Hypertriton with S' State and the Λ-N Interaction*

A. R. BODMER
Argonne National Laboratory, Argonne, Illinois
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The hypertriton $_\Lambda He^3$ is considered with the inclusion of an S' state $\Psi_{S'}$ in the total wave function. For $\Psi_{S'}$ the nucleons are in a singlet spin state and the space part is correspondingly antisymmetric with respect to exchange of the nucleons. Only a spin dependence of the Λ-N interaction can give a nonzero admixture of $\Psi_{S'}$ to the dominant component Ψ_S, which is space-symmetric under exchange of the nucleons and which is the only component that has been considered in previous investigations. Central, spin-dependent Yukawa potentials were used. The most flexible trial function used was one with 16 parameters for which Ψ_S has the same 6-parameter form as used by Downs and Dalitz and $\Psi_{S'}$ has a corresponding flexibility. In particular, for an intrinsic range $b=1.5$ F (corresponding to a Yukawa interaction appropriate to two-pion exchange), the effect of $\Psi_{S'}$ is quite appreciable; the singlet strength is reduced, the triplet strength slightly increased, and the spin dependence reduced by about a third. For a range corresponding to K-meson exchange ($b=0.84$ F), the effect of $\Psi_{S'}$ is considerably less. With inclusion of $\Psi_{S'}$ (for a given b), the singlet strength is found to be quite insensitive to the value of the triplet strength and is therefore almost entirely determined by $B_\Lambda(_\Lambda H^3)$. The resulting total Λ-N cross sections at low energies ($\lesssim 20$ MeV) are compared with the experimental values σ_{exp}. If it is assumed that the singlet scattering length a_s and effective range r_s are most reliably determined from hypernuclei, then for $b=1.5$ F (for which the estimated values with a hard core of radius 0.42 F are $a_s\approx -2.3$ F, $r_s\approx 3.6$ F), acceptable agreement with σ_{exp} can be obtained with only a modest increase of $|a_t|$ ($a_t\approx -1.3$ F, $r_t\approx 2.9$ F) above the value obtained from hypernuclei. (The maximum value consistent with the hypernuclear results is $a_t\approx -0.9$ F together with $r_t\approx 3.3$ F.) It is shown that an increase of this order of magnitude could be obtained through suppression of the coupling with the ΣN channel in $_\Lambda He^5$. Results are also given for a Yukawa potential with $b=2.07$ F, which is the intrinsic range for an interaction with a hard core of radius 0.42 F and an attractive Yukawa tail appropriate to an exchanged boson with mass $3m_\pi$. Finally, it is argued that there is a tentative indication for the existence of a repulsive core in the Λ-N interaction.

1. INTRODUCTION

THE hypertriton $_\Lambda H^3$ is the lightest hypernucleus that is bound and is of basic importance for knowledge about the Λ-N interaction. The lifetime of $_\Lambda H^3$ is also of considerable interest—in particular as a test of the correctness of one's ideas about its structure and of the use of the Λ-N-π decay amplitudes for calculations of hypernuclear decays. (In fact, the experimental lifetime[1] seems to be significantly less than the calculated one.[2])

The basic analysis of $_\Lambda H^3$ is that of Downs and Dalitz[3] who considered charge-independent and central, but spin-dependent, Yukawa interactions. The ground-state wave function is then a pure s state with the nucleons in a triplet spin state. Downs and Dalitz used a flexible 6-parameter trial function, the space part of which has the product form

$$\psi_S(r_1,r_2,r_3)=f(r_1)f(r_2)g(r_3)\,, \tag{1}$$

where r_1, r_2, and r_3 are triangular coordinates: r_3 is the neutron-proton separation and r_1 and r_2 are the Λ-nucleon separations. The functions $f(r)$ and $g(r)$ are each 3-parameter trial functions of the form

$$f(r)=e^{-a_1r}+xe^{-a_2r}\,, \tag{1a}$$

$$g(r)=e^{-b_1r}+ye^{-b_2r}\,. \tag{1b}$$

Here $f(r)$ refers to the Λ-N pairs and $g(r)$ to the n-p pair. The function ψ_S is symmetric with respect to interchange of the space coordinates of the neutron and proton, i.e., with respect to $r_1\leftrightarrow r_2$, $r_3\leftrightarrow r_3$. The total ground-state wave function Ψ_S thus has the isobaric spin $T=0$ since the nucleons are in a triplet spin state. The experimental value of the total binding energy is $B=B_d+B_\Lambda=2.525\pm0.15$ MeV, where $B_d=2.225$ MeV is the deuteron binding energy and $B_\Lambda=0.3\pm0.15$ MeV is the Λ separation energy with respect to the deuteron.[4] The energy B then uniquely determines an effective volume integral U_2 of the Λ-N interaction for any assumed shape of this interaction. In particular, if the singlet is more attractive than the triplet Λ-N interaction as is known to be the case,[5] then the total spin of $_\Lambda H^3$ is $J=\frac{1}{2}$ and one has $U_2=\frac{3}{2}U_s+\frac{1}{2}U_t$, where U_s and U_t are the singlet and triplet volume integrals, respectively, of the Λ-N interaction.

Subsequent investigations[6] of $_\Lambda H^3$ have been made for Λ-N interactions with a hard core. These investigations have all used trial functions that are wholly

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[1] See, for example, R. J. Prem and P. H. Steinberg, Phys. Rev. **136**, B1803 (1964), where references to other work are also given.

[2] R. H. Dalitz and G. Rajasekaran, Phys. Letters **1**, 58 (1962).

[3] B. W. Downs and R. H. Dalitz, Phys. Rev. **114**, 593 (1959).

[4] R. Levi-Setti, *Proceedings at the International Conference on Hyperfragments, St. Cergue, Switzerland, 1963*, edited by W. O. Lock (CERN, Geneva, 1964).

[5] R. H. Dalitz and L. Liu, Phys. Rev. **116**, 1312 (1959); M. M. Block, R. Gessaroli, J. Kopelman, S. Ratti, M. Schneeberger, L. Grimellini, T. Kukuchi, L. Lendinara, L. Monari, W. Becker, and E. Harth, *Proceedings of the International Conference on Hyperfragments, St. Cergue, Switzerland, 1963*, edited by W. O. Lock (CERN, Geneva, 1964).

[6] B. W. Downs, D. R. Smith, and T. N. Truong, Phys. Rev. **129**, 2730 (1963); D. R. Smith and B. W. Downs, *ibid.* **133**, B461 (1964); R. C. Herndon, Y. C. Tang, and E. W. Schmid, Nuovo Cimento **33**, 259 (1964); K. Dietrich, H. J. Mang, and R. Folk, Nucl. Phys. **50**, 177 (1964).

spatially symmetric with respect to interchange of the nucleons and most of them have also used the product form (1).

If, with purely attractive interactions the results for U_2 obtained from ${}_\Lambda H^3$ are combined with the values of the spin-averaged volume integral $\bar{U}=\frac{1}{4}(U_s+3U_t)$ which are obtained[7] from ${}_\Lambda He^5$ and also[8,9] from ${}_\Lambda Be^9$ and ${}_\Lambda C^{13}$, then for a given shape of the Λ-N interaction both U_s and U_t are determined. Thus, for Yukawa interactions with a Yukawa range $\mu_{2\pi}^{-1}=0.7$ F and a corresponding intrinsic range $b=1.484$ F, appropriate to the two-pion-exchange mechanism, one has $U_s=379_{-24}^{+18}$ MeV F^3 and $U_t=220_{-23}^{+25}$ MeV F^3. For $\mu_{2\pi}$ there is thus a strong spin dependence characterized by the volume integral $\Delta=U_s-U_t=159\pm40$ MeV F^3. For a Yukawa range $\mu_K^{-1}=0.4$ F ($b=0.84$ F), appropriate to the exchange of a K meson, one has $U_s=219\pm10$ MeV F^3 and $U_t=190\pm17$ MeV F^3, and the spin dependence is much smaller, namely, $\Delta=29\pm27$ MeV F^3.

With hard-core interactions of the same intrinsic range ($b\approx1.5$ F) as for $\mu_{2\pi}^{-1}$, the scattering length and effective ranges[10] turn out to be not very different from those obtained with purely attractive interactions.

For the bound-state nuclear three-body problem (H^3 or He^3) and spin-dependent central forces, it is well known that in addition to the dominant, spatially symmetric S state there is also some admixture of another s state, the so-called S' state.[11] In this paper we consider the effect of the analog of this state for ${}_\Lambda H^3$ which we denote by $\Psi_{S'}$.

Thus the total wave function of ${}_\Lambda H^3$ which we consider is

$$\Psi=(1-p^2)^{1/2}\Psi_S+p\Psi_{S'}\,, \tag{2}$$

where Ψ_S and $\Psi_{S'}$ are individually normalized. For $J=\frac{1}{2}$ and $T=0$, appropriate to the ground state, the nucleons will be in a singlet spin state for Ψ_S and correspondingly the space part of $\Psi_{S'}$ must then be antisymmetric with respect to the interchange of the nucleons.

