Relativistic Three-Pion Calculation. I*

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(Received 4 August 1965)

In order to do a reasonably simple relativistic three-meson calculation, we examine a relativistic analog to the Faddeev equations proposed by Alessandrini and Omnes. An explicit angular-momentum decomposition, following the method of Omnes, is given. Some remarks are made about suitable approximations for the relativistic off-shell two-body amplitude, and a separable form of this is used to reduce the equations to their final form. As an example, the explicit equations are written in the three-pion $I=0$, $J^p=1^-$ case.

I. INTRODUCTION

It is our purpose in this series of papers to examine the usefulness of the Faddeev formulation of the three-body problem in studying meson states. In fact, systems of three pseudoscalar mesons are quite interesting. There has been recently a great deal of experimental data showing enhancements in the invariant mass distributions of such particles. If we look at the three-pion system, for instance, there is some evidence[1] for the existence of the so-called $H(975)$, $A_1(1080)$, and $A_2(1230)$ mesons, besides the well-known ω. In the $K\bar{K}\pi$ system, we can find the $D(1285)$ and $E(1415)$ mesons, and in the $K\pi\pi$ system the $C(1220)$ meson. In many cases, the experimental situation is not very clear, and obviously it is highly desirable to have a model which allows us to understand these three-body resonances—or enhancements. Also, in a general understanding of all strongly interacting particles as composite, it is well known that the lowest-lying channels communicating with the π and K mesons themselves consist of three pseudoscalar mesons.

The existence of three-particle resonances and the fact that most of them appear to decay primarily via two-particle modes in which one of the products is itself unstable have led very naturally to the necessity of understanding states consisting in a stable particle and an unstable one. From a "two-body" point of view, however, this is a very difficult problem. Such a scattering amplitude contains many singularities which are not encountered in the scattering of two stable particles, and only a serious calculation using S-matrix techniques can be believed at present.[2] Thus it has become more and more important to study three-particle states as such. This has been explained very thoroughly by Lovelace,[3] and in nonrelativistic and semi-nonrelativistic cases the Faddeev equations,[4] which are well known by now, have led to very encouraging results.[5]

All the systems that we have mentioned are highly relativistic and therefore we will not use the Faddeev equations themselves, but their relativistic generalization, which has been given by Alessandrini and Omnes.[6] These equations have all the important properties and also all the drawbacks of the Faddeev equations, except that their solution is relativistically invariant and that they have relativistic kinematics and unitarity built in. Let us summarize briefly the properties of these equations.

(a) They deal with the scattering of three stable particles and exhibit exact three-body elastic unitarity.

(b) They describe systems of particles interacting by pairs and do not take into account three-body forces. By three-body forces, we mean the contribution of three-body graphs which cannot be separated into successions of two-body graphs, such as the one shown in Fig. 1. The fact that in these equations the particles are treated as interacting by pairs is very appealing, since, as said previously, many three-body resonances seem to decay in states where the two-body subsystems themselves resonate. Also such a model enables one to study systems where a virtual state is important, as it is in the π-π s wave. As the particles are treated symmetrically, the coupled-channel treatment, which would be necessary in a quasi-two-particle approach, is embodied in three-body equations.

* Work done under auspices of the U. S. Atomic Energy Commission.

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[1] See, for instance, G. Goldhaber, Second Coral Gables Conference, *Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, 1965), p. 34.

[2] See, for instance, I. T. Drummond, Phys. Rev. **140**, B482 (1965).

[3] C. Lovelace, Phys. Rev. **135**, B1225 (1964).

[4] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[5] See, for instance, Lovelace (Ref. 3); R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **136**, B650 (1964) and Phys. Rev. Letters **13**, 579 (1964); H. A. Bethe, Phys. Rev. **138**, B804, (1965); M. Bander, Phys. Rev. **138**, B322 (1965); T. L. Trueman, Phys. Rev. **137**, B1605 (1965).

[6] V. A. Alessandrini and R. L. Omnes, Phys. Rev. **139**, B167 (1965).

(c) They are off-the-energy-shell equations and involve as input quantities the off-shell two-body amplitudes. The fact that the original potentials do not appear at all must be considered as important, since very little is known about them, but at least the physical amplitude is known. On-shell equations have been recently derived by Mandelstam,[7] and their mathematical complication is greater than that of Faddeev-type equations, although it is undeniable that they have more physical content.

