mesons have been seen in three-pion states dominated by π-ρ. Work is now in progress on these calculations.

ACKNOWLEDGMENTS

We are greatly indebted to Dr. V. A. Alessandrini and Dr. R. L. Omnes for their constant interest in this work and for many enlightening discussions. We are grateful to Dr. I. T. Drummond for several crucial remarks.

We also wish to thank Dr. D. L. Judd for his hospitality at the Theoretical Group of the Lawrence Radiation Laboratory.

PHYSICAL REVIEW VOLUME 141, NUMBER 4 JANUARY 1966

Relativistic Three-Pion Calculation. III*

JEAN-LOUIS BASDEVANT†
Lawrence Radiation Laboratory, University of California, Berkeley, California

AND

RODNEY E. KREPS‡
Department of Physics, University of California, Berkeley, California
(Received 25 August 1965)

A model for three-spinless-meson relativistic resonances, based on the Alessandrini-Omnes generalization of the Faddeev equations, which was explained in two previous papers, is studied in all three-pion states with angular momentum $J<3$. The only part of the two-body amplitude which is taken into account is the ρ resonance. The separation of isospin is shown explicitly, and it is explained how the momentum-space equations reduce to only one when Bose statistics is taken into account. The results do not suggest the dynamical origin of the A_2, and many isoscalar resonances are found in a small energy region.

I. INTRODUCTION

In two previous papers[1,2] we have described a model for relativistic three-pion resonances. This model uses a version of the Faddeev equations given by Alessandrini and Omnes,[3] and assumes that the two-body amplitude is separable in the initial and final momenta. The calculation was carried out in II for the specific case $I=0$, $J^P=1^-$, by assuming that the π-π amplitude was dominated by the ρ. The results were encouraging in that they exhibited an independence from the detailed structure of the form factors used to characterize the off-shell dependence of the two-body amplitude. The energy-behavior of the eigenvalues of the kernel was shown, and the strong influence of the π-ρ normal threshold allowed one to understand qualitatively the features of the model.

In this paper, we shall keep the assumption that the two-body amplitude is entirely given by the $I=1$, $J=1$ interaction, and we shall examine the results of the same model for all isospin and parity states and for $J=0$, 1, and 2. It must be noted that the isoscalar part of the π-π interaction will contribute to the $I=1$ three-pion states. However, both because the three-pion resonance decays seem experimentally dominated by π-ρ, and because we wish to keep the calculation relatively simple, we have neglected these isoscalar contributions.

In Sec. II, we examine the consequences of having an $I=1$ two-body amplitude on the isospin structure of the three-body scattering amplitude. In Sec. III, we show that the identity of the particles and the application of Bose statistics reduce the number of equations that need to be considered from three to one. Finally, Sec. IV is a statement and discussion of the results.

II. ISOSPIN

In a three-pion system one can construct one state of isospin $I=0$, three independent $I=1$ states, two independent $I=2$ states, and one $I=3$ state. Since we are considering π-ρ systems, $I=3$ is impossible. We shall show that it is possible to remove the isospin dependence (up to the inhomogeneous term and a factor in the kernel) from the Faddeev equations for each definite total isospin.

Case $I=0$

There is just one total isospin state, which may be written variously as

$$|I_1=1, I_{23}=1, I=0\rangle = |I_2=1, I_{31}=1, I=0\rangle = |I_3=1, I_{12}=1, I=0\rangle, \quad (2.1)$$

* Work done under the auspices of the U. S. Atomic Energy Commission.
† On leave of absence from Laboratoire de Physique Théorique et Hautes Energies, Orsay, France. Work supported in part by a NATO fellowship. Present address: Centre de Recherches Nucléaires, Strasbourg-Cronenbourg, Bas-Rhin, France.
‡ Present address: University of Pittsburgh, Pittsburgh, Pennsylvania.
[1] J. L. Basdevant and R. E. Kreps, this issue, Phys. Rev. **141**, 1398 (1966). Hereinafter referred to as I.
[2] J. L. Basdevant and R. E. Kreps, preceding paper, Phys. Rev. **141**, 1404 (1966). Hereinafter referred to as II.
[3] V. A. Alessandrini and R. L. Omnes, Phys. Rev. **139**, B167 (1965).

where 1, 2, 3 are the labels of the particles. It is therefore obvious that the matrix elements of all the transition amplitudes involved in the Faddeev equations are all the same as far as isospin is concerned, and that the Faddeev equations can be written simply by omitting the isospin dependence.

