Linear and Nonlinear Mass-Difference Effects in a Model of the Baryon Multiplets*

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The breaking of $SU(3)$ symmetry is studied in two models: (a) the $J^P=\frac{3}{2}^+$ decuplet states formed as composites of baryons and mesons through baryon exchange, and (b) the $J^P=\frac{1}{2}^+$ octet states formed also as composites of baryons and mesons through decuplet exchange. In both models symmetry-breaking effects are introduced as mass splittings within the component baryon and meson octets. Expansion of dispersion integrals in powers of mass differences is employed to discuss sum rules for masses and coupling constants. In first order the expected results are obtained when the Gell-Mann–Okubo mass rule holds for the component particle masses; i.e., the GMO rule also holds for the composite particle masses, and the coupling constants of composite to component particles obey the known sum rules. However, a comparison of exact and first-order dispersion integrals in model (a) shows that the first-order approximation is quite poor. Therefore, the approximate propagation of the GMO rule from component to composite masses that was found in earlier work cannot be explained as merely a reflection of the first-order result. Significant higher order effects conspire to preserve nearly equal spacing of the decuplet masses. A mass-difference perturbation theory of eigenvalues and eigenvectors of the scattering matrix is presented, together with a review of the decomposition of perturbations by irreducible tensor operators. In the case of model (b), "octet enhancement" is verified (i.e., the intrinsic preference of the bootstrap equations for octet-type perturbations). This discussion of octet enhancement, and a similar one of Dashen and Frautschi, is criticized on two grounds: (i) linearization in mass differences is not justified, and (ii) the neglect of decuplet states in external lines might be a serious error. Inclusion of the external decuplet states is necessary for vertex symmetry. It is found that vertex symmetry is difficult to reconcile with other general requirements in approximations based on the ND^{-1} method.

1. INTRODUCTION

IN an earlier paper we studied breaking of $SU(3)$ symmetry in a model of the $\frac{3}{2}^+$ baryon decuplet.[1] The model described the B^* states as B-P composites formed by B exchange, where B^* is the $\frac{3}{2}^+$ decuplet, B the $\frac{1}{2}^+$ octet, and P the 0^- octet. Symmetry breaking was introduced by taking the observed masses of the B and P states. It turned out that the B^* masses approximately obeyed the Gell-Mann–Okubo[2] (GMO) equal-spacing rule, in the sense that the deviations from equal spacing were about 10% of the average level spacing. The B^*BP coupling constants were calculated, and found to deviate substantially from their pure-symmetry limits. The question of whether the coupling constants obeyed sum rules appropriate to an octet-type symmetry perturbation was left unsettled.

In this paper we try to explain and clarify the results of Ref. 1 by systematic use of first-order expansions in mass differences. With this linearized treatment of mass differences we also study the B octet formed as a B-P composite through B^* exchange. As other authors[3–5] have shown in models like ours, there is a precise "propagation of the GMO rule" if only first-order terms in the mass-difference expansion are retained. That is to say, if the GMO rule holds for the constituent B and P particles, it will also hold for the composite B or B^*. We show that the coupling strengths of the composite particles to their constituents also obey octet-type sum rules if the constituent masses satisfy the GMO rule; again, this result holds only to first order in mass differences. In the linear approximation we also study "octet enhancement"; i.e., the idea of Tarjanne and Cutkosky[6] that the bootstrap equations contain an intrinsic preference for octet-type symmetry perturbations.

Although we have given much attention to the first-order mass expansion, the most noteworthy result of the paper is that the first-order approximation is actually very poor. A comparison of exact dispersion integrals with their first-order expansions reveals that the sum of second- and higher order terms is often quite comparable with first-order terms. It follows that the approximate

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equal spacing rule of the B^* levels in Ref. 1 cannot be understood as merely a manifestation of the first-order theorem on propagation of the GMO rule. Some more interesting mechanism is at work, in which the higher order effects conspire to retain nearly equal spacing. Study of this phenomenon may lead to some hints as to why the GMO rule (as well as other consequences of first-order octet-type perturbations) seem to work so well in fitting experimental facts. It seems reasonable to approach the problem by a study of second-order effects, for which the present paper is good preparation.