(d) The drawbacks of the equations we use are well known. In the first place, no attention is paid to crossing symmetry, although this is believed to be a very important ingredient in relativistic theories. Secondly, only elastic processes are considered, both in the two-body and three-body systems, and this might be a great handicap in treating very energetic states.

In this paper, we will explain the general method that we use, and particularize it to the three-pion channel with $I=0$ and $J^p=1^-$, since the practical calculations and numerical solutions of this channel will be dealt with in the following paper. This channel is interesting for many reasons. First, a certain number of resonances have the same quantum numbers; the ω, the φ, and also the $E(1415)$ meson whose existence is not well established. Secondly, we want to know if our model exhibits the idea of Chew[8] that the strong forces which are responsible for the existence of the ρ in the two-pion system should also be responsible for an isoscalar three-pion resonance as the ω. Lastly, the problem has the advantage of a great symmetry; as we will show, the three-body equations reduce to a relatively simple form, and this calculation can be considered as an example of three-meson calculations. In future papers, we shall examine other three-pion states without going into the details contained here.

In Sec. II, we recall briefly the Alessandrini–Omnes equations and explain how we use them; we show how they reduce when angular momentum, parity, and isospin are separated, in Sec. III. Finally, in Sec. IV, we explain which off-the-energy-shell extension we take for the two-body scattering amplitude. As remarked previously, this is the physical input quantity entering our equations. It is not a quantity available through experiment, and therefore it is defined through an integral equation. For many reasons, the most important of which is the ignorance of the two-body potentials, we cannot use the exact form of the two-body off-shell amplitude, but an approximate one. We will follow Lovelace[3] in using the fact that poles dominate the physical amplitude in order to represent the off-shell dependence by phenomenological form factors. Thus we will exploit our experimental knowledge of the two-body systems to perform semiphenomenological calculations in the three-particle sector.

[7] S. Mandelstam, Phys. Rev. **140**, B375 (1965).

[8] G. F. Chew, Phys. Rev. Letters **4**, 142 (1960).

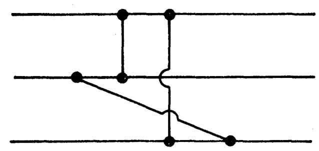

FIG. 1. A diagram not included in Faddeev-type equations.

It should be emphasized that this is far from being a fundamental relativistic theory of three-body processes; we have pointed out several drawbacks of the method that we use. Therefore, these calculations must be understood as models which will be successful only if the effects of overlapping resonances together with three-body unitarity are sufficient to build up three-body resonances and bound states in the channels in which they are observed experimentally.

II. STATEMENT OF THE EQUATIONS

Alessandrini and Omnes[6] have recently proposed a set of equations which are a relativistic analog of the Faddeev equations. These equations—(11.12) of Ref. 6—involve six amplitudes instead of the usual three in the Faddeev equations. Another difficulty is that the two-body amplitudes appear to depend on the energy of the third particle in a complicated way. Alessandrini[9] has pointed out that this latter problem can be eliminated as follows: Let us define our two-body amplitude to be relativistically invariant and unitary in the two-body Hilbert space. For example, we shall use a Blankenbecler–Sugar[10] type of equation to define it. We choose to normalize one-particle states by

$$\langle \mathbf{p}|\mathbf{p}'\rangle=(2\pi)^3 2\omega\delta(\mathbf{p}-\mathbf{p}'), \qquad (2.1)$$

where $\omega=(p^2+m^2)^{1/2}$. Then we define the two-body amplitude in the three-body Hilbert space by

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|t_1(\sigma)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle =(2\pi)^3 2\omega_1\delta(\mathbf{p}_1-\mathbf{p}_1')\langle \mathbf{p}_2\mathbf{p}_3|\hat{t}_1(\sigma)|\mathbf{p}_2'\mathbf{p}_3'\rangle. \qquad (2.2)$$

The derivation of the three-body equations proceeds exactly as before, and the final amplitude still satisfies three-body unitarity exactly. However, now we can choose a set of variables such that the energy of the third particle appears in the two-body amplitude only through the expression of the two-body invariant energy σ in terms of the three-body invariant energy s, by means of the equation

$$\sigma_{23}=s-2\omega_1 s^{1/2}+m_1^2. \qquad (2.3)$$

To return to the first difficulty, the reason that there are six amplitudes is that the kernel of the three-body equations involves all the two-body amplitudes with one initial interacting particle and one final one off the mass shell. For each noninteracting particle, there are four such amplitudes, which are in general all distinct.