Thus for $J=\frac{1}{2}$, $T=0$ one has

$$\Psi_S=\psi_S(r_1,r_2,r_3)\bar{\chi}_{1/2}{}^m\,,\qquad \psi_S(r_2,r_1,r_3)=+\psi_S(r_1,r_2,r_3)\,, \tag{3}$$

$$\Psi_{S'}=\psi_{S'}(r_1,r_2,r_3)\chi_{1/2}{}^m\,,\qquad \psi_{S'}(r_2,r_1,r_3)=-\psi_{S'}(r_1,r_2,r_3)\,, \tag{4}$$

where $\bar{\chi}_{1/2}{}^m$ and $\chi_{1/2}{}^m$ are orthonormal spin functions with $S=\frac{1}{2}$ and $S_3=m$. The functions $\bar{\chi}_{1/2}{}^m$ and $\chi_{1/2}{}^m$ correspond to the nucleons in a singlet and triplet spin state, respectively. (Explicit expressions may be found in Refs. 12 and 13.) Thus $\bar{\chi}_{1/2}{}^m$ is symmetric with respect to interchange of the nucleon spins whereas $\chi_{1/2}{}^m$ is antisymmetric. The functions Ψ_S and $\Psi_{S'}$ are the only independent S states since for $J=\frac{1}{2}$ there are only two independent spin functions for three spin-$\frac{1}{2}$ particles.

Similarly as for H^3 and He^3, one expects $p\neq0$ only if the interactions are spin-dependent. In fact, as will be shown below, one has $p\neq0$ only if the Λ-N interaction is spin-dependent. Since this spin dependence is indicated to be large for $b=1.484$ F, the amplitude p may not be negligible and there may be an appreciable effect on the strengths deduced for the Λ-N interaction.

If the component $\Psi_{S'}$ is included in a variational calculation for ${}_\Lambda H^3$, then the resulting value obtained for the volume integral U_2 will now depend on Δ. In particular, to lowest order in Δ one expects $p\propto\Delta$. Thus if Δ is not too large one expects a relation of the form

$$U_2(\Delta)=U_2(0)-c\Delta^2\,. \tag{5}$$

The volume integral $U_2(0)$ is the value for $\Delta=0$ or, equivalently, the value that is obtained for $p=0$ as is the case for all previous investigations.

The central object of this paper is then to obtain the relation between U_2 and Δ appropriate to $B_\Lambda({}_\Lambda H^3)$. This relation, together with the value of $\bar{U}$, will then determine U_s and U_t. Only (purely attractive) Yukawa Λ-N potentials are considered for the hypertriton calculation.

2. THE VARIATIONAL CALCULATION

For the variational calculation, to be described below, we use unnormalized functions $\psi_S(r_1,r_2,r_3)$ and $\psi_{S'}(r_1,r_2,r_3)$. The total wave function which thus is also unnormalized, is

$$\Psi=\psi_S(r_1,r_2,r_3)\bar{\chi}_{1/2}+A\psi_{S'}(r_1,r_2,r_3)\chi_{1/2}\,. \tag{6}$$

The normalization integral is

$$N[\Psi]=\int d\tau\,\Psi^2\,. \tag{7}$$

In Eq. (7) and below, $d\tau=8\pi r_1r_2r_3dr_1dr_2dr_3$ denotes the volume element appropriate to the triangular coordinates r_1, r_2, r_3. The triangular inequalities $r_1+r_2\geqslant r_3$, etc., must be satisfied for the integrations. The spin summations are implied in Eq. (7) and also in Eqs. (8) and (9) below.

For an S state, the kinetic energy integral is most conveniently used in the form[14]

$$T[\Psi]=\int d\tau\sum_{i=1}^{3}K_i\left[\left(\frac{\partial\Psi}{\partial r_i}\right)^2-\Psi\frac{\partial^2\Psi}{\partial r_i^2}-\frac{2\Psi}{r_i}\frac{\partial\Psi}{\partial r_i}\right], \tag{8}$$

[7] R. H. Dalitz and B. W. Downs, Phys. Rev. **111**, 967 (1958); A. R. Bodmer and S. Sampanthar, Nucl. Phys. **31**, 25 (1962).
[8] A. R. Bodmer and Shamsher Ali, Nucl. Phys. **56**, 657 (1964).
[9] A. R. Bodmer and J. W. Murphy, Nucl. Phys. **64**, 593 (1965).
[10] R. C. Herndon, Y. C. Tang, and E. W. Schmid, Phys. Rev. **137**, B294 (1965).
[11] See, for example, G. Derrick and J. M. Blatt, Nucl. Phys. **8**, 310 (1958); J. M. Blatt and L. M. Delves, Phys. Rev. Letters **12**, 544 (1964).
[12] R. G. Sachs, *Nuclear Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1953).
[13] L. I. Schiff, Phys. Rev. **133**, B802 (1964).
[14] B. W. Downs, D. R. Smith, and T. N. Truong, Phys. Rev. **129**, 2730 (1963).

where $K_i=\frac{1}{4}\hbar^2/\mathfrak{M}_i$ are the inertial parameters in which $\mathfrak{M}_i$ is the reduced mass of the ith pair. Thus

$$K_1=K_2=K=\hbar^2(M_\Lambda+M_N)/(4M_\Lambda M_N)$$

and

$$K_3=K_N=\hbar^2/2M_N.$$

The values $M_N=938.9$ MeV/c^2 and $M_\Lambda=1115.4$ MeV/c^2 have been used.

The potential energy integral is

$$W=\int d\tau\, V\Psi^2$$

with

$$V=V_{\Lambda p}+V_{\Lambda n}+V_{np}. \tag{9}$$

Here $V_{\Lambda p}$, $V_{\Lambda n}$, and V_{np} are the relevant potentials. These we take to be charge-independent and central but spin-dependent. Thus, the spin dependence is given by

$$V_{\Lambda n}=V_{\Lambda p}=V_{\Lambda N}$$

$$=\frac{1-\boldsymbol{\sigma}_\Lambda\cdot\boldsymbol{\sigma}_N}{4}V_s(r)+\frac{3+\boldsymbol{\sigma}_\Lambda\cdot\boldsymbol{\sigma}_N}{4}V_t(r), \tag{10}$$

$$V_{np}=\frac{1-\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_p}{4}\mathcal{V}_s(r)+\frac{3+\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_p}{4}\mathcal{V}_t(r), \tag{11}$$

where $V_s(r)$ and $V_t(r)$ are the singlet and triplet Λ-N potentials, respectively, and $\mathcal{V}_s(r)$ and $\mathcal{V}_t(r)$ are the relevant singlet and triplet n-p potentials, respectively. The coefficients of the singlet and triplet potentials in Eqs. (10) and (11) are the singlet and triplet spin-projection operators, respectively.

With the wave function (6), one gets for the normalization integral (7)

$$N[\Psi]=N_S[\psi_S]+A^2N_{S'}[\psi_{S'}], \tag{12}$$

where $N_S[\psi_S]$ and $N_{S'}[\psi_{S'}]$ are given by Eq. (7) but without the spin summations. For the normalized wave function (1), the amplitude p in Eq. (2) is then given by

$$p=\left(1+A^2\frac{N_{S'}}{N_S}\right)^{-1/2}. \tag{13}$$

Similarly for $T[\Psi]$ one has

$$T[\Psi]=T_S[\psi_S]+A^2T_{S'}[\psi_{S'}], \tag{14}$$

where $T_S[\psi_S]$ and $T_{S'}[\psi_{S'}]$ are given by Eq. (8)—again without the spin summation.

The potential energy integral (9) for the function (6) is obtained by noting that

$$V\chi_{1/2}{}^m=[\bar{V}(r_1)+\bar{V}(r_2)+\mathcal{V}_s(r_3)]\chi_{1/2}{}^m-\tfrac{3}{2}[V_\sigma(r_2)-V_\sigma(r_1)]\bar{\chi}_{1/2}{}^m \tag{15a}$$

and

$$V\bar{\chi}_{1/2}{}^m=[\tfrac{1}{2}V_2(r_1)+\tfrac{1}{2}V_2(r_2)+\mathcal{V}_t(r_3)]\bar{\chi}_{1/2}{}^m-\tfrac{3}{2}[V_\sigma(r_2)-V_\sigma(r_1)]\chi_{1/2}{}^m, \tag{15b}$$

where $\bar{V}(r)=\frac{1}{4}V_s(r)+\frac{3}{4}V_t(r)$, $V_2(r)=\frac{3}{2}V_s(r)+\frac{1}{2}V_t(r)$ and where $V_\sigma(r)=V_s(r)-V_t(r)$ is the spin-dependent part of the Λ-N interaction. [Equations (15a) and (15b) are readily obtained with the use of the explicit expressions for $\bar{\chi}_{1/2}{}^m$ and $\bar{\chi}_{1/2}{}^m$ and the properties of the Pauli spin operators.]