Section 2 begins with a review of notation and the formulation of the model by means of the ND^{-1} representation. After these preliminaries the mass difference expansion is treated. The discussion takes the form of a straightforward perturbation theory for the eigenvector and eigenvalue of the scattering matrix corresponding to a bound or resonant state. First- and second-order formulas for mass perturbations and first-order formulas for coupling-constant perturbations are derived. Section 3 contains a review of the expansion of mass and coupling constant perturbations as linear sums of matrix elements of irreducible tensor operators. The first-order result on propagation of GMO rule is obtained, and the general form of mass and coupling-constant sum rules is stated. The formulation for the discussion of octet enhancement is set up. Section 4 contains the comparison of exact dispersion integrals with their first-order expansions, for the case of the decuplet model. In Sec. 5 the coupling-constant sum rules are stated explicitly, and numerical evaluations of coupling-constant perturbations are carried out. In the decuplet case, the first order B^*BP coupling perturbations are compared with the exact perturbations of Ref. 1. As one might have expected from Sec. 4, they do not compare very well. However, both calculations show that the **27** representation is the one that mixes most strongly with the **10**, and they agree closely on the amount of **27**. The predominant **27–10** mixing is associated with the suppression of the decay mode $Y_1^* \to \Sigma+\pi$. In the case of the baryon octet, only the first-order calculation has been performed so far. A serious defect of our model is encountered; viz., there is a failure of vertex symmetry. If the residue of the B' baryon pole in the B-P scattering amplitude is compared with the residue of the B pole in the B'-P amplitude, it does not agree as it should. This may be traced to our omission of the B^*-P channels in the ND^{-1} representation. The impossibility of ensuring both vertex symmetry and T matrix symmetry in ND^{-1} procedures based on single-particle exchanges is pointed out. Octet enhancement in the linear bootstrap equations for the baryon masses is verified. However, this treatment of octet enhancement, as well as a similar one by Dashen and Frautschi,[4,7] is criticized on two grounds: (i) the linearization in mass differences is not well justified; (ii) the omission of B^*-P channels, which spoils vertex symmetry, is a doubtful procedure. Some of the B^*-P thresholds are quite comparable to B-P thresholds. The effects of mass differences on the positions of B^*-P thresholds is expected to have an important bearing on the octet enhancement question. Section 6 is concerned with conclusions and the outlook for future work. The Appendix is concerned with formulas for derivatives of dispersion integrals with respect to masses.

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2. FORMULATION OF THE MODEL AND THE MASS DIFFERENCE EXPANSION

As far as possible, the notation will follow that of Ref. 1. The partial-wave P-B scattering matrix is denoted by $T=[T_{ij}{}^{\gamma}(z)]$, $\gamma=(J,I,Y)$, where z is the (complex) energy in the zero-momentum frame, and the indices i, j label the channels. T describes both orbital states ($l=J\mp\frac{1}{2}$) for a given J. For a physical value of the energy w, $T(w+i0)$ represents the physical amplitude for one state, while $-T(-w-i0)$ is the amplitude for the other. Thus, there are two "physical cuts": $(-\infty, -w_0)$ and (w_0, ∞). Which orbital state is found as the limit from above the right cut is a matter of choice. For notational convenience later on, we take it to be the p state in both the $J=\frac{1}{2}$ and $J=\frac{3}{2}$ problems. In order to avoid threshold branch points and to insure correct momentum dependence of our model T matrix, we work with an associated matrix G. In the $p_{1/2}$ state, G is defined in the following way:

$$G_{ij}(z)=[(z-w_{0i})(z-w_{0j})]^{-1/2}T_{ij}(z). \tag{2.1}$$

Here w_{0i} is the threshold energy of channel i, and the square root is defined as positive for z real and greater than w_{0i} and w_{0j}. In the $p_{3/2}$ state we have

$$G_{ij}(z)=[q_i{}^2(z)q_j{}^2(z)(z+w_{0i})(z+w_{0j})]^{-1/2}T_{ij}(z), \tag{2.2}$$

where q_i is the momentum of channel i. If G is nonzero and finite at thresholds (as it will be in our model), then the T matrices of Eqs. (2.1) and (2.2) will have the proper momentum dependence at thresholds:

$$\begin{aligned} J=\tfrac{1}{2}\text{:}\quad T_{ij}\sim &\begin{cases} q_iq_j; & w=w_{0i}, w_{0j}(p_{1/2}) \\ 1; & w=-w_{0i}, -w_{0j}(s_{1/2}); \end{cases} \\ J=\tfrac{3}{2}\text{:}\quad T_{ij}\sim &\begin{cases} q_iq_j; & w=w_{0i}, w_{0j}(p_{3/2}) \\ (q_iq_j)^2; & w=-w_{0i}, -w_{0j}(d_{3/2}). \end{cases} \end{aligned} \tag{2.3}$$

These statements follow from the formula for $q_i{}^2$:

$$\begin{aligned} q_i{}^2(w)&=[(w-w_{0i})(w+w_{0i})(w-w_{1i})(w+w_{1i})]/4w^2, \\ w_{0i}&=M_i+m_i, \quad w_{1i}=M_i-m_i. \end{aligned} \tag{2.4}$$