Case $I=1$

Here there are three independent states and we shall choose the symmetric nonorthogonal basis

$$\begin{aligned} |1\rangle &\equiv |I_1=1, I_{23}=0, I=1\rangle, \\ |2\rangle &\equiv |I_2=1, I_{31}=0, I=1\rangle, \\ |3\rangle &\equiv |I_3=1, I_{12}=0, I=1\rangle. \end{aligned} \tag{2.2}$$

The matrix elements of the three-body amplitude between these states are written

$$\langle\alpha|T|\beta\rangle \equiv T_{\alpha\beta}. \tag{2.3}$$

We need to know the matrix elements between these states of, for example, the transition amplitude t^1 where particle 1 does not interact. To this end we define a new basis in which t^1 is diagonal:

$$\begin{aligned} |0\rangle_1 &\equiv |I_1=1, I_{23}=0, I=1\rangle, \\ |1\rangle_1 &\equiv |I_1=1, I_{23}=1, I=1\rangle, \\ |2\rangle_1 &\equiv |I_1=1, I_{23}=2, I=1\rangle. \end{aligned} \tag{2.4}$$

The other basis is related to this one through the equation

$$|\alpha\rangle = \textstyle\sum_\beta S_{\alpha\beta}|\beta\rangle_1, \tag{2.5}$$

where

$$S=\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1/\sqrt{3} & \frac{1}{3}\sqrt{5} \\ \frac{1}{3} & -1/\sqrt{3} & \frac{1}{3}\sqrt{5} \end{bmatrix}. \tag{2.6}$$

As our original basis is not orthogonal, the identity has the representation

$$\mathbf{1} = \textstyle\sum_{\alpha\beta}|\alpha\rangle I_{\alpha\beta}\langle\beta|, \tag{2.7}$$

where

$$I=\tfrac{3}{10}\begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}. \tag{2.8}$$

The Faddeev equation for the amplitude T^1 is

$$T^1 = t^1 - t^1 G_0[T^2+T^3]. \tag{2.9}$$

We take matrix elements of this equation with respect to the basis (2.2), and in the particular case considered here where the subsystem is in $I=1$, upon using (2.5) and (2.7), the resulting equation is

$$(T^1)=\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} t^1 - t^1 G_0 \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} [(T^2)+(T^3)], \tag{2.10}$$

where t^1 no longer has isospin dependence. It is clear from (2.10) that

$$T_{1\beta}{}^1 = 0 \tag{2.11a}$$

and

$$T_{2\beta}{}^1 = -T_{3\beta}{}^1 \equiv G_\beta{}^1. \tag{2.11b}$$

A similar treatment can be given to T^2 and T^3, so that the final Faddeev equations are

$$G_\beta{}^1 = \tfrac{1}{3}t^1(\delta_{\beta 2}-\delta_{\beta 3}) + \tfrac{1}{2}t^1 G_0[G_\beta{}^2 + G_\beta{}^3]. \tag{2.12}$$

The equations for $G_\beta{}^2$ and $G_\beta{}^3$ are obtained by cyclic permutation of the indices. The subscript β has no essential role in the structure of these equations as it is a final variable and appears only in the inhomogeneous term. The kernel of these equations is the same as for $I=0$ except for the numerical factor $-\frac{1}{2}$.[4]

Case $I=2$

There are two independent states and no basis symmetric in the three particles, so we choose

$$\begin{aligned} |1\rangle_1 &\equiv |I_1=1, I_{23}=1, I=2\rangle, \\ |2\rangle_1 &\equiv |I_1=1, I_{23}=2, I=2\rangle. \end{aligned} \tag{2.13}$$

