Lee Model with an Unstable V-Type Particle*†

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The solution is given to a modified Lee model in which there is a resonance having the quantum numbers of the usual V particle. It is demonstrated that all channels having the same quantum numbers are coupled by the integral equations; in particular, "unphysical" amplitudes involving unstable external particles are coupled to the physical amplitudes through solutions to homogeneous integral equations of the Omnes-Muskhelishvili type.

I. INTRODUCTION

THE purpose of this paper is to present and discuss the solution of a simple model in which an unstable baryon is present. We consider the Lee model,[1] modified by adding to the usual V, N, and θ particles an unstable particle, N^*, having the quantum numbers of a V particle; this N^* corresponds roughly to the (3,3) resonance in pion-nucleon scattering. From this case, one may readily generalize to the solution in the presence of several unstable baryons.

Such a model is of some interest for at least two reasons. One is that its solution may be used to test and illuminate various predictions and approximations made in less tractable models which, however, purport to be more closely descriptive of the real world. Thus the present model, being one in which both elastic scattering amplitudes and production amplitudes may be calculated exactly, proves to be a useful model for studying such things as the predictions of the isobar model of the higher resonances in pion-nucleon scattering,[2] and for investigating the validity of the resonance approximation used by various authors to simplify three-particle states.[3]

A second interest is in the actual solution of the equations of the model themselves. It is with this aspect that the present paper is concerned. The solution is obtained in two ways: one, by calculating the solution in the presence of a stable N^* and then analytically continuing in the mass of the N^* to an unstable value; the other, by calculating directly with N^* unstable. It is interesting that there are apparently major differences in the dispersion relation calculations between the stable case and unstable case as a result of the fact that in the latter case, states containing N^* do not, of course, appear in complete sets of states. The solutions, however, are closely similar. The way this comes about is that when N^* is stable amplitudes involving it occur explicitly in inhomogeneous terms of the Omnes-Muskhelishvili integral equations,[4,5] whereas when N^* is unstable these amplitudes turn up in previously inadmissible solutions to the homogeneous integral equations. Thus, stable and unstable particles in this model appear on very much the same footing; the integral equations couple not only the "physical" channels containing stable particles but also the unstable particle channels.

In the following, Sec. II recalls some of the properties of the Lee model and indicates precisely the modifications. In Sec. III, the solution is given for the $N\theta$ and $V\theta$ sectors when N^* is stable. Since the Lee model has been much discussed in the literature, many calculational details have been omitted; most of these details may be found in the references, particularly Refs. 6 and 7. Section IV describes in some detail the analytic continuation in the mass of the N^* of the $V\theta$ elastic amplitude; and Sec. V considers the nature of the equations and solutions when we attempt to calculate the $V\theta$ elastic amplitude directly with N^* unstable.

II. THE MODEL

The Lee model customarily describes a world containing three types of particles: two heavy, spinless fermions, N and V, and one light spinless boson, the θ. There are no antiparticles; further, N and V are static while θ is treated relativistically. The basic interaction is of the form $V\rightleftharpoons N\theta$. The present modification involves the addition of a particle N^* identical to V except in mass; we have thus the additional basic interaction $N^*\rightleftharpoons N\theta$.

The model may be defined more precisely by specifying the following currents:

$$\frac{(2\omega)^{1/2}(2\pi)^{3/2}}{u(\omega)}\left(-i\frac{d}{dt}+\omega\right)a_k(t)=j(t)\,, \qquad (1)$$

$$\left(-i\frac{d}{dt}+M_A\right)\psi_A(t)=f_A(t)\,, \qquad (2)$$

$$A=V \text{ or } N^*\,,$$

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[1] T. D. Lee, Phys. Rev. 95, 1329 (1954).

[2] R. F. Peierls, Phys. Rev. Letters 6, 641 (1961).

[3] See, for example, P. G. Federbush, M. T. Grisaru, and M. Tausner, Ann. Phys. (N. Y.) 18, 23 (1962).

[4] R. Omnes, Nuovo Cimento 8, 316 (1958).

[5] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff, Ltd., Groningen, The Netherlands, 1953).

[6] R. D. Amado, Phys. Rev. 122, 696 (1961).

[7] M. L. Goldberger and S. B. Treiman, Phys. Rev. 113, 1663 (1959).

with the properties:

$$[j,a_k^*]=[j,a_k]=\{j,\psi_A\}=0;\quad j\equiv j(0),\text{ etc.}\quad (3a)$$

$$[f_A,a_k^*]=C_A\psi_N u(\omega)/(2\omega)^{1/2}(2\pi)^{3/2},\quad (3b)$$

$$\langle 0|j=0,\quad (3c)$$

where $a_k(t)$, $\psi_N(t)$, $\psi_*(t)$, $\psi_V(t)$ are the Heisenberg field operators (the last two being renormalized) for the θ, N, N^*, and V particles. The static baryons have energies equal to their masses while for the θ we have $\omega=(k^2+M_\theta^2)^{1/2}$. $u(\omega)$ is a form factor chosen to remove ghosts and to make all integrals converge. The C_A are related to renormalization constants; they do not appear in any physical scattering amplitude. The properties (3) are satisfied, for example, by the usual Lee model trilinear interaction described by the Hamiltonian:

