\equiv 1$) are shown along with the phase shifts obtained by Roper[1]. The δ's we find,

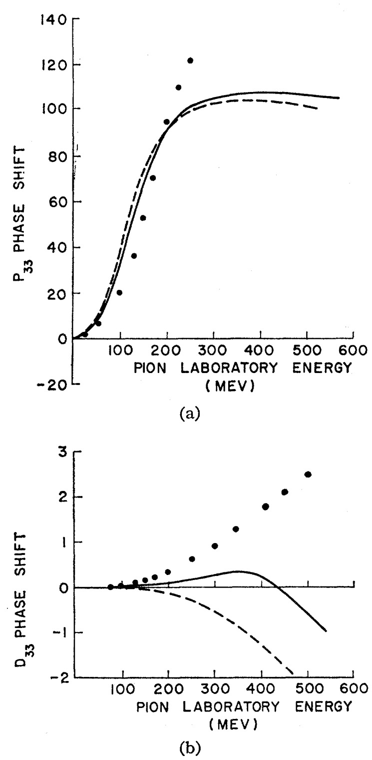

FIG. 3. (a) P_{33} phase shift as a function of E_L. The inelastic factors used here and in Fig. 4 are taken from Ref. 2. $W_c=17.3$ and 19.0 for the elastic and inelastic unitarity calculations, respectively. (b) Same as (a) for the D_{33} partial wave.

[18] J. Ball and D. Wong, Phys. Rev. **130**, 2112 (1963).
[19] T. Spearman, Phys. Rev. **129**, 1847 (1963).
[20] A smooth cutoff of the type used in Ref. 4 gives about the same results.

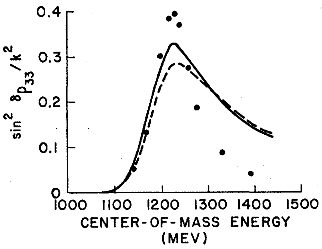

FIG. 4. A plot of $(1/k^2)\sin^2\delta$ for the P_{33} partial wave as a function of center-of-mass energy W. The cutoff is adjusted so that the peak in the cross section occurs at the observed position. $W_c=16.0$ and 17.8 for the elastic and inelastic calculations, respectively.

with the exception of the S_{11} and P_{11} partial waves, are in reasonably good agreement with Roper's results up to $E_L\sim$400–500 MeV. We observe that including inelastic effects produces better agreement with Roper's phases with the exception of the S_{31} and P_{13} partial waves. It seems likely that an improved treatment of the inelastic effects at high energy will produce better agreement here also.

According to Roper,[1] the inelastic factors in the $I=\frac{3}{2}$, $J=\frac{3}{2}$ partial waves are very nearly unity up to 700 MeV. However, Auvil *et al.*[2] did find values of

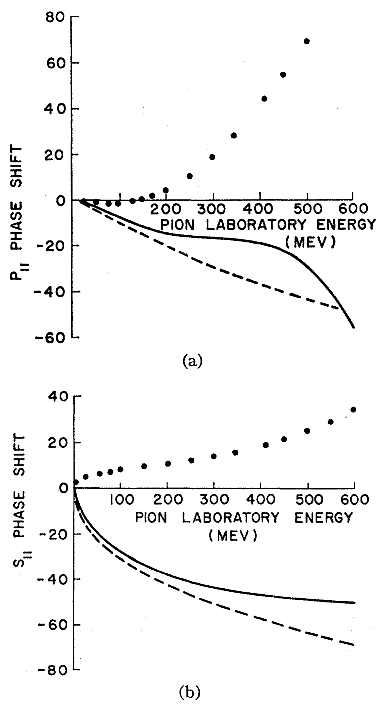

FIG. 5. (a) P_{11} phase as a function of E_L where the nucleon pole is forced to appear as a dynamical bound state at the correct energy. For the elastic unitarity calculation, $W_c=16.5$ giving an output $g_{\bar{N}N\pi}^2/4\pi=29.0$ and the computed scattering length is -0.278. For the inelastic unitarity calculation, $W_c=17.2$ giving an output $g_{\bar{N}N\pi}^2/4\pi=25.1$ and the scattering length of -0.222. (b) Same as (a) for the S_{11} partial wave. The computed scattering lengths are -0.517 and -0.476 for the elastic and inelastic unitarity calculations, respectively.