The baryon and meson masses of channel i are labeled M_i and m_i, respectively. We employ the matrix ND^{-1} representation for G; $G=ND^{-1}$, where N is analytic in a neighborhood of the physical cuts, and D is analytic

elsewhere. D has the representation

$$D_{ij}(z)=\delta_{ij}-\frac{z-\hat{w}}{\pi}\int_{-\infty}^{\infty}\frac{\rho_i(w)N_{ij}(w)dw}{(w-\hat{w})(w-z)}, \quad (2.5)$$

where the momentum factor ρ_i is defined so as to exclude the region of integration $(-w_{0i}, w_{0i})$. The subtraction point $\hat{w}$ lies inside the interval $(-w_{0i}, w_{0i})$, and ρ_i is defined by

$$\begin{aligned}\rho_i(w)&=|w-w_{0i}|q_i(|w|)\theta(|w|-w_{0i}), \quad J=\tfrac{1}{2},\\ &=|w+w_{0i}|q_i^3(|w|)\theta(|w|-w_{0i}), \quad J=\tfrac{3}{2}.\end{aligned} \quad (2.6)$$

To specialize on the model that we study, N is set equal to the single-baryon exchange approximation for G. In accordance with the bootstrap picture,[8] the baryons exchanged are the $\frac{3}{2}^+$ decuplet states B^* in the case of the $J=\frac{1}{2}$ amplitude, and the $\frac{1}{2}^+$ octet states B in the $J=\frac{3}{2}$ case. Thus, the B^* exchange produces B as a B-P composite, and the B exchange produces B^* as a B-P composite; P denotes the 0^- meson octet. The N matrix is computed assuming degenerate masses within the B, B^*, and P multiplets. Nondegenerate P masses obeying the Gell-Mann–Okubo rule are introduced only in the centrifugal barrier factor ρ of Eq. (2.5). Of the various elements of the ND^{-1} representation, the ρ factor is most sensitive to mass differences. For this reason, the simplest, if not the most refined, way of accounting for mass differences is the one we have chosen. The procedure is not entirely satisfactory, but it at least has the advantage of yielding a symmetric T matrix [see the remarks following Eq. (2.14) below]. The obvious proposals for improving the method lead to difficulties. For instance, if N is set equal to the Born matrix B computed with nondegenerate masses, the resulting T matrix is not symmetric.[9] Also, the integration over $\rho_i N_{ij}$ in the D matrix may then encounter singularities of N_{ij} if the threshold of channel j is lower than that of channel i. Symmetry of the T matrix is ensured[10] if one solves the integral equation of the ND^{-1} method with the kernel determined by the singularities of the nondegenerate Born matrix. Unfortunately, that method spoils the symmetry of the vertex function. The $B'BP$ coupling constant defined as the residue of the B' pole in the $BP\to BP$ amplitude would not agree with the residue of the B pole in the $B'P\to B'P$ amplitude.[11] Further remarks on the problem of vertex symmetry are included in Sec. 5.

We also take coupling constant relations implied by strict $SU(3)$ invariance in calculating N. This approximation can be dropped without too much difficulty, and in some future publication we intend to account for perturbations of coupling constants (i.e., the coupling strengths of the crossed-channel P-B states to the exchanged B or B^*). For the present we are concerned mostly with the perturbations of the direct-channel coupling constants due to mass differences. A formalism for ensuring consistency of the crossed- and direct-channel coupling perturbations is presented in the following.

To put Eq. (2.5) in convenient form, we use Eq. (2.6) and make the change of variable $w\to -w$ in the range of integration where w is negative. Then the D matrix element takes the form

$$D_{ij}(z)=\delta_{ij}-\frac{z-\hat{w}}{\pi}\int_{w_{0i}}^{\infty}dw\, q_i^{2J}(w)\times\left[\frac{(w\mp w_{0i})h_{ij}^{(-)}(w)}{(w-\hat{w})(w-z)}-\frac{(w\pm w_{0i})h_{ij}^{(+)}(w)}{(w+\hat{w})(w+z)}\right]. \quad (2.7)$$

The upper sign is to be taken for $J=\frac{1}{2}$, and the lower for $J=\frac{3}{2}$. Superscripts $(-)$ and $(+)$ indicate odd and even orbital states. In our model $h_{ij}^{(-)}$ is the degenerate, single-baryon exchange amplitude for the p wave, divided by $(w\mp w_0)q^{2J-1}$; (w_0 and q are the threshold energy and momentum of the degenerate P-B system). $h_{ij}^{(+)}$ is the corresponding amplitude for the s or d wave, divided by $(w\pm w_0)q^{2J-1}$. Because of degeneracy of the masses and the presence of only one type of exchange graph for a given J, the energy dependence of h_{ij} comes out as a factor. Thus,

$$h_{ij}^{(\pm)}(w)=h^{(\pm)}(w)(N_0)_{ij}, \quad (2.8)$$

where N_0 is a constant matrix and $h^{(\pm)}$ is a scalar multiplier. The following formulas define $h^{(\pm)}$.[12]

$$\begin{aligned}J=\tfrac{1}{2}:\quad &16\pi w(w\pm w_0)h^{(\pm)}(w)\\ &=(E+M)[A_{0,1}+(w-M)B_{0,1}]\\ &\quad+(E-M)[-A_{1,0}+(w+M)B_{1,0}],\\ J=\tfrac{3}{2}:\quad &16\pi w(w\mp w_0)q^2(w)h^{(\pm)}(w)\\ &=(E+M)[A_{1,2}+(w-M)B_{1,2}]\\ &\quad+(E-M)[-A_{2,1}+(w+M)B_{2,1}].\end{aligned} \quad (2.9)$$