[9] V. A. Alessandrini (private communication).

[10] R. Blankenbecler and R. Sugar, Phys. Rev. (to be published).

However, if one makes assumptions about the symmetry of the defining equation (for example, by taking a symmetric potential), one can obtain relations between some of these amplitudes. We shall make the much more drastic assumption that the completely off-shell amplitude is a symmetric function of the c.m. off-shell initial momenta. This implies that the amplitude, when expressed as a function of the relative initial and final momenta and the invariant energy, is independent of which initial particle is off the mass shell. This reduces the six coupled equations to three. Our reasons for making this assumption, aside from the increased simplicity attained, stem from our general ignorance of the off-shell amplitude and our reluctance to introduce more than an absolute minimum of arbitrary functions. We shall also make a similar assumption on the final variables, so that our three coupled equations reduce to a Faddeev-type system:

$$\begin{aligned} T^1&=t_1-t_1E(T^2+T^3)\,,\\ T^2&=t_2-t_2E(T^1+T^3)\,,\\ T^3&=t_3-t_3E(T^1+T^2)\,, \end{aligned} \tag{2.4}$$

where E is the three-body propagator used in Ref. 6. In the momentum-space representation, we have

$$\begin{aligned} \langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|T^1(s)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle &= \langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|t_1(\sigma)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle \\ &-\frac{1}{(2\pi)^8}\int d\mathbf{q}_1d\mathbf{q}_2d\mathbf{q}_3\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|t_1(\sigma)|\mathbf{q}_1\mathbf{q}_2\mathbf{q}_3\rangle \\ &\times\frac{\pi^2}{\omega_1\omega_2\omega_3}\frac{(\sum\omega_i)}{(\sum\omega_i)^2-s}\langle \mathbf{q}_1\mathbf{q}_2\mathbf{q}_3|T^2(s)+T^3(s)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle\,, \end{aligned} \tag{2.5}$$

where the momentum-conservation δ functions have not been factored out.

In sum,[9] Eq. (2.5) and the associated equations for T^2 and T^3 are a system of equations that (a) are relativistically invariant, (b) guarantee three-body elastic unitarity, (c) have the same relationship to the three-body multiladder diagrams as the Blankenbecler-Sugar equation does to the two-body ladder diagrams, (d) reduce the Faddeev equations in the nonrelativistic limit.

III. REDUCTION OF THE EQUATIONS

In order to reduce the equations to a usable form, we write them in terms of eigenstates of the total angular momentum, J. To do this, we can either use the techniques developed by Omnes,[11] or we can use the more common method of coupling two particles first and then coupling the third to the pair. However, in the general case, the latter method has several defects not shared by the former: It does not explicitly preserve any inherent symmetry of the interactions; it leads to complicated kinematics; and it involves recoupling angular momenta. For these reasons, we shall follow Omnes, even though in the present case, where it will turn out that we keep only one partial wave in each two-body subsystem, the other method is not more difficult.

We use his notation for the angular momentum states and his convention for their normalization, so that

$$\delta(\mathbf{P})\langle\omega JM\mu\mathbf{P}|\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3\rangle = A\,\delta(\mathbf{P})\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\mathfrak{D}_{\mu M}{}^{*J}(\alpha\beta\gamma)\times\prod_{i=1}^{3}\delta(\omega_i-(p_i^2+1)^{1/2})\,, \tag{3.1}$$

where

$$A=\left[\frac{(2J+1)}{\pi^2}(2\pi)^9\right]^{1/2}, \tag{3.2}$$

where ω_i is the c.m. energy of particle i, $\mathbf{P}$ is the total three-momentum, J is the total angular momentum, μ is the projection of J on the space-fixed z axis, and M is the projection of J on the body-fixed z axis. In this and all that follows, the three masses are taken equal to unity. The body-fixed axes with respect to which the Euler angles are defined have the z axis parallel to $\mathbf{p}_1\times\mathbf{p}_2$ and the x axis parallel to $\mathbf{p}_1$. The two-body interaction thus has the representation for $M=M'=0$

