direct calculation of the unstable case in the sense that arbitrary functions turn up which can be determined only by calculating the stable case solution and analytically continuing in the mass of the unstable particle. The main interest in the direct calculation thus remains in the demonstrated coupling of the stable and unstable particle channels.

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An Analysis of Some Pion-Pion Scattering Models

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A numerical investigation is made on some model calculations based on a suggestion of Blankenbecler. The models satisfy threshold conditions, elastic unitarity, and nearest crossed-channel singularities exactly. The internal parameters are chosen so as to minimize crossing dissatisfaction. No physically sensible result emerges.

I. INTRODUCTION

THIS paper is a report on an attempt to eliminate certain unsatisfactory properties of a model calculation first suggested by Blankenbecler[1] and later modified by Brehm.[2] The essential philosophy here is the same as in I. It consists of setting up a model which, *a priori*, satisfies certain general requirements for a scattering amplitude (such as threshold behavior and elastic unitarity) and then varying the free parameters present in the model in order to satisfy crossing symmetry as well as possible. The central differences between this calculation and the preceding ones are that the complete crossing relations are used, that the nearest crossed-channel singularities are given correctly, and that the model is purely elastic.

Section II is a description of the general model form chosen. Section III contains a discussion of crossing symmetry and the criteria for its satisfaction. Section IV contains the specific models and the results of the calculations based upon them. And Sec. V is an overall discussion of the calculation.

II. GENERAL FORM OF THE MODEL

We take the usual Mandelstam variables for the pion-pion problem,[3] and begin as in I by writing a partial-wave sum in N/D form for the scattering amplitude:

$$M(s,t)=\sum(2j+1)P_j(z)N_j(s)/D_j(s)\,, \tag{1}$$

where

$$z=1+2t/(s-4)\,, \tag{2}$$

and

$$D_j(s)=1-\frac{1}{\pi}\int_4^\infty\frac{\rho(s')N_j(s')ds'}{s'-s}\,, \tag{3}$$

and

$$\rho(s)=[(s-4)/s]^{1/2}. \tag{4}$$

Constant multiplicative factors in the phase-space function have been absorbed in N, and the sum is over even or odd j as the isospin T is even or odd. What now remains to be determined is the exact form of $N_j(s)$. We shall choose for large j the form

$$N_j(s)=f(s)g(j,s)[h(s)]^j\,, \tag{5}$$

where $g(j,s)$ is a rational function of j. As before, this choice will lead to the correct branch point in t.

The value of $z(s)$ [and hence of $t(s)$] for which the sum diverges and the manner in which it diverges depend only on the large j form of N_j, since as j becomes infinite D_j goes to one. For N_j of the form (5), we can do the sum using the generating function of the Legendre polynomials, and the curve of singularity is given by

$$0=1+h(s)^2-2zh(s) \tag{6}$$

or

$$t(s)=(1-h^2)(s-4)/4h. \tag{7}$$

If we want a Mandelstam-type cut-plane analyticity, then $t(s)=t_0$ is independent of s. This requires

$$h(s)=[(s-4+t_0)^{1/2}-(t_0)^{1/2}]/[(s-4+t_0)^{1/2}+(t_0)^{1/2}]. \tag{8}$$

This can be rewritten in the more familiar Khuri[4] form

$$h=e^{-\xi}\,,$$
$$\cosh\xi=1+2t_0/(s-4). \tag{9}$$

For the particular case at hand $t_0=4$ and

$$h(s)=(s^{1/2}-2)/(s^{1/2}+2). \tag{10}$$

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[1] R. Kreps, L. F. Cook, J. J. Brehm, and R. Blankenbecler, Phys. Rev. **133**, B1526 (1964). See this paper for further references. Hereafter referred to as I.
[2] J. J. Brehm, Phys. Rev. **135**, B1065 (1964).
[3] All quantities are in units of (pion mass)^2.
[4] N. N. Khuri, Phys. Rev. **130**, 429 (1963).

Note that the form (8) gives the correct threshold behavior in s for any value of t_0, and also introduces a left-hand s cut into N. It is possible to extend the preceding discussion, as in I, to give the higher singularities as well, and in particular to give the Mandelstam spectral boundary.

It is the function $g(j,s)$ which governs the type of singularity at t_0. In particular, if we want a singularity like $(t_0-t)^{1/2}$, we must choose

$$g(j,s)=[2j(h^2-1)-(h^2+3)]/(2j+3)(2j+1)(2j-1). \quad (11)$$

In I, the choice $g(j,s)=1$ was made, which resulted in a $(t_0-t)^{-3/2}$ singularity.