In terms of this basis the Faddeev equations become

$$\begin{aligned} (T^1) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} [t^1 - t^1 G_0][(T^2)+(T^3)], \\ (T^2) &= \begin{pmatrix} \frac{1}{4} & -\sqrt{3}/4 \\ \sqrt{3}/4 & \frac{3}{4} \end{pmatrix} [t^2 - t^2 G_0][(T^1)+(T^3)], \\ (T^3) &= \begin{pmatrix} \frac{1}{4} & \sqrt{3}/4 \\ \sqrt{3}/4 & \frac{3}{4} \end{pmatrix} [t^3 - t^3 G_0][(T^1)+(T^2)]. \end{aligned} \tag{2.14}$$

We define

$$\begin{aligned} G_\beta{}^1 &= -\tfrac{1}{2}T_{1\beta}{}^1, \\ G_\beta{}^2 &= T_{1\beta}{}^2, \\ G_\beta{}^3 &= T_{1\beta}{}^3, \end{aligned} \tag{2.15}$$

and upon inserting (2.15) into (2.14), we obtain equations identical to (2.12) except for the inhomogeneous terms.

III. STATISTICS

For a system of three identical bosons, the statistics requires complete symmetry upon particle interchange. If we call S the symmetrized amplitude and abbreviate the variables $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$ by (123) then, for $I=0$, we have

$$6S(123) = T(123)+T(231)+T(312)-T(132) - T(213)-T(321), \tag{3.1}$$

---

[4] Let us remark that the factors are the usual isospin crossing coefficients for a single isovector exchange, despite the iteration implicit in the equations. For an isoscalar two-body interaction, however, the factor is $\frac{1}{3}$.

since the isospin part is antisymmetric. If we define

$$\Sigma(123)=T^1(123)+T^2(312)+T^3(231)\,, \qquad (3.2)$$

then we have

$$6S(123)=\Sigma(123)+\Sigma(231)+\Sigma(312)-\Sigma(132) -\Sigma(213)-\Sigma(321)\,. \qquad (3.3)$$

For $I=1$, we have, for example,

$$6S_{1\beta}(123)=T_{1\beta}(123)+T_{2\beta}(213)+T_{3\beta}(231) +T_{1\beta}(132)+T_{2\beta}(312)+T_{3\beta}(321)\,. \qquad (3.4)$$

If we define

$$\Sigma(123)=G^1(123)+G^2(312)+G^3(231) \qquad (3.5)$$

and use Eq. (2.11) and its cyclic counterparts, we have

$$6S_{1\beta}(123)=\Sigma(213)+\Sigma(312)-\Sigma(231)-\Sigma(321)\,. \qquad (3.6)$$

Similar equations hold for S_2 and S_3.

For $I=2$, we again define Σ by (3.5) and we have

$$6S_{1\beta}(123)=-2\Sigma(123)+2\Sigma(132)+\Sigma(231) -\Sigma(213)+\Sigma(312)-\Sigma(321)\,, \qquad (3.7)$$

and an analogous form holds for S_2. The important feature of Eqs. (3.3), (3.6), and (3.7) is the physical amplitude depends only on the function Σ. In all three isospin states, because the particles have identical dynamics, the equations for G^i have the form

$$G^1(123)=I_1(123)+K(123,123)[G^2(123)+G^3(123)]\,,$$
$$G^2(123)=I_2(123)+K(231,231)[G^1(123)+G^3(123)]\,,$$
$$G^3(123)=I_3(123)+K(312,312)[G^1(123)+G^2(123)]\,. \qquad (3.8)$$

A trivial permutation of the variable involved in the second and third equations, and a summation, yield

$$\Sigma(123)=I(123) +[K(123,231)+K(123,312)]\Sigma(123)\,, \qquad (3.9)$$

which is our final momentum-space equation. Thus the statistics allows us to consider only one equation instead of the original three, and we do not need to impose any symmetry requirements on Σ.