With the degenerate B and P masses denoted by M and m, the baryon energy E satisfies the relations

$$\begin{aligned}E\pm M&=[(w\pm M)^2-m^2]/2w,\\ (E+M)(E-M)&=q^2.\end{aligned} \quad (2.10)$$

For the $J=\frac{1}{2}$ amplitude and $\frac{3}{2}^+$ baryon exchange, the partial-wave projections A_l, B_l of the invariant amplitudes are

$$\begin{aligned}A_l(w)&=-4\pi\phi(w)[(\bar{M}+M)\alpha(w)+(\bar{M}-M)\beta]Q_l(x)/q^2,\\ B_l(w)&=4\pi\phi(w)[\alpha(w)-\beta]Q_l(x)/q^2,\\ \alpha(w)&=\bar{q}^2+(x-1)q^2,\quad \beta=(\bar{E}+M)^2/3,\\ x&=1+[M^2+m^2-(\bar{M}^2+w^2)/2]/q^2,\\ \phi(w)&=[(M+m^2)+a^2]/[w^2+a^2].\end{aligned} \quad (2.11)$$

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The factor Q_l is the Legendre function of the second kind. The bar over q or E indicates that the quantity is to be evaluated at the mass $\bar{M}$ of the exchanged $\frac{3}{2}^+$ particle. The function $\phi(w)$ is an arbitrarily chosen cutoff factor, which is included to obtain convergence of (2.7). The "cutoff energy" a is an adjustable parameter to be determined later. For the $J=\frac{3}{2}$ amplitude and $\frac{1}{2}^+$ baryon exchange, the partial-wave projections are[12]

$$\begin{aligned} A_l(w)&=(\bar{M}-M)Q_l(x)/q^2, \\ B_l(w)&=Q_l(w)/q^2, \end{aligned} \tag{2.12}$$

where again $\bar{M}$ represents the exchanged particle's mass.

We discuss now the expansion in mass differences. Since we are concerned only with $p_{1/2}$ and $p_{3/2}$ states, we work with $G(w+i0)$, $w>0$. After division of the numerator and denominator by the constant matrix N_0, the G matrix has the form

$$G(w+i0)=h(w)E(w+i0)^{-1}, \tag{2.13}$$

where

$$h(w)=h^{(-)}(w), \quad E(z)=N_0^{-1}+I(z). \tag{2.14}$$

$I(z)$ is a diagonal matrix, obtained from the second term of Eq. (2.7) by replacing $h_{ij}^{(\pm)}$ by $h^{(\pm)}$. Since N_0 is symmetric, the symmetry of G follows from (2.13). The Taylor expansion in mass differences refers to the real part of $I(w+i0)$. Branch points prevent a similar expansion of the imaginary part of this quantity. This causes no difficulty, since the real part suffices to construct the K matrix, from which all physical results may be derived. We denote the ith diagonal entry of ReI by $J(w; M_i,m_i^2)$ and expand it in powers of M_i and m_i^2 about the degenerate values M and m^2.

$$\begin{aligned} J(w; M_i,m_i^2)=&J(w; M,m)+J_M(w; M,m)(M_i-M) \\ &+J_{m^2}(w; M,m)(m_i^2-m^2)+\cdots. \end{aligned} \tag{2.15}$$

The choice of m^2 rather than m as the expansion variable stems from the superiority of squared meson masses in satisfying the GMO rule. In terms of diagonal matrices J, δM, δm^2 and scalars J_0, J_M, J_{m^2}, Eq. (2.15) takes the form

$$\begin{aligned} J&=J_0+J_M\delta M+J_{m^2}\delta m^2+\cdots \\ &=J_0+\Delta J. \end{aligned} \tag{2.16}$$

The first-order mass derivatives are easily evaluated, although some care should be taken to justify differentiation of the principal value integral under the integral sign. This justification and also the question of higher order mass derivatives are treated in the Appendix. If the integral is written

$$J=P\int_{w_0}^{\infty}\frac{\phi(w'; M,m)dw'}{w'-w}, \quad w_0=M+m, \tag{2.17}$$

then the first mass derivative is

$$\frac{\partial J}{\partial M}=P\int_{w_0}^{\infty}\frac{\partial\phi}{\partial M}\frac{dw'}{w'-w}. \tag{2.18}$$

Higher mass derivatives are not necessarily obtained correctly by carrying out further differentiations under the integral sign.