$$\langle\omega J0|t_1|\omega'J'0\rangle = \frac{\delta_{JJ'}\delta(\omega_1-\omega_1')}{4p_1(2\pi)^6}\sum_\nu \mathfrak{D}_{\nu0}{}^{*J}(0,\pi/2,0)\times\mathfrak{D}_{\nu0}{}^{J}(0,\pi/2,0)\int_0^{2\pi}d\theta\, e^{i\nu u}F(\theta)\,, \tag{3.3}$$

where $F(\theta)$ is the two-body scattering amplitude, θ is the scattering angle, and u is the angle between the planes defined by ($\mathbf{p}_1$, $\mathbf{p}_2$) and ($\mathbf{p}_1$, $\mathbf{p}_2'$). Equation (3.3) is easily obtained by following Ref. 11, allowing for the different normalizations and kinematics.

In order to do the integration over u, we expand F in partial waves and make use of the well-known identity[12]

$$\int_0^{2\pi} e^{i\nu u}P_l(\cos\gamma\,\cos\gamma'+\sin\gamma\,\sin\gamma'\,\cos u)du = \frac{8\pi^2}{2l+1}Y_l{}^\nu(\gamma,0)Y_l{}^\nu(\gamma',0)\,. \tag{3.4}$$

These formulas essentially give the projections of a two-body state of definite angular momentum onto the various possible three-body angular momentum states.

We also need to known how to express the total parity of the state in terms of the angular momentum eigenstates, and, in the case of identical particles, how to take into account the proper statistics. This can easily be derived from the known effects of the corresponding operators in momentum space.

[11] R. Omnes, Phys. Rev. **134**, B1358 (1964).

[12] Our conventions follow A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

For parity, for example, if the Euler angles in Eq. (3.1) are (α,β,γ), then it is known that

$$\begin{aligned}\delta(\mathbf{P})\langle\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3|\mathcal{P}|\omega JM\mu\mathbf{P}\rangle &= \delta(\mathbf{P})\langle-\mathbf{p}_1,-\mathbf{p}_2,-\mathbf{p}_3|\omega JM\mu\mathbf{P}\rangle\epsilon_1\epsilon_2\epsilon_3\\ &= A\,\delta(\mathbf{P})\delta(\textstyle\sum\mathbf{p}_i)\mathfrak{D}_{\mu M}{}^J(\alpha,\beta,\gamma+\pi)\\ &\quad\times\prod_{i=1}^{3}\delta(\omega_i-(p_i^2+1)^{1/2})\epsilon_1\epsilon_2\epsilon_3\\ &= (-1)^M\delta(\mathbf{P})\langle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|\omega JM\mu\mathbf{P}\rangle\epsilon_1\epsilon_2\epsilon_3\,,\end{aligned}\tag{3.5}$$

where $\epsilon_1\epsilon_2\epsilon_3$ are the intrinsic parities of the particles. So, finally,

$$\mathcal{P}|\omega JM\mu\mathbf{P}\rangle=\epsilon_1\epsilon_2\epsilon_3(-1)^M|\omega JM\mu\mathbf{P}\rangle\,.\tag{3.6}$$

Note that the parity depends only on M for this choice of body-fixed axes.

The statistics can be treated in the same way, and for the isospin zero, $J=1^-$ three-pion state the statistics imply that $\langle\omega10|T|\omega'10\rangle$ is symmetric separately in the primed and unprimed energies. Parity requires that only the $M=0$ component is nonzero. The exchange of two particles then gives a factor (-1) from isospin, and a factor $(-1)^J$ from angular momentum. Combining these statements, we have, in our case,

$$\langle\omega_1\omega_2\omega_3|T|\omega_1'\omega_2'\omega_3'\rangle=-(-1)\langle\omega_2\omega_1\omega_3|T|\omega_1'\omega_2'\omega_3'\rangle\,,\tag{3.7}$$

where T is the total three-body amplitude. This statement and the two others corresponding to other exchanges, give the stated symmetry.