We now want the representation of (3.9) in terms of states of definite angular momentum. We shall follow the techniques of Omnes[5] as was done in I and II, so that the μth component of an object of angular momentum J can be written

$$f_\mu{}^J(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) =\frac{2J+1}{8\pi^2}\sum_M f_{\mu M}{}^J(\omega_1,\omega_2,\omega_3)\mathfrak{D}_{\mu M}{}^J(\alpha,\beta,\gamma)\,, \qquad (3.10)$$

where $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0$, $\omega_i{}^2=p_i{}^2+1$, and (α,β,γ) are the Euler angles which take the space-fixed into a body-fixed system whose z axis is along $\mathbf{p}_1\times\mathbf{p}_2$, and whose x axis is along $\mathbf{p}_1$.[6]

After separation of angular momentum in this fashion, our Eq. (3.9) becomes

$$\Sigma_M(\omega_1\omega_2\omega_3) =I_M(\omega_1\omega_2\omega_3)+\sum_{M'=-J}^{M'=+J}[K_{MM'}(\omega_1\omega_2\omega_3,\omega_2'\omega_1'\omega_3') \times e^{-iM'\theta_{12}'}+K_{MM'}(\omega_1\omega_2\omega_3,\omega_3'\omega_1'\omega_2')e^{iM'\theta_{13}'}] \times\Sigma_{M'}(\omega_1',\omega_2',\omega_3')\,, \qquad (3.11)$$

where θ_{ij}' is the angle between $\mathbf{p}_i'$ and $\mathbf{p}_j'$. The inhomogeneous terms depend on the final variables, which have been suppressed. The summation over M' will contain only even or odd terms according to the parity. If the two-body interaction is only in the lth partial wave, then $K_{MM'}$ has the form[1]

$$K_{MM'}(\omega,\omega')=\frac{-1}{2\pi^2}t(\omega,\omega';\sigma)G_0X_{MM'}\delta(\omega_1-\omega_1')\,, \qquad (3.12)$$

where $t(\omega,\omega';\sigma)$ is the two-body lth partial-wave amplitude, G_0 is the Green's function, and

$$X_{MM'}=\sum_\nu \Delta_{M\nu}{}^J\Delta_{M'\nu}{}^{J*}Y_l{}^\nu(\gamma,0)Y_l{}^\nu(\gamma',0)\,; \qquad (3.13)$$

the notations here are the same as in I and II. For the various angular momentum and parity states, assuming that the two-body interaction is in $l=1$, and defining

$$S\equiv\sin\gamma\,\sin\gamma'\,, \qquad (3.14)$$
$$C\equiv\cos\gamma\,\cos\gamma'\,,$$

the matrices are

$$J^P=0^-\,,\qquad X_{00}=(3/4\pi)C\,; \qquad (3.15a)$$

$$J^P=1^-\,,\qquad X_{00}=(3/8\pi)S\,; \qquad (3.15b)$$

$$J^P=1^+\,,\quad X_{MM'}=\frac{3}{16\pi}(S+2MM'C)\,; \qquad (3.15c)$$

$$J^P=2^-\,,\quad X_{MM'}=\frac{3}{32\eta}\{\tfrac{1}{2}MM'S +C[2-\tfrac{1}{4}(2+\sqrt{6})(M^2+M'^2) +(5+2\sqrt{6})M^2M'^2/16]\}\,; \qquad (3.15d)$$

$$J^P=2^+\,,\quad X_{MM'}=\frac{3}{16\pi}MM'S\,. \qquad (3.15e)$$

In the preceding equations, if the parity is even (odd) the possible values for M and M' are odd (even).

The final equations obtained by combining (3.15),

[5] R. L. Omnes, Phys. Rev. **134**, B1358 (1964).

[6] A little omission must be noted in Eq. (37) of Ref. 5 which relates states differing by a rotation of the body-fixed z axis into the plane. As is clear from the representation (3.10), in general there are phase factors present.