Our next concern is the expansion of the mass of the B-P composite state in terms of the mass differences of its constituents. In our model the latter occur only in the expansion (2.16) of J. The mass of the composite state is the energy w for which

$$\det[\mathrm{Re}E(w)]=\det[N_0^{-1}+J(w)]=0. \tag{2.19}$$

In order to study Eq. (2.19) to first order in mass differences, it is convenient to make use of a basis in which N_0 is diagonal. The diagonalization is accomplished by an orthogonal transformation to states that transform by irreducible representations of $SU(3)$; viz., the representations contained in the direct product $8\otimes 8$. The transformation to the diagonal matrix Λ reads as follows:

$$\begin{aligned} (N_0)_{ij}&=\langle B_iP_i; IY|N_0|B_jP_j; IY\rangle \\ &=\sum_{\mu,\gamma}\begin{pmatrix} 8 & 8 & \mu & \gamma \\ (I_1Y_1)_i & (I_2Y_2)_i & I & Y \end{pmatrix} \\ &\qquad\times\lambda_{\mu\gamma}\begin{pmatrix} 8 & 8 & \mu & \gamma \\ (I_1Y_1)_j & (I_2Y_2)_j & I & Y \end{pmatrix} \\ &=\sum_{\alpha}U_{i\alpha}\lambda_\alpha U_{j\alpha}=(U\Lambda U^T)_{ij}. \end{aligned} \tag{2.20}$$

The quantity

$$\begin{pmatrix} 8 & 8 & \mu & \gamma \\ I_1Y_1 & I_2Y_2 & I & Y \end{pmatrix}$$

is the isoscalar factor of the $SU(3)$ Clebsch-Gordan coefficient for forming the representation μ from the product of two 8's. (We employ de Swart's notation.[13]) The index γ distinguishes the two states transforming according to the two equivalent 8-dimensional representations that occur in $8\otimes 8$. These states must be chosen appropriately to obtain the diagonalization (2.20). For the case of B exchange in B-P scattering, the correct states are the usual antisymmetric and symmetric tensor products, $\phi^{(8_a)}$ and $\phi^{(8_s)}$. For B^* exchange in B-P scattering, one must take states $\phi^{(8_1)}$, $\phi^{(8_2)}$, which are obtained by a rotation[14]:

$$\begin{aligned} \phi^{(8_1)}&=(\cos\theta)\phi^{(8_s)}-(\sin\theta)\phi^{(8_a)}, \\ \phi^{(8_2)}&=(\sin\theta)\phi^{(8_s)}+(\cos\theta)\phi^{(8_a)}, \end{aligned} \tag{2.21}$$

$$\tan\theta=(\sqrt{5})/(\sqrt{6}+1). \tag{2.22}$$

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In the last line of Eq. (2.20) and in the following work, the states transforming by $SU(3)$ representations are denoted briefly by Greek indices α, β, $\cdots$, while states of a definite particle type are labeled by Latin indices i, j, $\cdots$. In general, not all of the representations of $8\otimes 8$ are actually present in the sum on α, since given I, Y values are not present in all such representations. In the new basis Eq. (2.19) reads

$$\det[\Lambda^{-1}+U^TJ(w)U] = \det[\Lambda^{-1}+J_0(w)+U^T\Delta J(w)U]. \quad (2.23)$$

The matrix $\Lambda^{-1}+J_0$ is diagonal, and the nondiagonal perturbation term has the form

$$(U^T\Delta JU)_{\alpha\beta}=(\Psi_0{}^\alpha,\Delta J\Psi_0{}^\beta), \quad (2.24)$$

where the vector $\Psi_0{}^\alpha$, the αth column of U, is an eigenvector of N_0. In the limit of degenerate masses, ΔJ tends to zero and (2.23) becomes

$$\det[\Lambda^{-1}+J_0(w)]=\prod_{\alpha=1}^{n}[\lambda_\alpha{}^{-1}+J_0(w)]=0. \quad (2.25)$$

In our problem only one of the factors of Eq. (2.25) vanishes—a factor corresponding to one of the 8 representations in the $J=\frac{1}{2}$ case, and the factor of the 10 representation in the $J=\frac{3}{2}$ case. If w_0 is the energy of the unperturbed level we have

$$\lambda_\alpha{}^{-1}+J_0(w_0)=0, \quad (2.26)$$

for the appropriate α.