In order to see what to choose for $f(s)$, let us look at the Born term for the choices (10) and (11):

$$M_B(s,t)=\sum_{j=0}^{\infty}(2j+1)P_j(z)N_j(s) \quad (12)$$

$$=[2f(s)(4-t)^{1/2}/(s^{1/2}+2). \quad (13)$$

The simplest choice for $f(s)$ which gives the Born term the correct analyticity and is also complicated enough not to be totally unreasonable is

$$f(s)=-\tfrac{1}{8}(s^{1/2}+2)A\,(1/(s+s_1)-R/(s+s_2)), \quad (14)$$

where A, R, s_1, and s_2 are undetermined constants which are, in general, different for each isospin channel.

We have considered so far only the large j behavior of $N_j(s)$; it is consistent with all the preceding comments to choose any finite number of partial waves to have any form desired, so long as the threshold behavior is kept correct. It is also possible, as in I, to introduce inelastic states and do a multichannel analysis, or to make special requirements between the parameters in order, for example, to insure the existence of a Pomeranchukon or the vanishing of a ghost residue.

Independent of the values of the parameters, any model of the type described above will, after appropriate symmetrization, have the correct threshold behavior in s, will have the nearest singularities in t and u correctly represented, and will satisfy elastic unitarity. There remains now the question of crossing symmetry.

III. CROSSING SYMMETRY

The well-known pion-pion crossing relations are

$$\begin{bmatrix} M^0(t,s) \\ M^1(t,s) \\ M^2(t,s) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} M^0(s,t) \\ M^1(s,t) \\ M^2(s,t) \end{bmatrix}, \quad (15)$$

where the superscript refers to the isospin. If we define

$$\Delta^T(s,t)=M^T(s,t)-M^T(t,s), \quad (16)$$

we may replace (15) by the equivalent set

$$-\tfrac{1}{2}\Delta^0(s,t)=\Delta^1(s,t)=\Delta^2(s,t) =\tfrac{1}{6}[5M^2(s,t)+3M^1(s,t)-2M^0(s,t)]. \quad (17)$$

In the earlier papers[1,2] only the first two of these equalities were used. When the complete equations (17) are used, one is faced with the necessity of varying all the parameters simultaneously, which makes the minimization much more difficult.

Our models will, of course, never satisfy (17) exactly, so we must decide upon a criterion of dissatisfaction. In order to do so, let us consider each quantity appearing in (17) as the coordinates of a vector in a Euclidean four-space. Then if (17) is satisfied exactly, the vector would lie on the ray passing through the origin and the point (1,1,1,1). In order to have a criterion independent of the actual magnitude of the vector, we choose the square of the sine of the angle between it and the ray as our function. Our procedure will be to evaluate this function for a series of values of s and t, and then to minimize the average with respect to the free parameters. Other functions are possible, but they lack the essential simplicity and symmetry between the elements of (17) of this function; and in fact the others tried did not work as well in the minimization procedure.

Where can we evaluate the crossing function? We would prefer to work in the physical s and t regions. Unfortunately, although our amplitude (1) is known for all values of s, it can be continued in t to $t>4$ only with great difficulty, and the convergence for $s<0$ is very slow. The symmetrization in t and u will also require $u\leq 4$, and thus the region where the amplitude can be evaluated relatively easily is the nonphysical region $0\leq s$, $t\leq 4$. Note that because of our choice of the correct singularity strength at $t=4$, we are actually able to go right up to the boundary, as opposed to the procedure in I.

There is, of course, no guarantee that our approximate satisfaction of crossing in the unphysical region means that physical-region crossing is satisfied and that the physical results obtained from this procedure (such as phase shifts) will be accurate. However, the true amplitude must have a partial-wave expansion in the unphysical region with the general properties of the model, and it must satisfy crossing there. We view the minimization as forcing the model into its best approximation of the true amplitude, and hope that the approximation will still be good in that section of the physical region nearest the unphysical region; i.e., at low energies. This hope is not unreasonable if the nearest singularities (which separate the regions) are given correctly, as they are; and if we actually can get a small and sharp minimum of the crossing function, which will depend on the specific models considered.