The interaction V_2 is twice the effective Λ-N interaction appropriate to Ψ_S (i.e., for the nucleons in a triplet state); $\bar{V}$ is the spin-averaged interaction, which is the effective interaction appropriate to $\Psi_{S'}$ (i.e., for the nucleons in a singlet state). The diagonal parts of (15a) and (15b) are then the corresponding total effective interactions for $\Psi_{S'}$ and Ψ_S, respectively. Then for the total wave function Ψ one obtains the expression

$$W[\Psi]=W_S[\psi_S]+2AW_{SS'}[\psi_S,\psi_{S'}]+A^2W_{S'}[\psi_{S'}], \tag{16}$$

where

$$W_S[\psi_S]=\int d\tau\,\psi_S{}^2[\tfrac{1}{2}V_2(r_1)+\tfrac{1}{2}V_2(r_2)+\mathcal{V}_t(r_3)], \tag{17}$$

$$W_{SS'}[\psi_S,\psi_{S'}]=\frac{3}{2}\int d\tau\,\psi_S\psi_{S'}[V_\sigma(r_2)-V_\sigma(r_1)], \tag{18}$$

$$W_{S'}[\psi_{S'}]=\int d\tau\times\psi_{S'}{}^2[\bar{V}(r_1)+\bar{V}(r_2)+\mathcal{V}_s(r_3)]. \tag{19}$$

From these expressions it is seen that an admixture of Ψ_S and $\Psi_{S'}$ can arise only through $W_{SS'}$. This depends only on the spin dependence of the Λ-N interaction (i.e., on the volume integral Δ for purely attractive interactions). It is to be noted that since $\Psi_{S'}$ is spatially antisymmetric with respect to interchange of the nucleons, the potential $\mathcal{V}_s$ in Eq. (19) must be the singlet potential appropriate to a spatially antisymmetric state for the nucleons. The potential $\mathcal{V}_t$ is of course the triplet potential for a spatially symmetric state, i.e., the triplet s-state potential.

For the *binding* energies appropriate to Ψ_S and $\Psi_{S'}$ individually, one has

$$B_S=-(T_S+W_S)/N_S,$$
$$B_{S'}=-(T_{S'}+W_{S'})/N_{S'}. \tag{20}$$

The binding-energy contribution due to $W_{SS'}$ is

$$B_{SS'}=-W_{SS'}/(N_SN_{S'})^{1/2}. \tag{21}$$

The total binding energy is then

$$B=(1-p^2)B_S+2p(1-p^2)^{1/2}B_{SS'}+p^2B_{S'}, \tag{22}$$

where p is given by Eq. (13).

We take the shapes of the singlet and triplet interactions to be the same. With purely attractive interactions the *normalized* Λ-N and n-p potentials are then denoted by $v_{\Lambda N}(r)$ and $v_{np}(r)$, respectively. The volume integrals of the singlet and triplet n-p interactions are denoted by $\mathcal{U}_s$ and $\mathcal{U}_t$, respectively.

For a given binding energy B, we then wish to obtain the volume integral U_2 as a function of Δ. The relevant variational principle is then

$$\Phi[\Psi]=\frac{T[\Psi]-W_1[\Psi]-BN[\Psi]}{w_2[\Psi]}\geqslant U_2, \quad (23)$$

where $T[\Psi]$ and $N[\Psi]$ are given by Eqs. (12) and (14), respectively, and where

$$W_1[\Psi]=\int d\tau\{[\mathcal{U}_t\psi_S^2+A^2\mathcal{U}_s\psi_{S'}^2]v_{np}(r_3) -3A\Delta\psi_S\psi_{S'}[v_{\Lambda N}(r_2)-v_{\Lambda N}(r_1)] +A^2\Delta\psi_{S'}^2[v_{\Lambda N}(r_1)+v_{\Lambda N}(r_2)]\}, \quad (24)$$

$$w_2[\Psi]=\frac{1}{2}\int d\tau(\psi_S^2+A^2\psi_{S'}^2)[v_{\Lambda N}(r_1)+v_{\Lambda N}(r_2)]. \quad (25)$$

Equation (23), together with Eqs. (24) and (25), is an immediate consequence of the variational principle and of the expressions previously given. The relation $\bar{U}=\frac{1}{4}U_s+\frac{3}{4}U_t=\frac{1}{2}(U_2-\Delta)$ has also been used. For any given n-p interaction, given shape $v_{\Lambda N}(r)$, and given values of B and Δ, the variational principle (23) then gives an upper bound for U_2.

The use of Yukawa shapes for the interactions leads to

$$v_{\Lambda N}(r)=\frac{\mu^2}{4\pi}\frac{e^{-\mu r}}{r},\quad v_{np}(r)=\frac{\nu^2}{4\pi}\frac{e^{-\nu r}}{r}. \quad (26)$$

The parameters taken for the triplet n-p interaction $\mathcal{V}_t$ are the ones used by Downs and Dalitz.[3] These are consistent with the low-energy scattering data and the binding energy of the deuteron. The inverse range is $\nu=1.428$ F^{-1} (corresponding to an intrinsic range $b=2.4995$ F) and $\mathcal{U}_t=1403.4$ MeV F^3. For the singlet potential $\mathcal{V}_s$ we used both an ordinary potential (i.e., without spatial exchange) which corresponds to the s-state interaction used by Downs and Dalitz, namely $\nu=1.428$ F^{-1} and $\mathcal{U}_s=951$ MeV F^3, and also, more realistically, a Serber potential. For the latter, $\mathcal{U}_s=0$ since $\mathcal{V}_s=0$ for a spatially antisymmetric state. In fact, the results turn out to be quite insensitive to $\mathcal{V}_s$ and are almost the same for the two potentials $\mathcal{V}_s$ considered.

The trial functions used for ψ_S are of the product form (1) and those for $\psi_{S'}$ are of the form

$$\psi_{S'}(r_1,r_2,r_3)=[F(r_1)H(r_2)-H(r_1)F(r_2)]G(r_3). \quad (27)$$

This has the required symmetry. In terms of the functions occurring in Eqs. (1) and (27) one readily obtains, for example,

$$W_{SS'}[\psi_S,\psi_{S'}]=3\int d\tau\, f(r_1)H(r_1)f(r_2)F(r_2) \times g(r_3)G(r_3)[V_\sigma(r_2)-V_\sigma(r_1)]. \quad (28)$$

The analogous expressions for the other quantities are obtained equally readily and will not be given.

The most flexible trial function used has 16 parameters and is denoted by $\Psi_{6,9}^{(16)}$, in which the left and right subscripts refer to the numbers of parameters in ψ_S and $\psi_{S'}$, respectively. Thus, for ψ_S we have used the same 6-parameter function (1) to (1b) as was used by Downs and Dalitz.[3] As emphasized by them, the 3-parameter functions (1a) and (1b) are sufficiently flexible to take adequate account, on the one hand, of the long tail of the wave function for large interparticle separations and, on the other hand, of the strong short-range correlations implied by the strong and short-ranged interactions—especially the quite short-range Λ-N interaction. Section 3 presents some results that are relevant in this connection. For the same reasons, in Eq. (27) we have used the three-parameter functions

$$F(r)=e^{-\alpha_1 r}+\xi e^{-\alpha_2 r}, \quad (29)$$

$$G(r)=e^{-\beta_1 r}+\zeta e^{-\beta_2 r}, \quad (30)$$

$$H(r)=e^{-\gamma_1 r}+\eta e^{-\gamma_2 r}. \quad (31)$$

The function $\psi_{S'}$ is thus a 9-parameter function. In addition to the 15 parameters entering through ψ_S and $\psi_{S'}$, the function $\Psi_{6,9}^{(16)}$ also contains the admixture parameter A as a variational parameter.

Some calculations were also made for the related 10-parameter function $\Psi_{6,3}^{(10)}$ which is obtained from $\Psi_{6,9}^{(16)}$ by imposing the constraints $F(r)\equiv f(r)$, $G(r)\equiv g(r)$. Since ψ_S will be the dominant component, the values of the six parameters occurring in $f(r)$ and $g(r)$ are variationally quite well determined, approximately independent of whether or not $\psi_{S'}$ is included in the calculation. For $\Psi_{6,3}^{(10)}$ the component $\psi_{S'}$ thus effectively depends only on the three parameters γ_1, γ_2, and η which enter through the use of Eq. (31) for $H(r)$.