To solve (2.19) we employ a straightforward perturbation method. The problem is equivalent to finding the energy w for which $N_0{}^{-1}+J(w)$ has an eigenvector Ψ with eigenvalue zero:

$$[N_0{}^{-1}+J(w)]\Psi=0. \quad (2.27)$$

We also wish to compute the vector Ψ, since its components are proportional to the coupling strength between the composite state and its constituents (see the discussion below). We replace the diagonal mass shift matrices δM, δm^2 by $x\delta M$, $x\delta m^2$, and develop w and Ψ as power series in the real parameter x.

$$\begin{aligned} w&=w_0+w_1x+w_2x^2+\cdots,\\ \Psi&=\Psi_0+\Psi_1x+\Psi_2x^2+\cdots. \end{aligned} \quad (2.28)$$

If H is defined by $H(x)=J[w(x);M+x\delta M,m^2+x\delta m^2]$, we have

$$[N_0{}^{-1}+H(0)+xH'(0)+\cdots]\times[\Psi_0+x\Psi_1+\cdots]=0, \quad (2.29)$$

or

$$[N_0{}^{-1}+H(0)]\Psi_0=0, \quad (2.30)$$

$$[N_0{}^{-1}+H(0)]\Psi_0=-H'(0)\Psi_0, \quad (2.31)$$

$$[N_0{}^{-1}+H(0)]\Psi_2=-H'(0)\Psi_1-(1/2!)H''(0)\Psi_0,\ \cdots. \quad (2.32)$$

With the notations

$$J_y=\partial J/\partial y(w_0;M,m) \quad \text{and} \quad J_0=J(w_0;M,m),$$

the first few derivatives of H are

$$\begin{aligned} H(0)&=J_0,\\ H'(0)&=J_ww_1+J_M\delta M+J_{m^2}\delta m^2,\\ H''(0)&=2J_w w_2+J_{ww}w_1{}^2+2(J_{Mw}\delta M+J_{m^2w}\delta m^2)w_1\\ &\quad +J_{MM}\delta M^2+2J_{Mm^2}\delta M\delta m^2+J_{m^2m^2}(\delta m^2)^2. \end{aligned} \quad (2.33)$$

Let $\Psi_0=\Psi_0{}^\alpha$ be the eigenvector of N_0 with the eigenvalue λ_α of (2.26), so that (2.30) is satisfied. Following the standard procedure of perturbation theory, one takes the scalar product of (2.31) with Ψ_0 to obtain w_1.

$$\begin{aligned} w_1&=-J_w{}^{-1}(\Psi_0{}^\alpha,\Delta\Psi_0{}^\alpha),\\ \Delta&=J_M\delta M+J_{m^2}\delta m^2. \end{aligned} \quad (2.34)$$

Substitution of the expansion $\Psi_1=\sum_{\beta\neq\alpha}c_\beta\Psi_0{}^\beta$ in (2.31) and formation of the scalar product with $\Psi_0{}^\gamma$ yields

$$\Psi_1=\sum_{\beta\neq\alpha}\Psi_0{}^\beta\frac{(\Psi_0{}^\beta,\Delta\Psi_0{}^\alpha)}{\lambda_\alpha{}^{-1}-\lambda_\beta{}^{-1}}. \quad (2.35)$$

$\Psi_0+\Psi_1$ has unit length, to first order. The second-order correction to the energy is

$$w_2=\sum_{\beta\neq\alpha}\frac{(\Psi_0,\Delta\Psi_0{}^\beta)^2}{\lambda_\beta{}^{-1}-\lambda_\alpha{}^{-1}}-\tfrac{1}{2}\frac{J_{ww}}{J_w{}^2}(\Psi_0,\Delta\Psi_0)^2+\frac{1}{J_w}(\Psi_0,\Delta'\Psi_0)(\Psi_0,\Delta\Psi_0) -\tfrac{1}{2}(\Psi_0,[J_{MM}\delta M^2+2J_{Mm^2}\delta M\delta m^2+J_{m^2m^2}(\delta m^2)^2]\Psi_0), \quad (2.36)$$

where $\Psi_0=\Psi_0{}^\alpha$. Thus, in first order the formulas resemble the ordinary perturbation formulas of the Hamiltonian formalism, but the analogy breaks down in higher orders. The procedure just described naturally fails if the unperturbed eigenvalues λ_β become nearly degenerate. One must resort to degenerate perturbation theory if $(\lambda_\alpha{}^{-1}-\lambda_\beta{}^{-1})J_w{}^{-1}$ is effectively of order δM. In our problem the nondegenerate theory applies.

The generalization to allow perturbations of the input coupling constants is not difficult. The most interesting case is that in which the same coupling constants occur in the direct and crossed channels. In that situation the requirement that the direct- and crossed-channel couplings be equal leads to definite values for the coupling-constant perturbations in terms of the mass perturbations. We have such a situation in the baryon problem if both parts of the reciprocal bootstrap are considered together. Then both the BBP and B^*BP

couplings appear in direct and crossed channels. Now the G matrix is a direct sum of the $J=\frac{1}{2}$ and $J=\frac{3}{2}$ G matrices, but the formalism described above still applies. The coupling constants are proportional to the components of Ψ, so the effect of perturbing the crossed-channel couplings is obtained by the replacement[14a]

$$N_0 \to N_0+M[\Psi]; \quad M_{ij}=\sum_{kl} \mu_{ij,kl}\Psi_k\Psi_l. \quad (2.37)$$

Since N_0 describes the unperturbed system, the perturbation series for M begins with the first order.