$$H=Z_V\psi_V^*\psi_V M_V+Z_*\psi_*^*\psi_* M_*+\psi_N^*\psi_N M_N$$
$$+\int d^3k\; a_k^*a_k\omega+[g_V\psi_V^*+g_*\psi_*^*]\frac{1}{(2\pi)^{3/2}}$$
$$\times\int\frac{d^3k u(\omega)a_k\psi_N}{(2\omega)^{1/2}}+\text{H.c.}$$
$$+Z_V\psi_V^*\psi_V\delta M_V+Z_*\psi_*^*\psi_*\delta M_*,\quad (4)$$

where Z_V, Z_* are renormalization constants; g_V, g_* are renormalized coupling constants; δM_V, δM_* are mass renormalization counterterms; M_V, M_* are physical masses.

III. SOLUTION FOR STABLE N^*

$N\theta$ sector. The solution for the $N\theta$ sector has been given elsewhere[8] and we simply quote the result. The $N\theta$ elastic scattering amplitude, $T_{N\theta}(\omega)$, defined by

$$\langle N\theta_{k'}^{\text{out}}|N\theta_k^{\text{in}}\rangle=\delta(k-k')$$
$$+2\pi i\delta(\omega-\omega')[u^2(\omega)T_{N\theta}(\omega)/(2\omega)(2\pi)^3]\quad (5)$$

has the form

$$T_{N\theta}(\omega)=-G^2/\omega/[(1-\beta(\omega))]\quad (6)$$

where

$$1-\beta(\omega)=1+\frac{G^2\omega}{(2\pi)^2}\int_1^\infty\frac{d\omega'(\omega'^2-1)^{1/2}u^2(\omega')}{\omega'^2(\omega'-\omega-i\epsilon)}-\frac{A_0\omega}{\omega-\omega_0}.\quad (7)$$

(For convenience, we have chosen $M_V=M_N$ and $M_\theta=1$.)

The amplitude has a pole at $\omega=0$ with residue $-G^2$ and a branch point at $\omega=1$ as in the usual Lee model. In addition, $1-\beta(\omega)$ has a pole at $\omega=\omega_0$ and a zero at $\omega=\Delta^*$ as in Fig. 1. Consequently, $T_{N\theta}(\omega)$ has a pole at $\omega=\Delta^*$ whose position and residue may be adjusted by varying the (positive) parameters A_0 and ω_0. This pole we interpret to represent the N^* particle so that $\Delta^*=M_*-M_N$ and the $N^*N\theta$ coupling constant is given by

$$\Gamma^2=\lim_{\omega\to\Delta^*}(\Delta^*-\omega)T_{N\theta}(\omega).\quad (8)$$

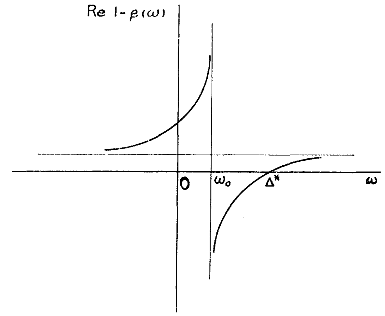

FIG. 1. Energy dependence of the real part of the function $1-\beta(\omega)$.

In the following, it is occasionally more convenient to exhibit the pole at Δ^* explicitly. We define a new function which has neither poles nor zeros:

$$1-\bar\beta(\omega)=[1-\beta(\omega)](\omega-\omega_0)\Delta^*/(\omega-\Delta^*)\omega_0\quad (9)$$

and write

$$T_{N\theta}(\omega)=-[G^2/\omega+B/(\omega-\Delta^*)]/[1-\bar\beta(\omega)],\quad (10)$$

where B, determined by the condition that $T_{N\theta}(\omega_0)=0$, is given by

$$B=G^2(\Delta^*-\omega_0)/\omega_0.\quad (11)$$

Further, note that the $N^*N\theta$ coupling may be written

$$\Gamma^2=B/[1-\bar\beta(\Delta^*)].\quad (12)$$

$V\theta$ sector. For the solution to the $V\theta$ sector, let us first define the following amplitudes:

$$T(\omega)=\langle V|j|V\theta_k^{\text{in}}\rangle(2\pi)^{3/2}(2\omega)^{1/2}/u(\omega),\quad (13a)$$

$$M(\omega)=\langle N^*|j|V\theta_k^{\text{in}}\rangle(2\pi)^{3/2}(2\omega)^{1/2}/u(\omega),\quad (13b)$$

$$K(\omega)=\langle 0|f_V|N\theta_k^{\text{in}}\rangle(2\pi)^{3/2}(2\omega)^{1/2}/u(\omega),\quad (13c)$$

$$L(\omega)=\langle 0|f_*|N\theta_k^{\text{in}}\rangle(2\pi)^{3/2}(2\omega)^{1/2}/u(\omega),\quad (13d)$$

$$F(\omega',\omega)=\langle N\theta_{k'}^{\text{in}}|j|V\theta_k^{\text{in}}\rangle(2\pi)^3$$
$$\times(2\omega 2\omega')^{1/2}/u(\omega)u(\omega').\quad (13e)$$