We can now write Eq. (2.5) after separation of total angular momentum; taking $J=1$, $M=M'=0$, and, for instance, $\mu=0$, we obtain

$$\begin{aligned}\langle\omega_1\omega_2\omega_3|T^1(s)|\omega_1'\omega_2'\omega_3'\rangle &= \frac{\mathcal{T}(\omega_1,\omega_2,\omega_3;\omega_1',\omega_2',\omega_3';s)}{4p_1(2\pi)^6}\delta(\omega_1-\omega_1')\\ &\quad-\frac{1}{2p_1(2\pi)^3}\int\frac{d\omega_1''d\omega_2''d\omega_3''(\sum\omega_i'')}{(\sum\omega_i'')^2-s}\\ &\quad\times\mathcal{T}(\omega_1,\omega_2,\omega_3;\omega_1'',\omega_2'',\omega_3'';s)\delta(\omega_1-\omega_1'')\\ &\quad\times\langle\omega_1''\omega_2''\omega_3''|T^2(s)+T^3(s)|\omega_1'\omega_2'\omega_3'\rangle\,,\end{aligned}\tag{3.8}$$

where the inhomogeneous term is the angular momentum projection of the two-body amplitude as defined in Eq. (3.3).

For our particular case, because the total isospin is zero, each pion pair must be in an $I=1$ state, and therefore only odd angular momenta contribute. If we assume that the $J=1$ part dominates, which seems to be essentially the case experimentally, we have

$$\mathcal{T}=3\sin\gamma_1\sin\gamma_1'f(\omega_1,\omega_2,\omega_3;\omega_1',\omega_2',\omega_3';s)\,,\tag{3.9}$$

where γ_1 is the angle between $(\mathbf{p}_2-\mathbf{p}_3)$ and $\mathbf{p}_1$, so that

$$\cos\gamma_1=\cos\gamma(\omega_1,\omega_2,\omega_3)=\frac{\omega_3^2-\omega_2^2}{(\omega_1^2-1)^{1/2}[2(\omega_2^2+\omega_3^2)-\omega_1^2-3]^{1/2}}\,.\tag{3.10}$$

The function f appearing in Eq. (3.9) is the $l=1$ partial wave of the two-body off-shell amplitude. It actually depends on three variables, which are the two-body off-shell energy σ defined in Eq. (2.3) and the magnitudes of the initial and final off-shell momenta

$$p_{23}=p(\omega_1,\omega_2,\omega_3)=[2(\omega_2^2+\omega_3^2)-\omega_1^2-3]^{1/2}/2\,.\tag{3.11}$$

The isospin dependence has yet to be extracted. Let (α,β,γ) and (α',β',γ') be the initial and final isospin indices of particles 1, 2, and 3. The two-body interaction is $I=1$, so the inhomogeneous term in the equation for T^1 has the dependence $\frac{1}{2}\delta_{\alpha\alpha'}(\delta_{\beta\beta'}\delta_{\gamma\gamma'}-\delta_{\beta\gamma'}\delta_{\gamma\beta'})$ while those for T^2 and T^3 have the indices interchanged. The projections of all these on the (totally antisymmetric) isospin zero state just give coefficients of one. In general, not only will T be a mixture of several states with the same isospin, but also the recoupling coefficients will not be one. However, for this particular case Eq. (3.8) is correct as it stands, even after the isospin extraction.

In order to decouple the equations, we define the kernel

$$\begin{aligned}K(\omega_1,\omega_2,\omega_3;\omega_1''\omega_2''\omega_3'';s) &= \frac{1}{2p_1(2\pi)^3}\frac{\sum\omega_i''}{(\sum\omega_i'')^2-s}\\ &\quad\times\mathcal{T}(\omega_1,\omega_2,\omega_3;\omega_2'',\omega_1''\omega_3'';s)\delta(\omega_1-\omega_2'')\,.\end{aligned}\tag{3.12}$$