TABLE I. Masses and widths in MeV of the resonances obtained in the various channels.

	$I=0$		$I=1, 2$	
J^P	M	Γ	M	Γ
0^-	1370[a]	140	none	
1^+	1390	80	1340	90
1^-	1470	60	none	
2^+	1350	85	none	
2^-	1480	60	1350	85

[a] It must be noted that in the 0^- channel there seem to be two resonances close together. In fact, two eigenvalues of the kernel, which are very nearly equal, approach 1. This explains the large width of the resonance curve obtained in this case.

(3.12), and (3.11) can be further reduced by using explicit symmetries of the kernel, and by using a separable approximation to the two-body amplitude.

IV. RESULTS AND DISCUSSION

The method used to analyze the equations is the same as in II. Again it should be emphasized that we are interested only in resonances, so that we study only the properties of the kernel and do not compute the full amplitude. Also, the only part of the two-body amplitude which we consider is the ρ state, so that the results are the same for $I=1$ and $I=2$, as explained in Sec. II. The complete results are given in Table I, where we give the masses and widths of the computed resonances for all isospins and for total angular momentum $J=0$, 1, 2 in the two parity states.

One immediate feature of this table is the absence of a $J^P=2^+$, $I=1$ resonance. In fact this is the channel of the A_2 particle,[7] whose existence and quantum numbers seem very well established[8] and which has a branching ratio of 60% in π-ρ. The only isovector resonances that we obtain are in $J^P=1^+$ or 2^-, which are impossible for the observed $K\bar{K}$ decay mode of the A_2. However, these resonances are not incompatible with the experimentally observed enhancements above 1 BeV (the A_1 meson,[7] for instance).

We also find a plethora of isoscalar resonances, all of which lie in a very limited energy region. Although this is not absolutely incompatible with the present experimental evidence, it does seem unlikely that all of these resonances actually do exist. However, as was pointed out in II, the E meson[7] seems a good candidate for one of them and at a lower energy there is some evidence for the so-called H meson.[7]

An interesting detail of the results is that this model predicts relatively small widths, which are in very good agreement with the size of observed widths of meson resonances in the energy range 1 to 1.4 BeV. Furthermore, it was seen in II that because of the nature of the eigenvalue curves, influenced essentially by the π-ρ normal threshold, the widths of resonances tend to increase as the mass decreases in this energy region. This is because the imaginary part of the eigenvalues is largest in the vicinity of the π-ρ threshold. It is interesting to remark that the widths and masses of experimentally observed resonances follow this qualitative rule.

In conclusion, we can say that the lack of an A_2 and the multitude of $I=0$ resonances seem to indicate serious deficiencies of the model. The simplest remedy that one could try would be to give more detailed information on the two-body scattering amplitude by including the f_0 and the ABC enhancement. This change would affect only the $I=1$ channels, and the ABC would not contribute to the 2^+ and 1^- amplitudes. The large branching ratio[8] of the A_2 into $K\bar{K}$ and π-η indicate that a good calculation of this particle should include these channels. In general, it seems certain that at the energies of our resonances, inelastic effects will often be important. Also, the inherent drawbacks of a Faddeev-type approach, among which are the omission of certain classes of diagram and the lack of consideration of crossing symmetry, may prevent this model from having any physical significance. However, the model does yield some interesting results and we consider that further theoretical and experimental information is necessary before either accepting or discarding it in whole or in part.

ACKNOWLEDGMENTS

We wish to thank Dr. David L. Judd for his hospitality at the Theoretical Group of the Lawrence Radiation Laboratory.

[7] For all experimental results, see, for example, G. Goldhaber, *Second Coral Gables Conference, Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, 1965), p. 34; and A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **36**, 977 (1964).

[8] For the determination of the existence and quantum numbers of the A_2, see Suh Urk Chung *et al.*, Phys. Rev. Letters **12**, 621 (1964); and M. Aderholz *et al.*, Phys. Letters **10**, 226 (1964).