IV. SPECIFIC MODELS AND RESULTS

For all the models under consideration, we must have some way of doing the integrals involved in D_j. One reason for the pole approximation of (14) is that the integrals can be represented explicitly as hypergeometric functions, and these can easily be continued into the physical region to compute quantities of interest. The relevant integral is

$$I(j,s)=\frac{s-4}{8}\int_4^\infty \frac{dx(x^{1/2}+2)}{(x-s)(x-4)}\left(\frac{x-4}{x}\right)^{1/2}\left(\frac{x^{1/2}-2}{x^{1/2}+2}\right)^j \quad (18)$$

$$=[y(1+y)/(2j+1)(y+1/2)] \times\{(1+y)F(1,1,j+\tfrac{3}{2},1+y) +yF(1,1,j+\tfrac{3}{2},-y)\}, \quad (19)$$

where

$$y=\tfrac{1}{4}(s^{1/2}-2). \quad (20)$$

For $s>4$, the real part of I is given by

$$\mathrm{Re}(I)=[y/(y+\tfrac{1}{2})]\{[(1+y)/(2j-1)] \times F(1,\tfrac{1}{2}-j,\tfrac{3}{2}-j,y/(1+y))[y/(2j+1)] \times F(1,\tfrac{1}{2}+j,\tfrac{3}{2}+j,y/(1+y))\}. \quad (21)$$

For esthetic reasons, for reasons of credibility, and for reasons of machine capability our guiding principle is to use the simplest model possible. Thus, for all the models we shall keep the pole positions the same in all three isospin channels and vary only the coupling constants A^T and R^T independently. Let us consider the models in order of increasing complexity.

A. Single-Pole Model

This corresponds to the choice $R^T=0$. If we keep the choice (11) for all j, we have

$$D_j(s)=1-\frac{A}{\pi(s+s_1)}\left\{\frac{I(j+2,-0_1)-I(j+2,s)}{2j+3}-\frac{I(j_1-s_1)-I(j,s)}{2j-1}\right\}. \quad (22)$$

The difficulty with this model becomes apparent if we ask for the width of a $T=j=1$ resonance at $s=29$, as a function of the pole position. For $s_1<220$, no resonance at $s=29$ is possible. At $s_1=220$, the width is 2.3×10^5 MeV, many orders of magnitude too large. As we increase s_1, the width gradually decreases, but never comes within several orders of magnitude of the experimental value, even for $s_1=10^4$.

If in spite of this fact we use this model with the incomplete crossing relations used in I, we discover that there is an excellent fit for small values of the coupling constants along the line $-\frac{1}{2}A^0=A^1=A^2$, and nowhere else. In particular, for $s_1=4000$ and $A_1=10$ we get a value of the crossing function 30 times better than that in I. One can view this result as indicating that a correct treatment of the singularities is of great assistance in satisfying the crossing relations, and that the use of the full crossing relations is necessary to obtain something physically sensible.

When we try to satisfy the complete crossing relations with this model, we find that there is no acceptable minimum. This is in accord with our expectation that one needs at least a short-range attractive pole and a long-range repulsive pole in order to get a decent resonance.

B. Two-Pole Model

Once we are forced into the form (14) for f, the simplest form to take for g is again (11) for all j, making no special treatment of the low partial waves. Besides simplicity, this choice has the advantage of explicit continuability in j, which in this case leads to both the possibility of computing Regge trajectories explicitly, and the possibility of doing a Sommerfeld-Watson transformation to show the existence of a Mandelstam representation for the amplitude. The unpleasant feature of this model can be phrased in a number of ways: (1) Although the amplitude itself has no pole at $j=\frac{1}{2}$, both N and D do. (2) The Regge trajectories must go to infinity near $j=\frac{1}{2}$. (3) The $j=0$ phase shifts will almost surely have a different sign from the higher partial waves. This type of result was also found in I, and although it is perhaps not totally impossible, it certainly is in disagreement with most of the current thinking on the subject.[5]

With two poles it is possible to obtain an exact ρ meson for most values of the pole positions. For $s_1=100$ and $s_2=40$, the corresponding values are $A^1=387$ and $R^1=0.533$. We shall use these numbers as a guide to the orders of magnitude for physically reasonable values when we look for a minimum of the crossing function.