$1-\eta\approx0.2$ in the D_{33} partial wave near 700 MeV and some slight inelastic effects in the P_{33} partial wave. We obtain the solid curves in Fig. 3 by including these inelastic effects in the D_{33} partial wave while using $\eta\equiv1$ in the P_{33} partial wave. The result is a slight improvement to Roper's analysis. Again, we expect the fit to be better when it becomes possible to include higher energy inelastic effects realistically. In Fig. 4 we adjusted the cutoff so that the peak in the cross section appears at the N^* mass (since the output widths are large). The result of including inelastic effects here is to reduce the computed width of the N^* by about 40 MeV (from $\sim$235 to $\sim$195 MeV).

Inelastic effects are very important in computing the D_{13} partial wave [Fig. 2(b)]. When elastic unitarity is used, the phase shifts are small and negative, remaining $>-0.5°$ up to 500 MeV. Using the inelastic factors found by Roper we obtain good agreement with his phases up to $\sim$450 MeV.

The only curves that are in clear disagreement with Roper's phases are for the S_{11} and P_{11} partial waves. Here W_c must be chosen so that the nucleon pole is produced at the correct energy. By computing the residue of this pole we obtain a value for $g_{\bar{N}N\pi}^2$ which is too large by about a factor of 2. The computed scattering lengths are in violent disagreement with the experimental values (1).[9] Including inelastic effects lead to only a slight improvement over the pure elastic calculations.[12] The results are not sensitive to γ_1.

Because of our failure to obtain agreement with experiment for the $J=\frac{1}{2}$, $I=\frac{1}{2}$ partial waves, we repeated the calculations with the nucleon pole in the

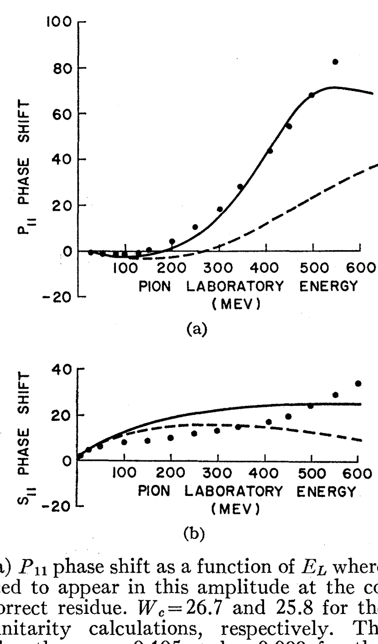

FIG. 6. (a) P_{11} phase shift as a function of E_L where the nucleon pole is forced to appear in this amplitude at the correct energy with the correct residue. $W_c=26.7$ and 25.8 for the elastic and inelastic unitarity calculations, respectively. The computed scattering lengths are -0.105 and -0.099 for the elastic and inelastic cases, respectively. (b) Same as (a) for the S_{11} partial wave. The computed scattering length in each case is 0.17.

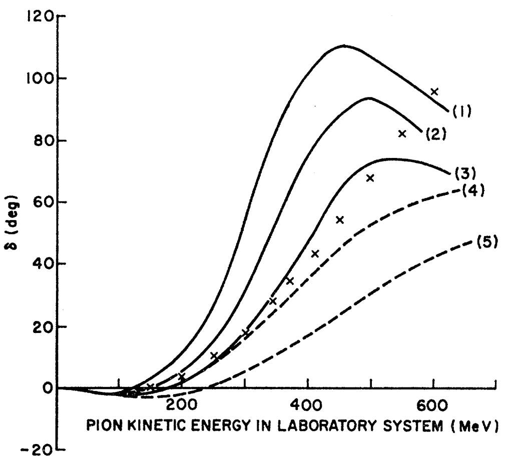

FIG. 7. Dependence of the P_{11} phase shift on γ_1. We use $g_{\bar{N}N\pi}{}^2/4\pi=15.0$, $\gamma_{33}=0.06$, and $\gamma_2=0.27\,\gamma_1$. The scattering lengths for curves 1–5 are -0.086, -0.094, -0.100, -0.095, and -0.106, respectively, with $W_c=26.3$, 26.2, 26.0, 27.2, and 26.9, respectively. (The scattering length in the S_{11} partial wave is 0.17 in all cases.) The solid and dashed curves are for inelastic and elastic unitarity calculations, respectively, and the crosses are the results of the phase-shift analysis of Ref. 1. Curves 1–5 correspond to $\gamma_1=-1.0$, -0.95, -0.90, -1.0, and -0.90, respectively.