Then, following Amado,[6] we contract the V and N^* from the left in T and M, respectively, and obtain the coupled equations:

$$T(\omega)=\frac{\langle 0|f_V|N^*\rangle M(\omega)}{\Delta^*}+\frac{1}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')K(\omega')F(\omega',\omega)}{2\omega'^2},\quad (14)$$

$$M(\omega)=-\frac{\langle 0|f_*|V\rangle T(\omega)}{\Delta^*}+\frac{1}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')L(\omega')F(\omega',\omega)}{2\omega'(\omega'-\Delta^*)}.\quad (15)$$

[8] P. K. Srivastava, Phys. Rev. 128, 2906 (1962).

In the above, the numbers $\langle 0|f_V|N^*\rangle$ and $\langle 0|f_*|V\rangle$ do not turn up in any physical scattering amplitudes so they are not considered further. Thus a knowledge of K, L, and F determines the physical scattering amplitudes T and M.

The vertex functions $K(\omega)$, $L(\omega)$ satisfy the following integral equations, obtained from Eqs. (13c), (13d) by contracting the θ particles:

$$K(\omega)=C_V+\frac{\Gamma\langle 0|f_V|N^*\rangle}{\Delta^*-\omega}+\frac{1}{(2\pi)^3}\int\frac{d^3k'K(\omega')T_{N\theta}^*(\omega')u^2(\omega')}{2\omega'(\omega'-\omega-i\epsilon)} \tag{16}$$

and a similar one for $L(\omega)$. These are standard Omnes-Muskhelishvili equations whose solutions are[4,7]

$$K(\omega)=\frac{G+\langle 0|f_V|N^*\rangle\Gamma\omega[1-\bar\beta(\Delta^*)]/\Delta^*(\Delta^*-\omega)}{[1-\bar\beta(\omega)]}, \tag{17}$$

$$L(\omega)=\frac{B/\Gamma+\langle 0|f_*|V\rangle G(\omega-\Delta^*)/\omega\Delta^*}{[1-\bar\beta(\omega)]}. \tag{18}$$

Note that, as will be discussed more fully later, these are solutions to the inhomogeneous equations; when N^* is stable, there is no admissible solution to the homogeneous equations.

Finally, $F(\omega',\omega)$ which is closely related to the production amplitude for the process $V\theta\to N\theta\theta$ satisfies the following equation:

$$F(\omega',\omega)=\frac{2\omega(2\pi)^3G\delta(k-k')}{u^2(\omega)}-GT(\omega)\left(\frac{1}{\omega'}-\frac{1}{\omega'-\omega-i\epsilon}\right)-\Gamma M(\omega)[1/(\omega'-\Delta^*)-1/(\omega'+\Delta^*-\omega-i\epsilon)]$$
$$+\frac{1}{(2\pi)^3}\int\frac{d^3k_1u^2(\omega_1)T_{N\theta}(\omega_1)F(\omega_1,\omega)}{2\omega_1}[1/(\omega_1-\omega'+i\epsilon)+1/(\omega_1-\omega+\omega'-i\epsilon)]. \tag{19}$$

This equation and its solution, which follows, are obtained in a straightforward way by the methods of Ref. 6[9]:

$$F(\omega',\omega)=\frac{2\omega G(2\pi)^3}{u^2(\omega)}\delta(k-k')+\frac{4\pi G\omega\,\mathrm{Im}[1-\bar\beta(\omega)]}{u^2(\omega)(\omega^2-1)^{1/2}\omega'(\omega'-\omega-i\epsilon)[1-\bar\beta^*(\omega')][1-\bar\beta(\omega-\omega')]}$$
$$+\frac{G\omega T(\omega)[1-\bar\beta(\omega)]}{\omega'(\omega'-\omega-i\epsilon)[1-\bar\beta^*(\omega')][1-\bar\beta(\omega-\omega')]}+\frac{\Gamma M(\omega)(\omega-2\Delta^*)[1-\bar\beta(\Delta^*)][1-\bar\beta(\omega-\Delta^*)]}{(\omega'+\Delta^*-\omega)(\omega'-\Delta^*)[1-\bar\beta^*(\omega')][1-\bar\beta(\omega-\omega')]}. \tag{20}$$

The homogeneous equation for $F(\omega',\omega)$ has a solution of the form[6,10]

$$F_0(\omega',\omega)=\frac{P(\omega',\omega)}{[1-\bar\beta\;(\omega')][1-\bar\beta(\omega-\omega')](\omega'-1)^n(\omega-\omega'-1)^n}, \tag{21}$$

where P is a polynomial in ω' and $n=0$ or 1. [For $n>1$, the integral of Eq. (19) diverges at the lower limit.] P and n may be determined only by the boundary conditions at the two singular points, $\omega'=1$ and $\omega'=\infty$.[4] If we require F to be finite at $\omega'=1$, we must have $n=0$. Then, in order for the integral equations (14), (15), (19) to be well defined, $F(\omega',\omega)\xrightarrow[\omega'\to\infty]{}1/\omega'^2$ so we have $P=0$.