The function itself deserves slightly more comment at this point. For randomly distributed points in four-space, its value is 0.75. For points where $s=t$, three of the four quantities in (17) are exactly zero, so its value at these points is also 0.75. Thus, for our grid of the twenty-five points s, $t=0, 1, 2, 3, 4$ the function cannot be less than 0.15. The average angle for points not on the symmetry line $s=t$ is given by

$$\sin\theta=[5(f-0.15)]^{1/2}, \quad (23)$$

where f is the value of the crossing function. We are looking for a minimum of the order of $f=0.2$.

During the first part of the calculation, the poles were chosen at $s_1=100$ and $s_2=40$, and a minimization was done on the other six parameters. It was found that in general f had a rather large value—between 0.5 and 0.9. However, there were several broad, deep minima near zero coupling. For example, $f=0.158$ ($\theta=6°$) at

$$A^0=-91.3,\quad R^0=0.414,$$
$$A^1=\ 57.3,\quad R^1=0.348, \quad (24)$$
$$A^2=\ 150,\quad R^2=0.402.$$

[5] See the comments at the end of I.

It was also found that away from this small-coupling region there was only one deep minimum, which was also very sharp. It occurred at

$$\begin{aligned} A^0&=-206.8, & R^0&=0.4437, \\ A^1&=\ 327.2, & R^1&=0.3755, \\ A^2&=\ 22.12, & R^2&=1.426, \end{aligned} \tag{25}$$

at which point $f=0.1911$. A 10% change in any of the $T=0$ or $T=1$ parameters put f up to 0.4 or more. We identify this minimum as the "physical solution" on the basis of the depth and sharpness of the minimum; and on the fact that since the ρ exists, we must have A^1 large enough to give a resonance. For these particular values, both the $T=j=0$ and $T=j=1$ partial waves have zeros of the D functions below threshold: the former at $s=-12.1$, the latter at $s=-23.6$. Since a variation of the pole positions was yet to come, the presence of these ghosts was not exceptionally alarming.

The preceding was based on a series of minimizations starting from points on a grid in parameter space. We also attempted to follow an analog of the procedure in I: We started from the exact values of the ρ, minimized on the other parameters, and then freed the isospin-one parameters. This procedure proved fruitless.

We next varied the pole positions and followed the physical solution as it moved under the minimization of the parameters. The change in f as s_1 and s_2 varied was very small, but there was a definite tendency for s_2 to increase. Unfortunately, as we moved in the s_1-s_2 plane the valley we were following became very broad and flat, so the pole localization was very poor: values of s_2 differing by ±15 gave essentially the same f. At the same time, A^0 and A^1 increased more or less proportionally to s_2, so they could not be determined to better than ±75 for A^0 and ±150 for A^1 (although for any given values of s_1 and s_2, A^0 and A^2 were determined to about 2%). A typical set of values is $s_1=110.7$, $s_2=80.41$, $f=0.1770$, and

$$\begin{aligned} A^0&=-511.9, & R^0&=\ 0.7447, \\ A^1&=\ 804.5, & R^1&=\ 0.7152, \\ A^2&=\ 0.857, & R^2&=33.73. \end{aligned} \tag{26}$$

All the values had the $j=0$ and $j=1$ ghosts previously mentioned.

At the same time, another solution appeared, which seemed to be the extension of the previous small-scattering solutions, although we were unable to obtain it from them. It was characterized by a $\sin\theta$ about $\frac{2}{3}$ of the above values, by a large A^0, and by a moderate A^1. It also had an actual minimum in s_1 and s_2. The minimum was again somewhat broad, so the localization of the pole positions was not very satisfactory, but it centered about the point $s_1=109.8$ and $s_2=78.7$. At these pole positions (and within ±5 of them) $f=0.1617$ and

$$\begin{aligned} A^0&=-522.9, & R^0&=0.7358, \\ A^1&=\ 350.0, & R^1&=0.7769, \\ A^2&=\ 5.907, & R^2&=3.795. \end{aligned} \tag{27}$$

The $T=j=0$ has a ghost at $s=-29.5$, but the $T=j=1$ D-function does not vanish; its minimum is about 0.60 at $s=50$. There is almost no $T=2$ scattering. This solution is unsatisfactory for two reasons—the ghost in $T=0$ and the lack of a resonance in $T=1$. One might hope that curing the former would help the latter.