Exploratory calculations were made with the 6-parameter function $\Psi_{4,1}^{(6)}$ which is characterized by $F(r)\equiv f(r)=e^{-a_1 r}$, $G(r)\equiv g(r)=e^{-b_1 r}+ye^{-b_2 r}$, and $H(r)=e^{-\gamma_1 r}$ [i.e., by putting $F=f$, $x=0$, $G=g$, and $\eta=0$ in Eqs. (1a), (1b), and (29)–(31)]. In contrast to $\Psi_{6,9}^{(16)}$ and $\Psi_{6,3}^{(10)}$, only 1-parameter functions are thus used for the Λ-N correlations. In particular, ψ_S is the 4-parameter function

$$\psi_S(r_1,r_2,r_3)=e^{-a_1 r_1}e^{-a_1 r_2}(e^{-b_1 r_3}+ye^{-b_2 r_3}). \quad (32)$$

Finally, some calculations were also made for the related 10-parameter function $\Psi_{4,5}^{(10)}$ which is obtained from $\Psi_{4,1}^{(6)}$ by letting $F(r)=e^{-\alpha_1 r}$ differ from $f(r)=e^{-a_1 r}$ and letting $G(r)$, now given by Eq. (30), differ from $g(r)$.

With Yukawa interactions and the above type of trial functions (which are effectively sums of products of exponentials), all the triangular integrations which occur may be done analytically and the results may all

TABLE I. Results for $\psi_S^{(6)}$, $\psi_S^{(4)}$, and $\psi_S^{(2)}$ with $B=2.526$ MeV.

ψ_S	b (F)	a_1 (F^{-1})	a_2 (F^{-1})	x	b_1 (F^{-1})	b_2 (F^{-1})	y	U_2 (MeV F^3)
$\psi_S^{(6)}$	1.484($\mu_{2\pi}$)	0.132	0.929	1.47	0.388	1.170	2.007	678.88
	0.84(μ_K)	0.165	1.408	1.955	0.396	1.164	2.022	423.85
$\psi_S^{(4)}$	1.484($\mu_{2\pi}$)	0.337	...	...	0.3665	1.190	2.015	745.83
	0.84(μ_K)	0.593	...	...	0.357	1.256	2.027	483.13
$\psi_S^{(2)}$	1.484($\mu_{2\pi}$)	0.383	...	...	0.65	...	...	781.5
	0.84(μ_K)	0.635	...	...	0.71	...	...	492.75

be expressed in terms of the algebraic expressions

$$I_{110}(x,y,z)=\int\int\int \exp(-xr_1-yr_2-zr_3)r_1r_2dr_1dr_2dr_3$$
$$=4[(x+y)(x+y+z)+(x+z)(y+z)]$$
$$\times[(x+y)^3(y+z)^2(z+x)^2]^{-1},$$
$$I_{111}(x,y,z)=\int\int\int \exp(-xr_1-yr_2-zr_3)r_1r_2r_3dr_1dr_2dr_3$$
$$=8[x(x+y)(x+z)+y(y+z)(y+x)$$
$$+z(z+x)(y+z)+2(x+y)(y+z)(z+x)]$$
$$\times[(x+y)^3(y+z)^3(z+x)^3]^{-1}.$$

These have obvious symmetries in the simultaneous interchange of x, y, z, and the corresponding indices. The procedures are entirely analogous to those used by Downs and Dalitz.[3] The final algebraic expressions are straightforward to obtain but are very lengthy, especially for the functions $\Psi_{6,9}^{(16)}$ and $\Psi_{6,3}^{(10)}$ and will not be given.

The parts of the various expressions which depend only on ψ_S (e.g., N_S, T_S, W_S, etc.) are clearly identical for $\Psi_{6,9}^{(16)}$ and $\Psi_{6,3}^{(10)}$ while the parts depending only on $\psi_{S'}$ are the same and differ only in that the parameters of $F(r)$ and $G(r)$ (i.e., α_1, α_2, ξ, β_1, β_2, ζ) that enter for $\Psi_{6,9}^{(16)}$ are replaced by the parameters of $f(r)$ and $g(r)$ (i.e., a_1, a_2, x, b_1, b_2, y) for $\Psi_{6,3}^{(10)}$. Of course, the terms depending on both ψ_S and $\psi_{S'}$ (e.g., $W_{SS'}$) have a different structure for $\Psi_{6,9}^{(16)}$ and $\Psi_{6,3}^{(10)}$. Most of the algebra is thus the same for $\Psi_{6,9}^{(16)}$ and $\Psi_{6,3}^{(10)}$ and comparatively little extra effort is required in using $\Psi_{6,9}^{(16)}$ instead of $\Psi_{6,3}^{(10)}$. Completely analogous remarks apply to $\Psi_{4,5}^{(10)}$ and $\Psi_{4,1}^{(6)}$.

In addition to the interchanges $a_1\leftrightarrow a_2$ with $x\leftrightarrow x^{-1}$, etc., the value of Φ is also invariant under $F(r)\leftrightarrow H(r)$ with $A\leftrightarrow -A$ for $\Psi_{6,9}^{(16)}$ and $\Psi_{4,5}^{(10)}$. These symmetries may of course be taken together in any combination, or in any number up to the maximum possible. More-restrictive obvious combinations of these symmetries apply to the expressions depending only on ψ_S or only on $\psi_{S'}$. All these symmetries provide a useful check both for the algebra and for the numerical calculations.

It is clear that associated with any particular local minimum of Φ there will thus be a whole class of equivalent local minima. However, in particular for the very flexible function $\Psi_{6,9}^{(16)}$, several nonequivalent local minima are to be expected and were indeed found.

The quantity $\Phi[\Psi]$ was minimized numerically with the aid of Davidon's metric-minimization procedure[15] for obtaining local minima. For this, the derivatives of Φ with respect to the variational parameters are required. Although these derivatives may also be obtained in closed algebraic form, it is much easier and less subject to error to obtain the derivatives numerically from the difference of Φ for neighboring values of the parameters. However, calculations using the analytic expressions for the derivatives were also made for $\Psi_{4,1}^{(6)}$ and $\Psi_{4,5}^{(10)}$ and gave the same results as with the use of the numerically obtained derivatives. The minimization procedure finds that local minimum which is "nearest," in the parameter space, to the point corresponding to the initial guesses of the parameters. These are a required input. The minimization program has a facility that permits imposition of linear constraints between the parameters. Thus, for example, the results for $\Psi_{6,3}^{(10)}$ may be obtained from a calculation for $\Psi_{6,9}^{(16)}$ together with the appropriate constraints.

The values of p, B_S, $B_{S'}$, and $B_{SS'}$ were calculated after the minimum was found. As a check, B was then calculated by use of Eq. (22) to see if this agreed with the input value of B.

3. RESULTS AND DISCUSSION

Calculations were made for Yukawa Λ-N interactions with the range $\mu^{-1}=\mu_{2\pi}^{-1}=0.7$ F ($b=1.484$ F), for $\mu^{-1}=\mu_K^{-1}=0.4$ F ($b=0.84$ F), and for $\mu^{-1}=0.977$ F ($b=2.07$ F). The last range was chosen so as to give the same intrinsic range as an interaction with a hard core of radius $r_c=0.42$ F and an attractive Yukawa part with a range $\mu^{-1}=\mu_{3\pi}^{-1}=0.47$ F corresponding to an exchanged meson of mass $3m_\pi=419$ MeV. Studies of one-boson-exchange models of the N-N interaction indicate a mass of about this value for the part of the interaction that is due to a spin-isospin scalar boson (σ meson). This interaction would be responsible in large part for the attractive tail of the Λ-N force (K meson exchange would also give a Yukawa tail of about

[15] W. C. Davidon, Argonne National Laboratory Report ANL-5990 Rev. 1959 (unpublished).