Combining the results (17), (18), (20) with Eqs. (14), (15), we see that we have a pair of coupled algebraic equations for $T(\omega)$, $M(\omega)$ which may be solved for these two amplitudes. Expressed in terms of four integrals which must be evaluated numerically, the amplitude of main interest $T(\omega)$ may be written

$$T(\omega)=N/D,$$

where

$$N=G^2/[\omega(1-\bar\beta(\omega))]+[G^2/\omega+B/(\omega-\Delta^*)]\omega I_1+(\omega-2\Delta^*)[1-\bar\beta(\omega-\Delta^*)][1-\bar\beta(\Delta^*)]$$
$$\times\left[\frac{BI_3/(\omega-\Delta^*)-G^2I_4/\omega}{1-\bar\beta(\omega)}+\omega[G^2/\omega+B/(\omega-\Delta^*)](I_2I_3-I_1I_4)\right], \tag{22a}$$

$$D=1-\omega[1-\bar\beta(\omega)]I_1+(\omega-2\Delta^*)[1-\bar\beta(\omega-\Delta^*)][1-\bar\beta(\Delta^*)]\{\omega[1-\bar\beta(\omega)](I_1I_4-I_2I_3)-I_4\}. \tag{22b}$$

[9] Note that $F(\omega',\omega)-[2\omega(2\pi)^3G\delta(k-k')/u^2(\omega)]$, as given by Eq. (19), is symmetrical under the interchange $\omega'\to\omega-\omega'$ so that the solution must exhibit this same symmetry.

[10] The criterion for an acceptable solution to the homogeneous equation, apart from the crossing requirement of footnote (9), is that it consist of the product $1/\{[1-\beta^*(\omega')][1-\beta(\omega-\omega')]\}$, which has the correct cuts, and a function which is analytic everywhere in the ω' plane with the possible exception of poles at the singular points $\omega-1$, 1, ∞.

The integrals are given by

$$I_1=\frac{G^2}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')}{2\omega'^3(\omega'-\omega-i\epsilon)|1-\bar{\beta}(\omega')|^2[1-\bar{\beta}(\omega-\omega')]}, \tag{23a}$$

$$I_2=\frac{B}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')}{2\omega'^2(\omega'-\Delta^*)(\omega'-\omega-i\epsilon)|1-\bar{\beta}(\omega')|^2[1-\bar{\beta}(\omega-\omega')]}, \tag{23b}$$

$$I_3=\frac{G^2}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')}{2\omega'^2(\omega'-\Delta^*)(\omega'-\omega+\Delta^*-i\epsilon)|1-\bar{\beta}(\omega')|^2[1-\bar{\beta}(\omega-\omega')]}, \tag{23c}$$

$$I_4=\frac{B}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')}{2\omega'(\omega'-\Delta^*)^2(\omega'-\omega+\Delta^*-i\epsilon)|1-\bar{\beta}(\omega')|^2[1-\bar{\beta}(\omega-\omega')]}. \tag{23d}$$

The expected analytic properties of $T(\omega)$, Fig. 2, are easily verified by the above. In particular, there is a pole at $\omega=0$ with residue G^2 [but no pole at $\omega=\Delta^*$, since $I_2(\Delta^*)=I_3(\Delta^*)$]. The function $1-\bar{\beta}(\omega)$ and the integrals I_1, I_2 contribute to the branch point at $\omega=1$ which is the threshold for elastic scattering while the function $1-\beta(\omega-\Delta^*)$ and the integrals I_3, I_4 contribute to the branch point at $\omega=1+\Delta^*$, the threshold for the reaction $V\theta\rightarrow N^*\theta$. Finally, each of the integrals has a branch point at $\omega=2$, the threshold for the process $V\theta\rightarrow N\theta\theta$. The singularity occurs when the branch point of $1-\bar{\beta}(\omega-\omega')$ at $\omega'=\omega-1$ moves up to the end point of the integration contour at $\omega'=1$.

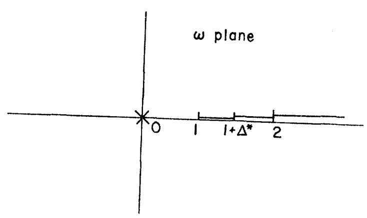

FIG. 2. Singularities of the $V\theta$ scattering amplitude, $T(\omega)$.

IV. ANALYTIC CONTINUATION

To obtain the $V\theta$ elastic scattering amplitude in the presence of an unstable N^*, we analytically continue $T(\omega)$ in the mass parameter, Δ^*. Increasing the real parameter A_0, for example,[11] will move the zero in $1-\beta(\omega)$ at Δ^* along the real axis to threshold at $\omega=1$ [where it will coalesce with a zero of $1-\beta(\omega)$ on the second sheet]. Further increase in A_0 will then cause one zero to move down onto the second sheet where it will produce a resonance in the $N\theta$ scattering. Correspondingly, in the $V\theta$ amplitude, the $N^*\theta$ threshold will move up to the $N\theta\theta$ threshold and thence down into an unphysical sheet.