$$M(x)=M[\Psi(x)]=M'(0)x+M''(0)x^2/2!+\cdots. \quad (2.38)$$

To modify (2.29) the expansion of $(N_0+M)^{-1}$ is required:

$$(N_0+M)^{-1}=N_0^{-1}-N_0^{-1}M'(0)N_0^{-1}+\cdots. \quad (2.39)$$

To express $M'(0)$ we use (2.28) and obtain

$$\begin{aligned} M_{ij}'(0)&=\sum_{kl} \mu_{ij,kl}(\Psi_{1k}\Psi_{0l}+\Psi_{0k}\Psi_{1l}), \\ M'(0)&=(\Phi,\Psi_1). \end{aligned} \quad (2.40)$$

Here Φ is a vector with matrices as components. It is given by

$$\Phi_{ij}{}^k=\sum_l(\mu_{ij,kl}+\mu_{ij,lk})\Psi_{0l}, \quad (2.41)$$

where k labels the vector components. The first-order perturbation equation that replaces (2.31) is

$$\begin{aligned}[N_0^{-1}+J_0(w_0)]\Psi_1=[&-J_w(w_0)w_1-J_M(w_0)\delta M]\Psi_0 \\ &+(N_0^{-1}\Phi N_0^{-1},\Psi_1)\Psi_0. \end{aligned} \quad (2.42)$$

The terms involving the meson mass shifts are suppressed, since they enter in the same way as the δM term. Let $\Psi_1=\sum_{\beta\neq\alpha} c_\beta\Psi_0{}^\beta$ and take the scalar product of (2.42) with $\Psi_0{}^\gamma$, $\gamma\neq\alpha$. The result is

$$\begin{aligned}(\lambda_\gamma^{-1}-\lambda_\alpha^{-1})c_\gamma=&-J_M(\Psi_0{}^\gamma,\delta M\Psi_0{}^\alpha) \\ &+\sum_{\beta\neq\alpha}\lambda_\gamma^{-1}\lambda_\alpha^{-1}[(\Psi_0{}^\gamma,\Phi\Psi_0{}^\alpha),\Psi_0{}^\beta]c_\beta. \end{aligned} \quad (2.43)$$

or

$$\sum_{\beta\neq\alpha} A_{\gamma\beta}c_\beta=J_M(\Psi_0{}^\gamma,\delta M\Psi_0{}^\alpha), \quad \gamma\neq\alpha. \quad (2.44)$$

After finding Ψ_1 by solving these equations, we can compute the first-order mass shift from (2.42):

$$w_1=-J_w{}^{-1}(\Psi_0{}^\alpha,[J_M\delta M-\lambda_\alpha{}^{-2}(\Phi,\Psi_1)]\Psi_0{}^\alpha). \quad (2.45)$$

In the vector-meson bootstrap problem, Fulco and Wong[15] have effectively carried out the procedure just outlined.

To derive the relation between the eigenvector Ψ and the coupling constants, we work with the following representation of $G(w+i0)$:

$$G=L(1-i\rho L)^{-1}; \quad L=N(\mathrm{Re}D)^{-1}. \quad (2.46)$$

The diagonal matrix ρ has elements defined by Eq. (2.6). The relation between the Hermitian matrix L and G is analogous to that between the usual reaction matrix K and T. According to Eq. (2.14),

$$L(w)=h(w)[N_0^{-1}+J(w)]^{-1}, \quad (2.47)$$

so L and $N_0^{-1}+J$ have the same eigenvectors. Near a resonance or bound state with energy w_*, L may be represented approximately as

$$L=[\gamma/(w_*-w)]\Psi\Psi^\dagger, \ \gamma>0, \ \Psi=\Psi^*, \ (\Psi,\Psi)=1. \quad (2.48)$$

The eigenvector Ψ satisfies Eq. (2.27) for $w=w_*$. The corresponding approximate expression for G is

$$G=\frac{\gamma/(w_*-w)}{1-i(\Psi,\rho\Psi)\gamma/(w_*-w)}\Psi\Psi^\dagger. \quad (2.49)$$

This formula may be ascertained by Taylor expansion of $(1-i\rho L)^{-1}$, and proved by multiplying (2.49) on the right by $1-i\rho L=1-i\rho\gamma/(w_*-w)\Psi\Psi^\dagger$. Equation (2.48) is exact at $w=w_*$ in the sense that $\lim_{w\to w_*}(w_*-w)L(w)=\gamma\Psi\Psi^\dagger$ is an exact equation. In the case of a resonance, Eq. (2.49) is not exact in the same sense; rather, a formula like (2.48) holds for G at a complex point w_* on a secondary sheet. Equation (2.49) asserts that only one eigenvalue of G is nonzero at $w=w_*$, which is not the case in general. The narrower the resonance, the better the approximation of Eq. (2.49). In the case of zero width (a bound state) we have $\rho=0$, $G=L$, and Eq. (2.49) is exact at w_*. The work of Ref. 1 indicates that (2.49) is sufficiently accurate for the $\Xi_{1/2}{}^*$ and $Y_1{}^*$ decuplet resonances, but that it begins to fail slightly for the broader $N_{3/2}{}^*$. For the case of a resonance we have

$$\begin{aligned}\mathrm{Im}G&=\gamma\frac{\Gamma/2}{(w_*-w)^2+\Gamma^2/4}\Psi\Psi^\dagger, \\ \Gamma/2&=(\Psi,\rho\Psi)\gamma. \end{aligned} \quad (2.50)$$