TABLE II. Results for $\Psi_{6,9}^{(16)}$ with $B=2.526$ MeV and with $B=2.226$ MeV (where indicated) and with $\mathfrak{U}_s=0$.

b (F)	Δ (MeV F³)	a_1 (F⁻¹)	a_2 (F⁻¹)	x	b_1 (F⁻¹)	b_2 (F⁻¹)	y	α_1 (F⁻¹)	α_2 (F⁻¹)	ξ	β_1 (F⁻¹)	β_2 (F⁻¹)	ζ	γ_1 (F⁻¹)	γ_2 (F⁻¹)	η	A	p	B_S (MeV)	$B_{S'}$ (MeV)	$B_{SS'}$ (MeV)	U_2 (MeV F³)
0.84[a]	50	0.168	1.468	1.881	0.372	1.120	2.271	0.737	2.550	1.359	0.108	0.721	1.874	0.445	1.201	0.585	−0.747	0.072	1.995	−100.48	7.39	410.67
	100	0.178	1.634	1.737	0.361	1.129	2.093	0.717	2.424	1.415	0.107	0.941	1.946	0.447	1.174	0.521	−1.163	0.121	0.77	−116.07	14.43	346.80
1.484	0	0.049	0.603	2.116	0.380	1.129	2.213	...	...	...	...	...	...	...	...	...	...	...	2.226	...	...	621.05
(B=2.226)	100	0.053	0.627	1.972	0.379	1.130	2.196	0.603	1.471	1.569	0.142	0.911	2.020	0.316	0.736	0.430	−0.768	0.032	2.16	−64.34	2.11	590.82
	200	0.057	0.664	1.845	0.374	1.129	2.203	0.493	1.479	1.576	0.145	0.801	2.028	0.297	0.704	0.470	−1.205	0.058	1.98	−70.44	4.20	513.77
1.484[a]	100	0.125	0.868	1.521	0.382	1.158	2.021	0.478	1.390	1.629	0.150	0.940	2.000	0.354	0.861	0.410	−0.608	0.059	2.27	−69.45	4.26	647.83
	200	0.133	0.951	1.380	0.368	1.157	1.996	0.445	1.282	1.620	0.137	1.014	1.985	0.368	0.889	0.414	−1.129	0.105	1.65	−76.93	8.35	569.78
2.07	0	0.109	0.648	1.356	0.385	1.159	2.007	...	...	...	...	...	...	...	...	...	...	...	2.526	...	...	909.89
	200	0.108	0.650	1.370	0.378	1.169	2.000	0.519	1.221	1.510	0.178	0.980	2.000	0.286	0.990	0.395	−0.634	0.071	2.24	−51.79	3.91	848.30
	400	0.117	0.764	1.238	0.351	1.156	2.114	0.438	1.158	1.298	0.148	0.990	1.914	0.352	1.063	0.472	−1.917	0.124	1.51	−62.49	8.12	689.37

[a] The results for $\Delta=0$ are given in Table I.

the same range[16]). It may be hoped that the results for the scattering length for a purely attractive interaction are not too different from those obtained with a hard core if both have the same intrinsic range $b\approx 2.0$ F. This is true for $b\approx 1.5$ F but clearly needs confirmation for a different value of b. It should be noted that an interaction with a hard core of radius $r_c=0.42$ F and $b=1.5$ F (as used in previous studies of hypernuclei, e.g., Ref. 10) corresponds to an attractive Yukawa part with a range of only $\mu^{-1}=0.23$ F (i.e., corresponding to an exchanged mass of $6.1m_\pi$).

We first discuss briefly some results obtained with only Ψ_S. These are of interest in connection with the question of the adequacy of various forms of trial functions [all, of course, of the general product form (1)]. Table I shows results for $B=2.526$ MeV and for $\mu_{2\pi}$ and μ_K for the 6-parameter function $\psi_S^{(6)}$ [Eqs. (1)–(1b)], for the 4-parameter function $\psi_S^{(4)}$ [Eq. (32)], and for the 2-parameter function

$$\psi_S^{(2)}=e^{-a_1r_1}e^{-a_1r_2}e^{-b_1r_3}, \qquad (33)$$

which was used by Dalitz and Downs[17] in their exploratory investigation of ${}_\Lambda H^3$. The present results for $\psi_S^{(6)}$ (and also those for $\Psi_{6,9}^{(16)}$ and $\Psi_{6,3}^{(10)}$ with small Δ, i.e., effectively with $p=0$) agree with those of Ref. 3 and the present results for $\psi_S^{(2)}$ agree with those of Ref. 16.

A comparison of the results for $\psi_S^{(6)}$ and $\psi_S^{(4)}$ shows that the (3-parameter) n-p function $g(r)$ is very similar for both. This is as expected in view of the relatively small distortion of the deuteron by the Λ. Especially for the shorter range μ_K^{-1}, the improvement in U_2 as a result of using $\psi_S^{(4)}$ instead of $\psi_S^{(2)}$ [i.e., as a result of the greater flexibility of $g(r)$], is seen to be considerably less than the improvement due to a corresponding greater flexibility in the Λ-N function $f(r)$, i.e., corresponding to the improvement in using $\psi_S^{(6)}$ instead of $\psi_S^{(4)}$.

This clearly shows the importance of using a Λ-N function that has enough flexibility to take adequate account of the strong short-range correlations as well as of the long tail due to the small Λ separation energy.[3] There seems less need for a flexible n-p function, presumably because of the longer range of the n-p interaction as well as of the larger separation energy of a nucleon.

Complete results for calculations with the full wave function $\Psi_{6,9}^{(16)}$ are shown in Table II for $B=2.526$ MeV (and for $B=2.226$ MeV for $\mu_{2\pi}$ only) and for $\mathfrak{U}_s=0$, i.e., for a Serber potential for $\mathcal{V}_s$. These are the results for the deepest local minimum found. The initial guesses for the parameters of ψ_S were kept in the region found for $\Delta=0$. This is because these parameters are not expected to change much as Δ is increased from zero, in view of the fact that Ψ_S is expected to be the

[16] B. W. Downs and R. J. N. Phillips, Nuovo Cimento **36**, 120 (1965), discuss one-boson-exchange models for the Λ-N interaction.

[17] R. H. Dalitz and B. W. Downs, Phys. Rev. **110**, 985 (1958).

TABLE III. Results for U_2 [Eq. (34)] and p [Eq. (35)].

B (MeV)	b (F)	$U_2(0)$ (MeV F^3)	c (MeV^{-1} F^{-3})	d (MeV^{-2} F^{-6})	u (MeV^{-1} F^{-3})	v (MeV^{-2} F^{-6})
2.526	0.84	423.85	1.004×10^{-2}	2.334×10^{-5}	1.676×10^{-3}	-4.68×10^{-6}
	1.484	678.88	3.482×10^{-3}	3.775×10^{-6}	6.633×10^{-4}	-6.83×10^{-7}
	2.07	909.89	1.701×10^{-3}	8.075×10^{-7}	$3.98\ \times10^{-4}$	-2.19×10^{-7}
2.226	1.484	621.05	3.364×10^{-3}	3.410×10^{-6}	3.488×10^{-4}	-2.93×10^{-7}

dominant component. Table II shows that this is indeed the case. Also, as expected, it is seen that the parameters of $g(r)$ depend only slightly on μ. Further, as expected, U_2 varies quadratically with Δ and p varies linearly with Δ for small Δ. Over the whole range of Δ considered and for all cases, an almost perfect fit was obtained with

$$U_2=U_2(0)-c\Delta^2+d\Delta^3\,, \tag{34}$$

$$p=u\Delta+v\Delta^2. \tag{35}$$

(Results were also obtained for other values of Δ not shown in Table II.) The results of these fits are shown in Table III.

It is seen that, although the admixture of $\Psi_{S'}$ is rather small, the value of U_2 is quite appreciably less than $U_2(0)$ for the values of Δ of interest, especially so for $\mu_{2\pi}$. For a given Δ, the reduction is proportionally larger the shorter the range. The rather large energies $E_{s'}=-B_{s'}$ result from the large kinetic energies $T_{S'}/N_{S'}$ of $\Psi_{S'}$ (typically of the order of 70 MeV). These are a consequence of the large curvature of the n-p space correlation function which is implied by the antisymmetry of $\Psi_{S'}$ with respect to the interchange of the nucleon coordinates.

For $\Psi_{6,9}{}^{(16)}$ a considerable number of nonequivalent local minima were found corresponding to distinct regions for A and the parameters of $\psi_{S'}$ (the initial guesses for the parameters of ψ_S were always kept in the region found for $\Delta=0$ in view of the comments already made). A number of these local minima are shown in Table IV for $\mu_{2\pi}$, $B=2.256$ MeV, and $\Delta=200$ MeV F^3. The results of Table IV were in fact obtained with the singlet s-state potential for $\mathcal{V}_s$. The topmost set of results is for the deepest local minimum found and corresponds to the local minima shown in Table II (which are for $\mathcal{V}_s=0$). A comparison of the appropriate entries in Tables II and IV shows that the results depend very little on $\mathcal{V}_s$ and that even the use of a quite strongly attractive $\mathcal{V}_s$ (as in Table II) reduces the value obtained for U_2 only slightly below that for $\mathcal{V}_s=0$. This is because the spatial antisymmetry of $\Psi_{S'}$ with respect to the nucleons keeps these apart and effectively outside the range of their interaction. Thus, the contribution to the expectation value of the potential energy of $\Psi_{S'}$ due to $\mathcal{V}_s$ (i.e., $[\int d\tau\, \psi_{S'}{}^2\mathcal{V}_s]/N_{S'}$) is only about 2.5 MeV for the singlet s-state potential (and is of course zero for the Serber potential). The corresponding value for Ψ_S is about 20 MeV.