In carrying out this continuation, it is convenient to use Eq. (9) to replace $1-\bar{\beta}(\omega)$ in the integrals by $1-\beta(\omega)$ since the latter remains a real analytic function when N^* goes unstable. When this is done, all the functions in Eq. (22) retain their same form with the exception of the integral, I_4. It may be written

$$I_4=\frac{B}{G^2}\frac{\omega_0^3}{\Delta^{*3}}\frac{1}{2\pi i}\int_{C_1}\frac{d\omega'\omega'}{(\omega'-\omega_0)^2(\omega-\omega'-\omega_0)[1-\beta(\omega')][1-\beta(\omega-\omega')]}. \tag{24}$$

The singularities of the integrand and the contour C_1 are indicated in Fig. 3(a).

The main point of interest is that this integrand, unlike the other three, has a pole at $\omega'=\Delta^*$. Consequently, when Δ^* moves past threshold onto the second sheet it drags the contour with it [Fig. 3(b)]. The continued

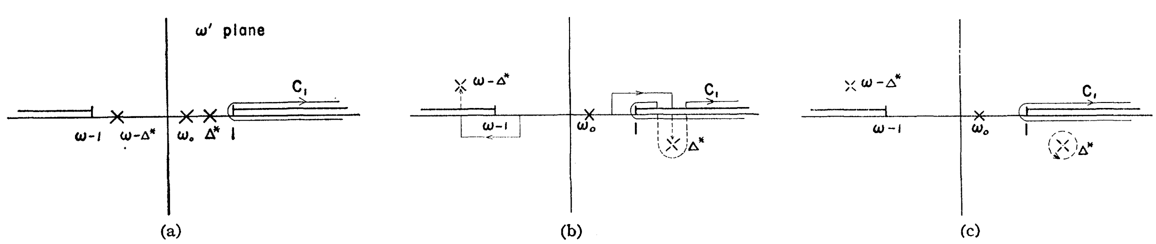

FIG. 3. Distortion of the contour of the integral I_4 that results from the motion of the pole in the integrand onto the second sheet.

[11] In Fig. 1, note that

$$1-\beta(\pm\infty)=1-\frac{G^2}{(2\pi)^2}\int_1^\infty\frac{d\omega'(\omega'^2-1)^{1/2}u^2(\omega')}{\omega'^2}-\frac{A_0}{\omega_0}.$$

I_4 thus consists of two terms, corresponding to the two contours of Fig. 3(c). One, I_4', has the form of the original I_4 with, however, the pole at $\omega'=\Delta^*$ on the second sheet; the other is simply the residue resulting from integrating around the pole. Thus we write

$$I_4\rightarrow I_4{}^C=I_4'+\Delta I_4,$$

where

$$\Delta I_4=\{[1-\bar{\beta}(\omega-\Delta^*)][1-\bar{\beta}^{\rm II}(\Delta^*)](\omega-2\Delta^*)\}^{-1}. \tag{25}$$

$1-\beta^{\rm II}(\Delta^*)$ is the continuation in Δ^* of $1-\bar{\beta}(\Delta^*)$; it is not calculated since it cancels from the amplitude. Returning to Eq. (22), we see that the first two terms of both N and D are exactly cancelled by the new ΔI_4 terms; further, both N and D contain the factor $(\omega-2\Delta^*)[1-\bar{\beta}(\omega-\Delta^*)][1-\bar{\beta}^{\rm II}(\Delta^*)]$ which may therefore be cancelled throughout. Finally, a few algebraic manipulations result in the cancellation of all explicit Δ^*s, so that we obtain the following form for $T(\omega)$:

$$T(\omega)=N'/D',$$

where

$$N'=\frac{G^2\omega_0(\omega I_7-I_5)}{\omega(\omega-\omega_0)[1-\beta(\omega)]}+G^4(\omega-\omega_0)\omega_0{}^2(I_5I_6-I_7I_8), \tag{26a}$$

$$D'=I_5-G^2\omega\omega_0{}^2(\omega-\omega_0)[1-\beta(\omega)](I_5I_6-I_7I_8). \tag{26b}$$

The integrals are given by

$$I_5=\frac{1}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')}{2\omega'(\omega'-\omega_0)^2(\omega-\omega'-\omega_0)|1-\beta(\omega')|^2[1-\beta(\omega-\omega')]}, \tag{27a}$$

$$I_6=\frac{1}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')}{2\omega'^3(\omega'-\omega-i\epsilon)(\omega'-\omega_0)^2(\omega-\omega'-\omega_0)|1-\beta(\omega')|^2[1-\beta(\omega-\omega')]}, \tag{27b}$$

$$I_7=\frac{1}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')}{2\omega'^2(\omega'-\omega_0)^2(\omega-\omega'-\omega_0)|1-\beta(\omega')|^2[1-\beta(\omega-\omega')]}, \tag{27c}$$

$$I_8=\frac{1}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')}{2\omega'^2(\omega'-\omega-i\epsilon)(\omega'-\omega_0)^2(\omega-\omega'-\omega_0)|1-\beta(\omega')|^2[1-\beta(\omega-\omega')]}. \tag{27d}$$

In this form, $T(\omega)$ is manifestly a real analytic function in the plane cut from 1 to ∞ except for the pole at $\omega=0$. There is also the production threshold at $\omega=2$ arising from pinches in the integrands as before when N^* was stable. Finally, one may readily check that the former $V\theta\rightarrow N^*\theta$ threshold lies below the physical region on the sheet reached by passing through the production threshold.