direct channel included in the generalized potential. (We add a term to ${}^{(3/2,1/2)}B^L$, Eq. (15), of the form (0,3) $g_{\bar{N}N\pi}{}^2/(m^2-s)$.) This procedure insures that the nucleon pole in our solution occurs at the right position with the correct residue; we must of course adjust the cutoff so that no zero appears in $D(W)$ below the elastic threshold. It is now possible to adjust W_c to obtain a good fit to the low energy S_{11} and P_{11} phase shifts. This is shown in Fig. 6 where our calculated scattering lengths are $a_1=0.17$ and $a_{11}=-0.099$. We found that $\delta_{P_{11}}(W)$ was sensitive to γ_1. Since γ_1 is not well determined, we treat it as an adjustable parameter, while keeping it close to -1.0 [using (17) to determine γ_2]. (The other partial waves were not sensitive to the exact input values of the parameters.) The dependence of $\delta_{P_{11}}$ on γ_1 is shown in Fig. 7. In all cases we see that $\delta_{P_{11}}$ becomes large at relatively low energies.[14]

This calculation should not be interpreted as evidence for the elementarity of the nucleon (see Appendix). We only conclude that in order to obtain agreement with the experimental value of the S_{11} and P_{11} phase shifts, the nucleon pole in our solution must have the correct position and residue: The generalized potential obtained from (15) and (16) and the η used were not good enough to do this.

APPENDIX

Consider the usual single-channel N/D equations in the s plane with elastic unitarity. (The result is easily generalized to include inelastic unitarity.) We will prove that if the D function is zero at some point $s=s_0<s_E$ (the elastic threshold) for a given generalized potential B then adding a pole to B of the form $g/(s-s_0)$ with g arbitrary does not change the solution $A=N/D$.[21] (Or, in other words, the residue of a dynamical pole or bound state is determined by the potential and cannot be arbitrarily changed.)

The unsubtracted N/D equations are

$$N(s)=B(s)+\frac{1}{\pi}\int_{s_E}^{\infty}\frac{[B(s')-B(s)]}{s'-s}\rho(s')N(s')ds'\,, \quad \text{(A1)}$$

$$D(s)=1-\frac{1}{\pi}\int_{s_E}^{\infty}\frac{\rho(s')N(s')ds'}{s'-s-i\epsilon}\,. \quad \text{(A2)}$$

We assume that $B(s)$ is such that (A1) is a Fredholm integral equation and thus has a *unique* solution. Now define

$$\bar{B}(s)=B(s)+g/(s-s_0)\,, \quad \text{(A3)}$$

where $s_0<s_E$. We may now compute new functions $\bar{N}$ and $\bar{D}$ such that

$$\bar{N}(s)=\bar{B}(s)+\frac{1}{\pi}\int_{s_E}^{\infty}\frac{[\bar{B}(s')-\bar{B}(s)]}{s'-s}\rho(s')\bar{N}(s')ds'\,, \quad \text{(A4)}$$

$$\bar{D}(s)=1-\frac{1}{\pi}\int_{s_E}^{\infty}\frac{\rho(s')\bar{N}(s')ds'}{s'-s-i\epsilon}\,. \quad \text{(A5)}$$

Equation (A4) is also a Fredholm equation with a unique solution. By expressing B in terms of $\bar{B}$ and rearranging (A1) we find

$$N(s)=\bar{B}(s)+\frac{1}{\pi}\int\frac{[\bar{B}(s')-\bar{B}(s)]}{s'-s}\rho(s')\times N(s')ds'-\frac{g}{s-s_0}D(s_0)\,. \quad \text{(A6)}$$

If $D(s_0)=0$, then (A4) and (A6) become identical and hence

$$\bar{N}(s)=N(s)\,, \quad \text{(A7)}$$

since the solutions of (A4) and (A6) are unique. Hence $\bar{D}(s)=D(s)$ and our solution is unchanged for arbitrary g.

Thus for $B(s)$ such that $D(s_0)=0$, the residue of the dynamical pole is determined by $B(s)$ alone, and one cannot change the solution of the N/D equations by adding a term of the form $g/(s-s_0)$ with g arbitrary. It is still possible, of course, to introduce a nondynamical pole in A at $s=s_0$ with arbitrary residue by readjusting B to a new B' so that $B'(s)+g/(s-s_0)$ does not generate a zero in the D function at $s=s_0$.[22] This is what we did in the calculations in Figs. 6 and 7 described in Sec. III.

[21] A similar result has been obtained by P. Kaus and F. Zachariasen, Phys. Rev. **138**, B1304 (1965).

[22] Neglecting inelastic effects, we could in principle use Levinson's Theorem to distinguish the two types of poles in A as $s=s_0$: (i) the "dynamical" pole resulting from a zero in D, and (ii) the "elementary particle" pole inserted in A (where no zero in D occurs). The quantity $\delta(s_E)-\delta(\infty)=\pi$ for case (i) and zero for case (ii).