The results of Table IV for the distinct local minima of $\Psi_{6,9}{}^{(16)}$ are shown in order of increasing U_2. It is seen that the values of U_2 for the minima shown are only slightly larger than for the deepest minimum (the topmost) which was found. This result is presumably due to the great flexibility of the function $\psi_{S'}$ which has 9 parameters and which might therefore be expected to lead to a number of nonequivalent minima with nearly the same value of U_2. This is borne out by the results for the two distinct minima of $\Psi_{6,3}{}^{(10)}$ shown in Table IV. As remarked earlier, for $\Psi_{6,3}{}^{(10)}$ the component $\psi_{S'}$ has effectively only 3 parameters whereas ψ_S is the same as for $\Psi_{6,9}{}^{(10)}$. Nevertheless, the value of $U_2(0)-U_2(\Delta)$ is only slightly less (by about 4% for $\mu_{2\pi}$, $\Delta=200$ MeV F^3) than for $\Psi_{6,9}{}^{(16)}$. The additional flexibility of $\psi_{S'}$ for $\Psi_{6,9}{}^{(16)}$ thus gives only a rather slight improvement, although the functions $F(r)$ and $G(r)$ [constrained to be the same as $f(r)$ and $g(r)$, respectively, for $\Psi_{6,3}{}^{(10)}$] are considerably different from $g(r)$ and $f(r)$, respectively. It should also be noted that the values of p are very similar for all the distinct local minima of $\Psi_{6,9}{}^{(16)}$ and $\Psi_{6,9}{}^{(10)}$ that are shown.

Table IV also shows, for interest, the results for $\Psi_{4,5}{}^{(10)}$ and $\Psi_{4,1}{}^{(6)}$. It is interesting to note that if the results for $U_2(\Delta)$, even for $\Psi_{4,1}{}^{(6)}$, are normalized to the value of $U_2(0)$ obtained with $\psi_S{}^{(6)}$, then the values obtained for U_2 are quite close to those obtained with $\Psi_{6,9}{}^{(16)}$.

To obtain the values of U_s and U_t (and hence also of U_2 and Δ), we use our results for $U_2(\Delta)$ together with the appropriate values of $\bar{U}$ and the relations $U_2=\frac{3}{2}U_s+\frac{1}{2}U_t$ and $\bar{U}=\frac{1}{2}(U_2-\Delta)$. From analysis of ${}_\Lambda He^5$ (Refs. 7, 9), one has $\bar{U}=260\pm12.5$ MeV F^3 for $\mu_{2\pi}$ and $\bar{U}=197.5\pm10$ MeV F^3 for μ_K. These values are also very close to those obtained from ${}_\Lambda Be^9$ (Ref. 8) and ${}_\Lambda C^{13}$ (Ref. 9). For $\mu^{-1}=0.977$ F, analysis of ${}_\Lambda He^5$ gave $\bar{U}=340\pm15$ MeV F^3. Then with $B_\Lambda({}_\Lambda He^3)=0.3\pm0.15$ MeV one gets the results of Table V for $\mathcal{V}_s=0$. (The results obtained with the singlet s-state potential for $\mathcal{V}_s$ are virtually identical with those of Tables V and VI.) The results for $p=0$ (i.e., with only Ψ_S) are also shown. The corresponding results for the low-energy scattering parameters are shown in Table VI. The scattering lengths, effective ranges, and well-depth parameters are denoted by a_s, r_s, and s_s, respectively, for the singlet case and by a_t, r_t, and s_t for the triplet case.

The effect of $\Psi_{S'}$ is seen to be largest and quite appreciable for $b=1.5$ F. For $b=2$ F the effect is some-

TABLE IV. Results for $B=2.526$ MeV, $\Delta=200$ MeV F^3, $b=1.484$ F and $\mathfrak{U}_s=951$ MeV F^3, $\nu=1.428$ F^{-1}.

Ψ	a_1 (F^{-1})	a_2 (F^{-1})	x	b_1 (F^{-1})	b_2 (F^{-1})	y	α_1 (F^{-1})	α_2 (F^{-1})	ξ	β_1 (F^{-1})	β_2 (F^{-1})	ζ	γ_1 (F^{-1})	γ_2 (F^{-1})	η	A	p	B_S (MeV)	$B_{S'}$ (MeV)	$B_{SS'}$ (MeV)	U_2 (MeV F^3)
$\Psi_{6,9}$[16]	0.133	0.963	1.373	0.368	1.157	1.992	0.449	1.278	1.634	0.154	1.012	1.990	0.363	0.885	0.409	−1.141	0.109	1.61	−74.32	8.37	565.70
$\Psi_{6,9}$[16]	0.133	0.944	1.361	0.368	1.156	1.997	0.502	1.442	1.545	0.162	1.098	1.984	0.367	1.095	−0.205	−1.070	0.1085	1.621	−74.267	8.336	565.90
$\Psi_{6,9}$[16]	0.132	0.945	1.394	0.375	1.170	1.887	0.248	0.620	1.640	0.313	2.095	−0.495	0.567	1.720	1.078	1.276	0.1107	1.6185	−70.349	8.132	566.09
$\Psi_{6,9}$[16]	0.131	0.965	1.458	0.366	1.158	2.027	0.195	0.717	1.897	0.189	0.505	1.941	0.867	2.294	0.989	0.416	0.1007	1.608	−86.415	9.037	566.82
$\Psi_{6,9}$[16]	0.132	0.938	1.382	0.374	1.1795	1.911	0.534	1.588	1.661	0.312	1.970	−0.645	0.300	1.414	0.496	−1.372	0.1077	1.635	−72.672	8.185	567.60
$\Psi_{6,9}$[16]	0.146	1.089	1.268	0.376	1.185	1.869	1.209	1.007	0.051	0.142	0.767	2.028	0.342	1.112	−0.483	−1.283	0.1056	1.575	−80.624	8.893	568.07
$\Psi_{6,3}$[10]	0.134	1.076	1.501	0.365	1.161	2.010	a_1	a_2	x	b_1	b_2	y	0.137	1.002	−0.154	−0.952	0.1079	1.628	−72.957	8.234	575.47
	0.132	0.953	1.427	0.366	1.161	2.008	a_1	a_2	x	b_1	b_2	y	0.812	1.938	1.056	0.861	0.1013	1.633	−85.321	8.856	570.28
$\Psi_{4,5}$[10]	0.321	...	...	0.339	1.186	2.029	0.531	...	...	0.107	0.966	1.963	1.030	...	...	0.870	0.133	1.08	−77.85	10.77	638.00
$\Psi_{4,1}$[6]	0.326	...	...	0.336	1.888	2.026	a_1	...	...	b_1	b_2	y	1.067	...	...	0.698	0.124	1.11	−88.11	11.32	643.25

what smaller, while for $b=0.84$ F it is quite small. There are in fact two competing effects; on the one hand, Δ increases (for $p=0$) as the range increases while, on the other hand, for a given Δ the value of $U_2(0)-U_2(\Delta)$ decreases as the range increases.

Inclusion of $\Psi_{S'}$ reduces the singlet strength and slightly increases the triplet strength. The effect is larger on U_s than on U_t since U_s enters much more strongly into U_2. In particular, for $\mu_{2\pi}$ the spin dependence Δ is seen to be considerably reduced (by about a third) below $\Delta(p=0)$. This will have a considerable effect on the excitation energy of an excited state in which the Λ spin is flipped with respect to the ground state. Thus, for example, the binding energy of the possible excited state (with $J=1$) of ${}_\Lambda He^4$ is increased (for $\mu_{2\pi}$) from about 0.55 MeV ($\Delta=160$ MeV F^3) to about 0.85 MeV ($\Delta=116$ MeV F^3).

It should also be remarked that when $\Psi_{S'}$ is included, the value of U_s is quite insensitive to the value of $\bar{U}$ (since the value obtained for U_2 decreases as $\bar{U}$ increases) and the error in U_s is largely due to the uncertainty in $B_\Lambda({}_\Lambda He^3)$. Thus, for example, for $\mu_{2\pi}$ and $B_\Lambda({}_\Lambda He^3)=0.3$ MeV one has $U_s=248$ MeV F^3 ($U_t=247$ MeV F^3) for $\bar{U}=272.5$ MeV F^3; and $U_s=346$ MeV F^3 ($U_t=215$ MeV F^3) for $\bar{U}=247.5$ MeV F^3. The singlet strength is thus almost entirely determined by $B_\Lambda({}_\Lambda He^3)$ and a better experimental value for this would be very valuable. It is also seen that the error in Δ (for a given b) is considerably reduced if $\Psi_{S'}$ is included.

The value of p is seen to be rather small (≈ 0.07 for $\mu_{2\pi}$). No detailed study has so far been made of the possible effects of $\Psi_{S'}$ on the properties of ${}_\Lambda H^3$, in particular on the lifetime. However, we may mention that for the magnetic moment $\mathfrak{M}$ of ${}_\Lambda H^3$ the cross term $(\Psi_S,\mathfrak{M}\Psi_{S'})$, which is linear in p, is zero and the effect of $\Psi_{S'}$ is therefore very small, of order p^2. (The term linear in p would no longer be zero for finite momentum transfers and would contribute to the magnetic form factor—if this form factor could be measured.)