V. DIRECT CALCULATION

It is interesting to try to calculate directly the amplitude for $V\theta$ elastic scattering in the presence of an unstable N^* without going through the stable case calculation and analytically continuing in the N^* mass as we have done. Note that since N^* is unstable, $|V\rangle$ and $|N\theta\rangle$ or $|V\theta\rangle$ and $|N\theta\theta\rangle$ now constitute complete sets of states in their respective sectors; therefore amplitudes involving N^* explicitly no longer appear in the equations and we might expect solutions to these equations to have different forms from those previously obtained.

The equations of main interest are those for $K(\omega)$ and $F(\omega',\omega)$ since, when these two functions are known, we may calculate the $V\theta$ elastic amplitude from

$$T(\omega)=\frac{1}{(2\pi)^3}\int\frac{d^3k'u^2(\omega')K(\omega')F(\omega',\omega)}{2\omega'^2} \tag{28}$$

[which corresponds to Eq. (14)].

Consider $F(\omega',\omega)$ first. It satisfies an equation identical to (19) except that the inhomogeneous term containing $M(\omega)$ is now missing. One may readily check that a solution is given by Eq. (20) except that the term containing $M(\omega)$ is again dropped. However, there now is a solution of the homogeneous equation of the form

$$F_0(\omega',\omega)=A(\omega)/[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0), \tag{29}$$

where $A(\omega)$ is, at this point, a completely arbitrary function. F_0 goes like $1/\omega'^2$ as $\omega'\rightarrow\infty$ and is finite everywhere in the ω' plane as desired. [Note that $1-\beta(\omega')$ has a pole at $\omega'=\omega_0$.] We may check that this is indeed a solution.

Inserting F_0 into the integral of Eq. (19), we have

$$I=\frac{A(\omega)}{2\pi i}\int_{C_1}\frac{d\omega_1}{[1-\beta(\omega_1)][1-\beta(\omega-\omega_1)](\omega_1-\omega_0)(\omega-\omega_1-\omega_0)}((\omega_1-\omega'+i\epsilon)^{-1}+(\omega_1-\omega+\omega'-i\epsilon)^{-1}), \quad (30)$$

where we have made use of the fact that, by Eqs. (6) and (7), $T_{N\theta}(\omega_1)$ may be written

$$T_{N\theta}(\omega_1)=-4\pi\,\mathrm{Im}[1-\beta(\omega_1)]/u^2(\omega_1)(\omega_1^2-1)^{1/2}[1-\beta(\omega_1)]. \quad (31)$$

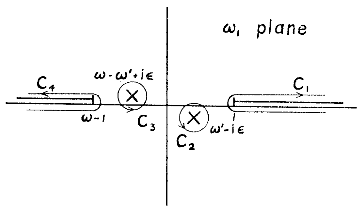

FIG. 4. Singularities in the integrand of I, Eq. (30). The contour C_1 is equivalent to the sum of C_2, C_3, C_4.

The behavior of the integrand at ∞ permits the replacement of the contour C_1 by the contours C_2, C_3, C_4 (Fig. 4).
Thus we write symbolically

$$\int_{C_1}=\int_{C_2}+\int_{C_3}+\int_{C_4}.$$

Then as a result of the symmetry of the integrand under the interchange $\omega_1\rightarrow\omega-\omega_1$, it turns out that $\int_{C_2}=\int_{C_3}$ and $\int_{C_4}=-\int_{C_1}$ so that I is simply $A(\omega)$ multiplied by the residue at $\omega'-i\epsilon$, say. Thus,

$$I=A(\omega)/[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0) \quad (32)$$

which indeed equals $F_0(\omega',\omega)$ itself.

For stable N^*, a function of this type does not satisfy the homogeneous equation since in that case F_0 has poles on the physical sheet at $\omega'=\Delta^*$, $\omega'=\omega-\Delta^*$ whose residues, in a calculation of the type just performed, yield an additional term. And of course if these poles are removed by including the factor $(\omega'-\Delta^*)(\omega-\omega'-\Delta^*)$ in the numerator of F_0, the new function will not have the correct behavior at ∞.