Let us note that this is not exactly the kernel which appears directly in Eq. (3.8), and that it is symmetric in (ω_2,ω_3) and in (ω_1',ω_3''). We also note that T^1 is symmetric in (ω_2,ω_3) and similarly for T^2 and T^3. We define new amplitudes V^i by

$$\begin{aligned}V^1(\omega_1,\omega_2,\omega_3)&=\langle\omega_1\omega_2\omega_3|T^1(s)|\omega_1'\omega_2'\omega_3'\rangle\,,\\ V^2(\omega_1,\omega_2,\omega_3)&=\langle\omega_2\omega_1\omega_3|T^2(s)|\omega_1'\omega_2'\omega_3'\rangle\,,\\ V^3(\omega_1,\omega_2,\omega_3)&=\langle\omega_3\omega_2\omega_1|T^3(s)|\omega_1'\omega_2'\omega_3'\rangle\,.\end{aligned}\tag{3.13}$$

The ω' variables are regarded as parameters, and occur in the same order in each of the T^i defining the V^i. We may now rewrite Eq. (3.8) as

$$V^1(\omega)=I_1(\omega)-\int d\omega_1''d\omega_2''d\omega_3''K(\omega,\omega'')\times[V^2(\omega'')+V^3(\omega'')]\,,\tag{3.14}$$

with V^2 and V^3 given by cyclic permutations. The inhomogeneous terms I_i from Eq. (3.8) are

$$\begin{aligned}I_1(\omega)&=\mathcal{T}(\omega_1\omega_2\omega_3;\omega_1'\omega_2'\omega_3';s)\delta(\omega_1-\omega_1')/4p_1(2\pi)^6\,,\\ I_2(\omega)&=\mathcal{T}(\omega_1\omega_2\omega_3;\omega_2'\omega_1'\omega_3';s)\delta(\omega_1-\omega_2')/4p_1(2\pi)^6\,,\\ I_3(\omega)&=\mathcal{T}(\omega_1\omega_2\omega_3;\omega_3'\omega_1'\omega_2';s)\delta(\omega_1-\omega_3')/4p_1(2\pi)^6\,.\end{aligned}\tag{3.15}$$

It is now trivial to decouple Eqs. (3.14), since they may be written as

$$(V)=(I)-\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}K(V). \tag{3.16}$$

The three-by-three matrix appearing in Eq. (3.16) has eigenvalues 2, -1, -1, and corresponding eigenvectors

$$\Sigma(\omega)=V^1+V^2+V^3\,,\quad \Delta^{ij}(\omega)=V^i-V^j. \tag{3.17}$$

These eigenvectors satisfy the decoupled equations

$$\Sigma(\omega)=\sum_{i=1}^{3} I_i-2\int d\omega''K(\omega,\omega'')\Sigma(\omega'')\,, \tag{3.18a}$$

$$\Delta^{ij}(\omega)=I_i-I_j+\int d\omega''K(\omega,\omega'')\Delta^{ij}(\omega''). \tag{3.18b}$$

Re-expressing T in terms of Σ and Δ^{ij}, we obtain

$$\begin{aligned}3\langle\omega_1\omega_2\omega_3|T(s)|\omega'\rangle\\ =\Sigma(\omega_1\omega_2\omega_3)+\Sigma(\omega_2\omega_1\omega_3)+\Sigma(\omega_3\omega_2\omega_1)\\ +\Delta^{12}(\omega_1\omega_2\omega_3)-\Delta^{12}(\omega_2\omega_1\omega_3)+\Delta^{13}(\omega_1\omega_2\omega_3)\\ -\Delta^{13}(\omega_3\omega_2\omega_1)+\Delta^{23}(\omega_2\omega_1\omega_3)-\Delta^{23}(\omega_3\omega_2\omega_1).\end{aligned} \tag{3.19}$$

Now it is obvious that the statistics, which require that T be a symmetric function of its variables, imply that the Δ^{ij} terms give no contribution, and only Eq. (3.18a) need be considered.

One could attempt to decouple the equations by observing that

$$\langle\omega_1\omega_2\omega_3|T^1(s)|\omega_1'\omega_2'\omega_3'\rangle =\langle\omega_2\omega_1\omega_3|T^2(s)|\omega_2'\omega_1'\omega_3'\rangle. \tag{3.20}$$

The trouble is that, unless one sets $\omega_1'=\omega_2'=\omega_3'$, the result of substituting Eq. (3.20) into Eq. (3.8) is not an integral equation, since the parameters ω' of T^1 are then different on the two sides. It happens, in the present case, that one is allowed to choose $\omega_1'=\omega_2'=\omega_3'$, and that this procedure does give the correct equation. However, in many channels this cannot be done (for example, because $T=0$ at this point).