We now turn to a discussion of the Λ-N total cross section σ at low energies ($\lesssim 20$ MeV) where the s-wave phase shift dominates. In the effective-range approximation,[18] which we use, one has

$$\sigma=\tfrac{1}{4}\sigma_s+\tfrac{3}{4}\sigma_t$$
$$=\frac{\pi}{k^2+[a_s^{-1}-\frac{1}{2}r_sk^2]^2}+\frac{3\pi}{k^2+[a_t^{-1}-\frac{1}{2}r_tk^2]^2}. \quad (36)$$

The experimental total cross sections σ_{exp} for Λ-p scattering recently obtained by Zechi-Zorn *et al.*[19] are

[18] For the scattering parameters of Ref. 10 (i.e., for an interaction with $b=1.5$ F and with a hard core of radius 0.42 F) and for c.m. energies $\lesssim 20$ MeV, the values obtained for σ with the use of Eq. (36) are within a few percent of the corresponding exact values given in Ref. 10.

[19] B. Zechi-Zorn, R. A. Burnstein, T. B. Day, B. Kehoe, and G. A. Snow, reported in an invited paper by R. A. Burnstein at the American Physical Society meeting, Washington, D. C., April 1965 (unpublished), and (private communication). Earlier results by these authors are given in Phys. Rev. Letters **13**, 282 (1964).

TABLE V. Results for volume integrals with $B_\Lambda({}_\Lambda H^3)=0.3\pm0.15$ MeV.

b (F)	$\bar{U}$[a] (MeV F^3)	p[b]	U_2 (MeV F^3)	U_s (MeV F^3)	U_t (MeV F^3)	Δ (MeV F^3)
$0.84(\mu_K)$	197.5 ± 10	0	$424_{-7.5}^{+6}$	219 ± 10	190 ± 16	29 ± 27
		$0.038_{-0.035}^{+0.025}$	419_{-12}^{+10}	215.5 ± 13	191.5 ± 18	23.5_{-22}^{+16}
$1.484(\mu_{2\pi})$	260 ± 12.5	0	679_{-24}^{+16}	379_{-24}^{+18}	220 ± 23	159 ± 40
		0.068 ± 0.015	637_{-24}^{+19}	348_{-24}^{+20}	231 ± 23	117_{-20}^{+14}
$2.07(\mu^{-1}=0.977$ F)	340 ± 15	0	910_{-40}^{+30}	512.5_{-37}^{+30}	282.5 ± 31	230_{-70}^{+60}
		0.065 ± 0.012	859.5_{-37}^{+30}	474.5_{-35}^{+30}	295 ± 31	179.5_{-49}^{+37}

[a] Values obtained from analysis of ${}_\Lambda He^5$. [b] Errors are estimates.

shown with errors bars and plotted against the c.m. energy in Fig. 1.

We consider first the (purely attractive) Yukawa interactions. The values of σ for the hypernuclear results with $b=1.484$ F ($\mu_{2\pi}$) are considerably less than σ_{exp}. (In Fig. 1, curves A′ and A are for the central values of the hypernuclear scattering parameters obtained respectively with and without the use of $\Psi_{S'}$.) If we consider a_s and r_s to be more reliably determined (for a given b) from hypernuclei (effectively from ${}_\Lambda H^3$) than a_t and r_t, then to obtain reasonable agreement with σ_{exp}, with due allowance for the errors in a_s and r_s, requires $-1.5 \lesssim a \lesssim -1.2$ F, 2.25 F$\lesssim r_t \lesssim 2.4$ F (see, for example, curve B), where the relation between a_t and r_t appropriate to a Yukawa interaction with $b=1.484$ F has been used. For $b=0.84$ F (μ_K) it does not seem possible to obtain a satisfactory fit to σ_{exp} at both the lower and higher range of energies with any reasonable values of the scattering parameters consistent with the hypernuclear results (see, for example, curve C) even if the triplet parameters are varied outside the hypernuclear values. This indicates that purely attractive interactions with $b\lesssim1$ F are ruled out.

For interactions with a hard core, we first consider the results for $b=1.5$ F ($r_c=0.42$ F). Curves D′ and D in Fig. 1 are for the central hypernuclear results obtained with and without $\Psi_{S'}$, respectively. (The results for the former case have been estimated by use of the results of Ref. 10 which are for Ψ_S only.) The values of σ are somewhat larger than for a Yukawa interaction with $b=1.5$ F, but are still considerably below σ_{exp}. Even for the maximum values of σ permitted by the errors of the hypernuclear results, σ is still below σ_{exp} although the discrepancy is no longer too bad. If we again suppose that a_s and r_s are most reliably determined from the analyses of hypernuclei, then one finds that the values -1.5 F$\lesssim a_t \lesssim -1.2$ F, $2.5\lesssim r_t\lesssim3.0$ are required (see curves E and F in Fig. 1) for an acceptable fit to σ_{exp} (the relation between a_t and r_t being appropriate to $b=1.5$ F, $r_c=0.42$ F). Thus for $b=1.5$ F, both with and without a hard core, only a rather moderate increase in $|a_t|$ above the hypernuclear values is required to obtain agreement with σ_{exp}, especially if it is remembered that with $\Psi_{S'}$ the errors allow values $a_t\approx-0.9$ F and -1.0 F, respectively, with and without a hard core. [That only a moderate increase in $|a_t|$ is

TABLE VI. Results for the low-energy scattering parameters. The notation (H.c.) indicates that the results are for an interaction with a hard core of radius $r_c=0.42$ F.

b (F)		Singlet parameters $-a_s$ (F)	r_s (F)	s_s	Triplet parameters $-a_t$ (F)	r_t (F)	s_t
0.84	only Ψ_S	$1.44_{-0.19}^{+0.21}$	$1.09_{-0.03}^{+0.04}$	0.68 ± 0.03	$0.99_{-0.20}^{+0.23}$	$1.21_{-0.07}^{+0.08}$	0.59 ± 0.05
	with $\Psi_{S'}$	$1.365_{-0.22}^{+0.27}$	$1.11_{-0.04}^{+0.05}$	0.67 ± 0.04	$1.005_{-0.21}^{+0.26}$	$1.20_{-0.07}^{+0.09}$	0.60 ± 0.05
1.484	only Ψ_S	2.45 ± 0.41	$1.93_{-0.06}^{+0.09}$	$0.675_{-0.04}^{+0.03}$	$0.78_{-0.12}^{+0.15}$	$2.88_{-0.22}^{+0.24}$	0.39 ± 0.04
	with $\Psi_{S'}$	1.93 ± 0.31	$2.05_{-0.08}^{+0.11}$	0.62 ± 0.04	$0.84_{-0.13}^{+0.16}$	$2.78_{-0.21}^{+0.24}$	0.41 ± 0.04
2.07	only Ψ_S	3.11 ± 0.56	$2.75_{-0.10}^{+0.16}$	$0.65_{-0.05}^{+0.04}$	$0.95_{-0.17}^{+0.18}$	$4.30_{-0.35}^{+0.30}$	0.36 ± 0.04
	with $\Psi_{S'}$	2.54 ± 0.42	$2.91_{-0.125}^{+0.17}$	0.60 ± 0.04	$1.015_{-0.15}^{+0.19}$	$4.25_{-0.40}^{+0.22}$	0.38 ± 0.04
1.5 (H.c.)	only Ψ_S [a]	$2.89_{-0.41}^{+0.59}$	1.94 ± 0.08	0.865 ± 0.017	0.71 ± 0.06	3.75 ± 0.22	0.675 ± 0.011
	with $\Psi_{S'}$ [b]	2.28 ± 0.5	2.06 ± 0.1	0.8 ± 0.04	0.77 ± 0.15	3.62 ± 0.3	0.7 ± 0.04

[a] These are the values given in Ref. 10. [b] Values estimated by use of the results of Ref. 10.

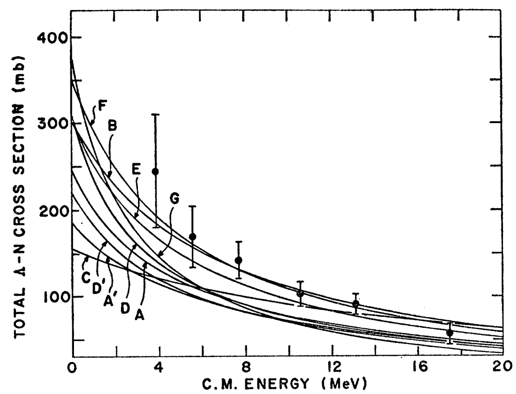

FIG. 1. Results for the Λ-N total cross section σ. The experimental values, marked with error bars, are those of Ref. 19. The parameters used for the various curves are the following.