The entire solution in the unstable case may now be written

$$\begin{aligned}F(\omega',\omega)=&\frac{2\omega G(2\pi)^3\delta(k-k')}{u^2(\omega)}\\&+\frac{4\pi G\omega\,\mathrm{Im}[1-\beta(\omega)](\omega'-\Delta^*)(\omega-\omega'-\Delta^*)\omega_0(\omega-\omega_0)}{u^2(\omega)(\omega^2-1)^{1/2}\omega'(\omega'-\omega-i\epsilon)[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0)\Delta^*(\omega-\Delta^*)}\\&+\frac{G\omega T(\omega)[1-\beta(\omega)](\omega'-\Delta^*)(\omega-\omega'-\Delta^*)\omega_0(\omega-\omega_0)}{\omega'(\omega'-\omega-i\epsilon)[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0)\Delta^*(\omega-\Delta^*)}\\&+\frac{A(\omega)}{[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0)}. \quad (33)\end{aligned}$$

The meaning of the arbitrary function, $A(\omega)$, is clear if we compare the last terms of Eqs. (20) and (33) which have the same ω' dependence; evidently $A(\omega)$ may be interpreted up to factors as the analytic continuation in Δ^* of $M(\omega)$ which we recall is the amplitude for the process $V\theta\rightarrow N^*\theta$. Thus the unstable N^* is introduced through this homogeneous solution in the direct calculation. Finally, we recall that in the case of the analytically continued amplitude all explicit Δ^*'s could be eliminated; this may be done again in the direct calculation for the appearance of explicit Δ^*'s is, in fact, spurious. This can be seen simply by splitting off two terms from the last term of $F(\omega',\omega)$ as follows:

$$\begin{aligned}&\frac{A(\omega)}{[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0)}\\&=\frac{A'(\omega)}{[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0)}\\&+\frac{4\pi\omega G\,\mathrm{Im}[1-\beta(\omega)]\omega_0(\omega-\omega_0)}{u^2(\omega)(\omega^2-1)^{1/2}[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0)\Delta^*(\omega-\Delta^*)}\\&+\frac{G\omega T(\omega)[1-\beta(\omega)]\omega_0(\omega-\omega_0)}{[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)\Delta^*(\omega-\Delta^*)(\omega-\omega'-\omega_0)}. \quad (34)\end{aligned}$$

Combining the last two terms of (34) with the second and third terms of (33), we get

$$F(\omega',\omega)=\frac{2\omega G(2\pi)^3\delta(k-k')}{u^2(\omega)}-\frac{4\pi\omega G\,\mathrm{Im}[1-\beta(\omega)]\omega_0(\omega-\omega_0)}{u^2(\omega)(\omega^2-1)^{1/2}\omega'(\omega'-\omega-i\epsilon)[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0)}$$
$$-\frac{G\omega T(\omega)[1-\beta(\omega)]\omega_0(\omega-\omega_0)}{\omega'(\omega'-\omega-i\epsilon)[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0)}$$
$$+\frac{A'(\omega)}{[1-\beta^*(\omega')][1-\beta(\omega-\omega')](\omega'-\omega_0)(\omega-\omega'-\omega_0)}. \quad (35)$$

The arbitrariness of $F(\omega',\omega)$ is thus entirely characterized by the function $A'(\omega)$.

Entirely similar arguments can be applied to the vertex function $K(\omega)$. Again there is a solution to the homogeneous equation that was not admissible in the stable-case calculation and the function turns out to have the form

$$K(\omega)=\frac{\{[G/\Delta^*+A]\omega_0\omega-G\omega_0\}}{[1-\beta(\omega)](\omega-\omega_0)} \quad (36)$$

compared to the previous result which may be written

$$K(\omega)=\frac{\{[G/\Delta^*-\langle 0|f_V|N^*\rangle\Gamma(1-\beta(\Delta^*))/\Delta^{*2}]\omega_0\omega-G\omega_0\}}{[1-\beta(\omega)](\omega-\omega_0)}. \quad (37)$$

The constant A, which is arbitrary in the direct calculation, is thus closely identified with the analytic continuation of $\langle 0|f_V|N^*\rangle$.

Note that at least one author[12] has argued that, since states involving resonances are not to be included in complete sets of states, one should be able to calculate the $V\theta$ elastic scattering amplitude in the presence of a resonance by using Amado's result, Eq. (38), directly:

$$T(\omega)=\frac{G^2/\omega}{\{[1-\omega C(\omega)]/[1+\omega C(\omega)]+\beta_1(\omega)\}}, \quad (38)$$

where

$$C(\omega)=\frac{P}{\pi}\int_1^\infty\frac{d\omega'\,\mathrm{Im}[1-\beta_1(\omega')]\beta_1(\omega-\omega')}{\omega'(\omega'-\omega)|1-\beta_1(\omega')|^2[1-\beta_1(\omega-\omega')]}. \quad (39)$$

The only modification would be to change the factor

$$\beta_1(\omega)=-\frac{G^2\omega}{(2\pi)^2}\int_1^\infty\frac{d\omega' u^2(\omega')(\omega'^2-1)^{1/2}}{\omega'^2(\omega'-\omega-i\epsilon)} \quad (40)$$

to

$$\beta(\omega)=\beta_1(\omega)+A_0\omega/(\omega-\omega_0). \quad (41)$$

This can be shown to be equivalent in the present calculation to choosing $A=A(\omega)=0$ and putting all explicit Δ^*'s equal to ω_0. Clearly this procedure is not correct. It ignores the fact, demonstrated in the foregoing, that although "unphysical states" such as $|N^*\theta\rangle$ are not included among sets of states, they are coupled to the "physical states" through solutions to homogeneous equations.