C. Ghost-Elimination Model

As mentioned previously, we can constrain A and R to make the N function vanish at the position of a ghost. If the ghost is at $s=-s_g$, then

$$R^0(s_g)=(s_2-s_g)/(s_1-s_g) \tag{28}$$

and $A^0(s_g)$ is determined by the equation

$$D_0(-s_g)=0. \tag{29}$$

Unfortunately, this model has nothing corresponding to the physical solution of the previous model; for $s_1=100$ and $s_2=40$ all values of f exceed 0.4. This can be understood in terms of the sharpness of the minimum of the model B in the parameters A^0 and R^0. The curve $A^0(s_g)$ versus $R^0(s_g)$ just does not come close enough to the physical-solution points to drop into the minimum.

Thus, we must try some other way to ameliorate the original model. The most obvious thing to do is to modify the low partial waves.

D. Modified S-Wave Model

If we are willing to give up the good behavior in j by changing the s wave, we can eliminate the awkward sign change discussed in Sec. IIIB. This can be done, for example, by adding to $g(j,s)$ a term of the form $C[\sin(j)/j]$. In order to keep the number of new parameters at a minimum, we introduce the form

$$g(0,s)=(h^2/3)-Q, \tag{30}$$

with $g(j,s)$ as before for $j\neq0$. $Q=-1$ is the previous model, and now we hope to have $Q\sim1$, in order to get a reasonable relation between the s- and d-wave phase shifts.

If we start from the solution (25) and let Q vary (keeping the poles fixed), we end up with a minimum at $Q=-1.644$, $f=0.1774$, and

$$\begin{aligned} A^0&=-128.8, & R^0&=0.4377, \\ A^1&=\ 308.1, & R^1&=0.3831, \\ A^2&=\ 2.310, & R^2&=6.327. \end{aligned} \tag{31}$$

However, we are interested in $Q\sim1$, and it turns out that the way to get to this region is to continue Q

through infinity. There is, in fact, a local minimum at $Q=0.9249$, $f=0.2053$, and

$$A^0=302.9\,,\quad R^0=0.4245\,,$$
$$A^1=327.9\,,\quad R^1=0.4403\,,\qquad (32)$$
$$A^2=-7.869\,,\quad R^2=5.059\,.$$

The $T=j=0$ again has a zero at $s=-1.026$. The $T=j=1$ D-function has a zero at $s=5.772$, but the N function is negative there, so the "width" has the wrong sign. It is, however, small: -0.065.

When the pole positions are varied, the same sorts of phenomena as observed in Sec. IIIB occur. The function decreases only slightly as s_1 and s_2 vary; s_2 and the coupling constants tend to increase; and the minimum is very broad with consequent poor pole localization. In fact, no true minimum was ever found. There was, however, a very local minimum at $s_1=105.2$ and $s_2=69.45$ for which $f=0.1953$ and $Q=1.368$. The other parameters were

$$A^0=382.0\,,\quad R^0=\ 0.6753\,,$$
$$A^1=643.4\,,\quad R^1=\ 0.6828\,,\qquad (33)$$
$$A^2=-3.169\,,\quad R^2=10.66\,.$$

Again, $T=j=0$ had a zero at $s=-1.33$, and $T=j=1$ had a zero at $s=5.99$ with a "width" of -0.060.

IV. DISCUSSION

The objection to all the above models is obvious: they give extremely nonphysical results, when they give results at all. The encouraging feature is that it is possible to find a minimum in the coupling constants which is sharp, deep, and strong, and which can thus be identified as the "physical solution."

Two features seem to be generally true. First, the $T=2$ state seems to have very little scattering, while the $T=0$ and $T=1$ states show strong scattering. Second, the variation of the pole positions seems to have little effect on the value of the crossing function, or on the general nature of the physical solution. This might be partially expected on the basis that for large values of s_1 and s_2, the essential dependence of the N function is on the ratios A/s_1 and AR/s_2 and not on the individual parameters.

There are three types of modification which could be used to improve the calculation. First, one could modify the low partial waves more drastically. This has the double disadvantage that not only will one probably have to introduce even more parameters, but also if one changes the s dependence of the N functions, one has to make explicit guesses with little ***a priori*** information. Second, one could introduce inelastic states and do a multichannel N/D calculation. Again, this introduces more parameters, and it is not clear that the presence of inelastic state is not reasonably well represented by simply having two (or more) poles in N. Third, one could go away from the pole approximation altogether, and introduce some other function such as a Q_j, as Brehm[2] has done. One feels, however, that a small number of poles should be a reasonably good approximation for most such functions.

In sum, although the general technique seems to hold promise, it is difficult to envision exactly what must be done to make it deliver.

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