Curve	a_s (F)	r_s (F)	a_t (F)	r_t (F)	Interaction
A′	−1.93	2.05	−0.84	2.78	Yukawa, $b=1.484$ F, with $\Psi_{S'}$
A	−2.45	1.93	−0.78	2.88	Yukawa, $b=1.484$ F, without $\Psi_{S'}$
B	−1.93	2.05	−1.4	2.26	Yukawa, $b=1.484$ F and singlet parameters with $\Psi_{S'}$
C	−1.365	1.11	−1.005	1.2	Yukawa, $b=0.84$ F, with $\Psi_{S'}$
D′	−2.28	2.06	−0.77	3.62	with hard core, $b=1.5$ F and with $\Psi_{S'}$ (estimated values)
D	−2.89	1.94	−0.71	3.75	with hard core, $b=1.5$ F and without $\Psi_{S'}$ (central values of Ref. 10)
E	−2.3	2.05	−1.2	3.0	with hard core, $b=1.5$ F and singlet parameters with $\Psi_{S'}$
F	−2.3	2.05	−1.4	2.8	with hard core, $b=1.5$ F and singlet parameters with $\Psi_{S'}$
G	−3	3	−1	5.45	with hard core, $b=2.07$ F and with $\Psi_{S'}$ (estimated values)

necessary is a result of expression (36) being heavily weighted in favor of the triplet contribution.]

We discuss an interesting possibility which can give an increase in the value of $|a_t|$ obtained from the analysis of hypernuclei. This arises from a possible difference, especially for the triplet case, between the free Λ-N interaction and the effective interaction in hypernuclei. Such a difference can arise from a modification of the coupling of the ΛN with the ΣN channel. In the pion-potential model of de Swart and Iddings,[20] which we use, there is, in particular, a strong coupling for the triplet case due to the strong one-pion-exchange tensor force. For the singlet case the coupling is quite weak and relatively unimportant (see also Dalitz).[21] With a hard core of radius $r_c \approx 0.4$ F, the triplet interaction is in fact effectively attractive only because of the strong coupling. If, then, this coupling is appreciably suppressed in hypernuclei, one may expect the effective triplet interaction to become less attractive than the free interaction, whereas the singlet interaction will not be appreciably affected. This is in the required direction since the free triplet interaction, relevant for σ, will then be larger than the effective interaction which is the one obtained from analyses of hypernuclei.

One might expect such a suppression to be appreciable especially for ${}_\Lambda He^5$ since for this the (virtual) process $He^4+\Lambda \rightarrow He^4+\Sigma$ is forbidden because of isospin conservation, and the coupling can occur only through $He^4+\Lambda \rightarrow {}_4Z(T=1)+\Sigma$, i.e., through $T=1$ states of the $A=4$ nuclei. The relevant mean excitation energy E^* of these states relative to He^4 may reasonably be expected to be between 20 and 30 MeV. This is an appreciable fraction of the Σ-Λ mass difference ($M_\Sigma - M_\Lambda = 76.9$ MeV) which (for a given ΛN,ΣN potential matrix) determines the importance of the coupling between the ΛN and ΣN channels. A proper calculation of the effect would imply a variational calculation for ${}_\Lambda He^5$ which includes ${}_4Z(T=1)-\Sigma$ components in the total wave function.

We have obtained an estimate for the effect in ${}_\Lambda He^5$ by simply doing a coupled-channel calculation for the free scattering case with the use of the potentials and procedure of de Swart and Iddings but with an effective Σ mass $M_\Sigma + E^*$ instead of with the actual mass M_Λ. The suppression of the coupling will then be determined by $\chi = (M_\Sigma + E^* - M_\Lambda)/(M_\Sigma - M_\Lambda)$. This has the values $1.26 \leqslant \chi \leqslant 1.39$ for 20 MeV $\leqslant E^* \leqslant 30$ MeV. A hard core of radius $r_c = 0.42$ F was used. This with the meson-theory potentials of Ref. 20 gives $b=1.5$ F for both the singlet and triplet interactions.[22] For $\chi = 1.3$, 1.5, 2, and 3, one obtains the "suppressed" scattering lengths $\bar{a}_t = -1.02$, -0.91, -0.71, and -0.48 F for a free scattering length (i.e., for $\chi=1$) of $a_t = -1.25$ F; and the values $\bar{a}_t = -1.3$, -1.15, -0.89, and -0.6 F for

[20] J. J. de Swart and C. K. Iddings, Phys. Rev. 128, 2810 (1962).

[21] R. H. Dalitz, Phys. Letters 5, 53 (1963).

[22] The precise values used for the Σ-Σ-π coupling constant $f_{\Sigma\Sigma}$ is not important if it is small ($|f_{\Sigma\Sigma}| \lesssim 0.15$). For a given value of r_c the value of the Λ-Σ-π coupling constant $f_{\Sigma\Lambda}$ is effectively determined by the free scattering length a_t.

$a_t = -1.6$ F. Even though, for plausible values of χ, the differences between a_t and $\bar{a}_t$ are not large (about 0.3 F), it is nevertheless seen that the differences are of the order of magnitude required to bring the hypernuclear results into agreement with σ_{exp} without excessive straining. Thus a value of $\bar{a}_t = -0.9$ F is equivalent to $a_t \approx -1.2$ F, $r_t \approx 2.8$ F for the free interaction. This would go some way towards reconciling the hypernuclear results with σ_{exp} as is shown by curve E in Fig. 1 (which is for the central values of a_s and r_s obtained with the use of $\Psi_{s'}$) and which gives a not unacceptable fit. (The values $a_t = -1.4$ F, $r_t = 2.8$ F, curve F, would lead to a reasonably good fit to σ_{exp}.) A better calculation of the effect of the suppression of the ΣN channel would clearly be of interest. The above discussion, which rests on the assumption that the hypernuclear analyses determine the singlet parameters more reliably than the triplet ones would imply that the spin dependence of the free Λ-N interaction is considerably less than previously believed.

As has already been pointed out, for interactions with a hard core and with quite plausible mechanisms for the interaction, one can readily obtain intrinsic ranges larger than $b = 1.5$ F. On the basis of the results for a Yukawa interaction with $b = 2.07$ F, one may guess that reasonable values for an interaction with $b = 2.07$ F and with a hard core of radius $r_c \approx 0.4$ F might be $a_s \approx -3$ F, $r_s \approx 3$ F, $a_t \approx -1$ F, and $r_t \approx 5.5$ F. Curve G in Fig. 1 shows σ for these values. The agreement with σ_{exp} is no better than for $b = 1.5$ F, $r_c = 0.42$ F. The failure to obtain any improvement for $b = 2.07$ F in spite of the larger values of $|a_s|$ and $|a_t|$ is due to the large effective ranges. A more acceptable fit for $b = 2.07$ F with the same singlet parameters is obtained for $a_t \approx -1.6$ F, $r_t \approx 3.95$ F. It would again seem possible to get a large part of such an increase in $|a_t|$ from the suppression of the ΣN channel. Larger values of $|a_s|$ ($\lesssim 4$ F, which may well be considered permissible in view of the uncertainties) do not significantly improve the fit to σ_{exp}. It would be desirable to have analyses of hypernuclei for interactions with a hard core and with larger values of b (about 2 F).

Our discussion of σ does not enable one to choose between interactions with and without a repulsive core from a comparison of the hypernuclear results with σ_{exp}. However, in this connection it is important to note that $b = 1.5$ F is probably very much an upper limit for *purely attractive* interactions on the basis of any plausible mechanism for the Λ-N interaction. In fact, without a hard core, either the uncorrelated two-pion mechanism or a one-boson-exchange model with boson masses $\gtrsim 400$ MeV (as seems likely), would give a value of b considerably smaller ($\lesssim 1$ F) than 1.5 F.[23] If one adopts this viewpoint, then our discussion shows that there is a tentative indication for the existence of a repulsive core in the Λ-N interaction.

Finally we emphasize the importance of even better Λ-p scattering data and also of an improved determination of $B_\Lambda({}_\Lambda H^3)$. Further analyses of hypernuclear binding energies would also seem very desirable. It seems likely that the results of analyses of hypernuclei, together with improved scattering data, could eventually lead to a much more detailed knowledge of the Λ-N interaction.

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[23] In fact, S. Ali, A. R. Bodmer, and J. W. Murphy, Phys. Rev. Letters **15**, 534 (1965), have found evidence for a quite short range ($\mu^{-1} \lesssim 0.4$ F) of the attractive part from a comparison of the calculated binding energy of the excited state of ${}_\Lambda Be^9$ with the recently obtained experimental value [R. J. Piserchio, J. J. Lord, and D. Fournet Davis, Bull. Am. Phys. Soc. **10**, 115 (1965)].