To continue, we see that Eqs. (35) and (36) together with (28) permit us to calculate $T(\omega)$ in terms of $A'(\omega)$ and A. The problem is now to determine A' and A when we do not have an equation corresponding to (15). One possible approach is to impose unitarity on $T(\omega)$, expressed in terms of these unknowns; another is to return to $F(\omega',\omega)$ and contract a θ from the right instead of from the left as was done previously. Both courses lead to what appear to be entirely intractable sets of equations for A' and A although these equations may, in principle, determine them.

Consequently, it appears that in practice Amado's method of solution for the $V\theta$ sector breaks down in the

[12] P. K. Srivastava, Phys. Rev. 131, 464 (1963).

direct calculation of the unstable case in the sense that arbitrary functions turn up which can be determined only by calculating the stable case solution and analytically continuing in the mass of the unstable particle. The main interest in the direct calculation thus remains in the demonstrated coupling of the stable and unstable particle channels.

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An Analysis of Some Pion-Pion Scattering Models

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A numerical investigation is made on some model calculations based on a suggestion of Blankenbecler. The models satisfy threshold conditions, elastic unitarity, and nearest crossed-channel singularities exactly. The internal parameters are chosen so as to minimize crossing dissatisfaction. No physically sensible result emerges.

I. INTRODUCTION

THIS paper is a report on an attempt to eliminate certain unsatisfactory properties of a model calculation first suggested by Blankenbecler[1] and later modified by Brehm.[2] The essential philosophy here is the same as in I. It consists of setting up a model which, *a priori*, satisfies certain general requirements for a scattering amplitude (such as threshold behavior and elastic unitarity) and then varying the free parameters present in the model in order to satisfy crossing symmetry as well as possible. The central differences between this calculation and the preceding ones are that the complete crossing relations are used, that the nearest crossed-channel singularities are given correctly, and that the model is purely elastic.

Section II is a description of the general model form chosen. Section III contains a discussion of crossing symmetry and the criteria for its satisfaction. Section IV contains the specific models and the results of the calculations based upon them. And Sec. V is an overall discussion of the calculation.

II. GENERAL FORM OF THE MODEL

We take the usual Mandelstam variables for the pion-pion problem,[3] and begin as in I by writing a partial-wave sum in N/D form for the scattering amplitude:

$$M(s,t)=\sum(2j+1)P_j(z)N_j(s)/D_j(s)\,, \tag{1}$$

where

$$z=1+2t/(s-4)\,, \tag{2}$$

and

$$D_j(s)=1-\frac{1}{\pi}\int_4^\infty\frac{\rho(s')N_j(s')ds'}{s'-s}\,, \tag{3}$$

and

$$\rho(s)=[(s-4)/s]^{1/2}. \tag{4}$$

Constant multiplicative factors in the phase-space function have been absorbed in N, and the sum is over even or odd j as the isospin T is even or odd. What now remains to be determined is the exact form of $N_j(s)$. We shall choose for large j the form

$$N_j(s)=f(s)g(j,s)[h(s)]^j\,, \tag{5}$$

where $g(j,s)$ is a rational function of j. As before, this choice will lead to the correct branch point in t.

The value of $z(s)$ [and hence of $t(s)$] for which the sum diverges and the manner in which it diverges depend only on the large j form of N_j, since as j becomes infinite D_j goes to one. For N_j of the form (5), we can do the sum using the generating function of the Legendre polynomials, and the curve of singularity is given by

$$0=1+h(s)^2-2zh(s) \tag{6}$$

or

$$t(s)=(1-h^2)(s-4)/4h\,. \tag{7}$$

If we want a Mandelstam-type cut-plane analyticity, then $t(s)=t_0$ is independent of s. This requires

$$h(s)=[(s-4+t_0)^{1/2}-(t_0)^{1/2}]/[(s-4+t_0)^{1/2}+(t_0)^{1/2}]. \tag{8}$$

This can be rewritten in the more familiar Khuri[4] form

$$h=e^{-\xi}\,,$$
$$\cosh\xi=1+2t_0/(s-4)\,. \tag{9}$$

For the particular case at hand $t_0=4$ and

$$h(s)=(s^{1/2}-2)/(s^{1/2}+2)\,. \tag{10}$$

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[1] R. Kreps, L. F. Cook, J. J. Brehm, and R. Blankenbecler, Phys. Rev. **133**, B1526 (1964). See this paper for further references. Hereafter referred to as I.

[2] J. J. Brehm, Phys. Rev. **135**, B1065 (1964).

[3] All quantities are in units of (pion mass)^2.

[4] N. N. Khuri, Phys. Rev. **130**, 429 (1963).