IV. TWO-BODY AMPLITUDE

The input to our kernel (3.12) is the off-energy-shell two-body $J=1$ partial-wave amplitude. This amplitude is, of course, not available directly from experiment. In the nonrelativistic case it is defined as the solution of the Lippmann-Schwinger equation. Similarly, in our case, it is defined to be the elastic part of the Bethe-Salpeter amplitude (in the ladder approximation this is just the Blankenbecler-Sugar[10] amplitude), and is taken to satisfy

$$f(p,p';\sigma)=V(p,p')-\frac{1}{\pi}\int_4^\infty dx\frac{\rho(x)}{x-\sigma}V(p,k)f(k,p';\sigma)\,, \tag{4.1}$$

where $k=\frac{1}{2}(x-4)^{1/2}$, σ is the invariant energy, p (or p') is the magnitude of the c.m. incoming (or outgoing) off-shell momentum, ρ is the phase-space factor, and V is the input for the particular partial wave.

Nonrelativistically, V is just the potential, and Lovelace,[3] among others has investigated the properties of the resulting amplitude. We shall use those of his results which seem appropriate to our relativistic problem.

Since we have little knowledge of V for our case, we shall construct an amplitude satisfying the more general requirements of Eq. (4.1), principal among which is elastic unitarity,

$$\operatorname{Im}f(p,p';\sigma)=-\frac{1}{16\pi}\left(\frac{\sigma-4}{\sigma}\right)^{1/2}f\left[p,\left(\frac{\sigma-4}{4}\right)^{1/2};\sigma\right]\times f^*\left[\left(\frac{\sigma-4}{4}\right)^{1/2},p';\sigma\right]. \tag{4.2}$$

We shall require our amplitude to satisfy Eq. (4.2), exactly, and not just in a limited region, for two reasons. First, we want our three-body amplitude to be exactly unitary to ensure, among other things, that residues have correct signs. Second, the region in energy where we want to know the amplitude runs from $(s^{1/2}-1)^2$ to $-\infty$, as can be seen from Eq. (2.3), since the energy of the third particle runs from 1 to ∞. If our amplitude were not unitary and had singularities in σ on the negative axis, our kernel would be completely wrong and would, for instance, have complex eigenvalues in the three-body bound-state region.

Perhaps the major approximation that we shall make in the two-body amplitude is separability. We shall assume

$$f(p,p';\sigma)=g(p)g(p')/D(\sigma). \tag{4.3}$$

It is known that in the vicinity of a resonance or bound-state pole, the amplitude is separable, and Lovelace[3] has argued that to the extent that we expect the amplitude to be dominated by a pole, Eq. (4.3) should be a good representation.

However, the main reason for the assumption of separability is that it reduces the three-dimensional equations (3.18) to a one-dimensional form.[13] It not only eliminates the δ functions of the kernel but also converts the equations from something impossible to do reliably on present computers, because of storage limitations, to something relatively tractable.

Our further reason for using Eq. (4.3) is that basically what we want is an off-shell two-body amplitude that is the best approximation, when used in the three-body equations, to the "correct" amplitude. There is no particular reason with our present knowledge to think that in this sense the separable form will be worse than any other arbitrary form we could pick. By giving up separability, one loses the benefit of a great simplification and gains only a hypothetical feeling of generality.

[13] J. L. Basdevant, Phys. Rev. 138, B892 (1965).

When we take the separable form and require that it satisfy Eq. (4.2), we find that $D(\sigma)$ has the form

$$D(\sigma)=P(\sigma)+\frac{1}{\pi}\int_4^\infty \frac{dx}{x-\sigma}\rho(x)g^2(k)\,, \tag{4.4}$$

where $P(\sigma)$ is any polynomial with real coefficients. The fact that $P(\sigma)$ is not determined by unitarity is essentially the well-known Castillejo-Dalitz-Dyson (C.D.D.) ambiguity. If the amplitude satisfied Eq. (4.1), then as $(\sigma)\to\infty$, $f(p,p';\sigma)\to V(p,p')$. Thus if our input function is not dependent on the off-shell energy, our polynomial must be a constant.

On the other hand, taking $P(\sigma)$ linear has certain advantages: It is the natural generalization of the Breit-Wigner form $D(\sigma)=\sigma-\sigma_r+i(\sigma_r)^{1/2}\Gamma$ for a resonance of width Γ and mass $(\sigma_r)^{1/2}$; and it allows us to put an explicit zero of $D(\sigma)$ anywhere on the second Riemann sheet. Further, if the compactness of the kernel restricts the effective range of ω in Eq. (2.3) enough so that the range of variation of σ where we need to know $D(\sigma)$ is relatively small, then one might argue that the asymptotic behavior in σ of $D(\sigma)$ would not matter, and one could use the linear form. Actual calculations will clarify this point.

We come now to the question of the function $g(p)$. Its most natural identification is as a resonance form factor, and if V were known, we could trivially extend Eq. (4.1) to the second sheet and obtain an equation for $g(p)$. Alternatively, if the N function of an N/D calculation were known, one could make the choice[3] $g(p)=(N)^{1/2}$, as then Eqs. (4.4) and (4.3) give f its desired on-shell form.

We also know[6] that, for a nonrelativistic problem with a Yukawa potential or for a Blankenbecler-Sugar amplitude in the lth partial wave, we have

$$g(p)\sim p^l \quad \text{as} \quad p\to 0\,, \tag{4.5a}$$

and

$$g(p)\sim p^{-(l+2)} \quad \text{as} \quad p\to\infty\,. \tag{4.5b}$$

Relativistically, while we still expect Eq. (4.5a) to be true, Eq. (4.5b) is in some doubt. For good convergence of our three-body equations, $g(p)$ must go to zero faster than $1/p$ at infinity, and what we shall actually do is to take a family of functions satisfying (4.5a, b). Since $g(p)$ gives the detailed structure of the resonance, we can hope that the gross three-body results will be more or less independent of the exact form of $g(p)$, and will depend only on the position and width of the two-body resonance. Alternatively, we may be able to find a $g(p)$ which gives consistent results in all the relevant three-body channels. If neither of these proves to be the case, a drastic re-examination of our model will be necessary.

Finally, having chosen a form factor that involves a certain number of free parameters, we will impose the condition that $D(\sigma)$ give us the experimentally observed resonance.

When we use the separable form of f and apply the methods of Ref. 13, we obtain from Eq. (3.18a) a one-dimensional equation. We define

$$\Sigma(\omega_1\omega_2\omega_3)=\frac{3\sin\gamma(\omega_1\omega_2\omega_3)g(p(\omega_1\omega_2\omega_3))\Phi(\omega_1)}{2(\omega_1^2-1)^{1/2}(2\pi)^3D(\sigma(s,\omega_1))}\,. \tag{4.6}$$

Then

$$\Phi(x)=I(x)+\int_1^\infty K(x,y)\Phi(y)dy\,, \tag{4.7}$$

where $I(x)$ is obtained from the inhomogeneous term of Eq. (3.18a) in the same way as Φ is obtained from Σ in Eq. (4.6). The kernel in Eq. (4.7) is given by

$$K(x,y)=-\frac{3}{(2\pi)^3}\frac{1}{(y^2-1)^{1/2}D[\sigma(s,y)]}\times\int_{a-}^{a+}da\frac{(x+y+a)}{(x+y+a)^2-s}\sin\gamma(y,x,a)\sin\gamma(x,y,a)\times g[p(x,y,a)]g[p(y,x,a)]\,, \tag{4.8}$$

where $\sin\gamma(x,y,a)$ and $p(x,y,a)$ can be read from (3.10) and (3.11), and where the limits of integration are given by the triangle inequality as

$$a\pm=[1+((x^2-1)^{1/2}\pm(y^2-1)^{1/2})^2]^{1/2}\,. \tag{4.9}$$

As mentioned previously, the following paper will be a numerical investigation of the properties of this kernel, aimed at finding eventual resonances. Other three-pion channels will yield very similar kernels, and we will study the mathematical difficulties associated with such kernels.

ACKNOWLEDGMENTS

We are greatly indebted to Dr. V. A. Alessandrini and Dr. Roland L. Omnes for their constant interest in this work and for many enlightening discussions.

We also wish to thank Dr. David L. Judd for his hospitality at the Theoretical Group of the Lawrence Radiation Laboratory.