We define the squared coupling constant between channel i and the composite state as $\gamma\Psi_i{}^2$, where Ψ_i is the ith component of Ψ. (This is not the same as the definition of the coupling constant via the Lagrangian. For the relation between the two definitions see Sec. 5.) Thus, the approximate equation (2.50) relates the experimental width and branching ratios to the coupling constants in the conventional way. In principle, the comparison between theory and experiment need not be based on the approximate equation. Ideally, one would compute L from the experimental G by the exact equation $L=(1+iG\rho)^{-1}G$, and evaluate the coupling constants from $\gamma\Psi_i{}^2=\lim_{w\to w_*}(w_*-w)L_{ii}(w)$. The coupling constant sum rules hold only for coupling con-

[14a] *Note added in proof.* The formalism described here does not allow for perturbations of the kinematical factors involved in the relation between Ψ and the coupling constants that appear in the Lagrangian. For a discussion of these factors see Eq. (5.22) ff and Eq. (5.36) ff.

[15] J. R. Fulco and D. Y. Wong, Phys. Rev. **137**, B1239 (1965).

stants defined as residues of L, essentially because no mass difference expansion of the momentum factor ρ exists. Some related remarks are to be found in Ref. 16.

To find the reduced width γ of Eq. (2.50) to first order in mass differences, one needs the eigenvalue of L near $w=w_*$ to first order. It is easy to check that to lowest order in δM the eigenvalue of $N_0^{-1}+J(w)$ for eigenvector Ψ is

$$\mu(w)=\lambda_\alpha^{-1}+J_0(w)+[\Psi_0{}^\alpha,\Delta(w)\Psi_0{}^\alpha]. \quad (2.51)$$

Therefore,

$$h(w)/\mu(w)=\gamma/(w_*-w); \quad \gamma=-h(w_*)/\mu'(w_*). \quad (2.52)$$

Now expand $\gamma(w_*)$ about $w_*=w_0$, and set $w_*-w_0=w_1$. The result to first order in mass differences is

$$\gamma=-\frac{h}{J_0'}\left[1+\left(\frac{h'}{h}-\frac{J_0''}{J_0'}\right)w_1-\frac{1}{J_0'}(\Psi_0{}^\alpha,\Delta'\Psi_0{}^\alpha)\right], \quad (2.53)$$

where prime denotes d/dw, and all quantities are evaluated at w_0. Of course, we have neglected perturbations of crossed-channel coupling constants in (2.53).

3. TENSOR OPERATOR DECOMPOSITION, GENERAL FORM OF SUM RULES, AND OCTET ENHANCEMENT

It is rewarding to expand the mass and coupling-constant perturbations as linear sums of matrix elements of irreducible tensor operators. The relevance of such expansions to the discussion of bootstrap equations was first shown by Glashow.[17] The usual sum rules come out when the sum over operators is reduced to an octet operator term alone. We briefly review the tensor operator formalism and its application to our model. The formalism gives the setting for the idea[4,6,7] of "octet enhancement," which is discussed at the end of this section.

For the most part we follow de Swart's notation.[13] Let a finite-dimensional vector space $\mathcal{C}$ be a representation space of $SU(3)$. The unitary operators $U(\alpha)$ represent the group on $\mathcal{C}$; α denotes a group element. In the cases of interest $U(\alpha)$ is reducible, so $\mathcal{C}$ breaks up into invariant subspaces corresponding to irreducible representations. If the vectors of an invariant subspace for irreducible representation μ are denoted by $\phi_\nu{}^{(\mu)}$, the following equation defines the unitary matrices $D_{\nu'\nu}{}^{(\mu)}(\alpha)$ constituting the representation μ:

$$U(\alpha)\phi_\nu{}^{(\mu)}=\sum_{\nu'} D_{\nu'\nu}{}^{(\mu)}(\alpha)\phi_{\nu'}{}^{(\mu)}. \quad (3.1)$$

The index ν stands for (I,I_3,Y) collectively. An irreducible tensor operator $T_\nu{}^{(\mu)}$ is an operator acting on $\mathcal{C}$. It is characterized by the equation

$$U(\alpha)T_\nu{}^{(\mu)}U(\alpha)^{-1}=\sum_{\nu'} D_{\nu'\nu}{}^{(\mu)*}(\alpha)T_{\nu'}{}^{(\mu)}. \quad (3.2)$$

Matrix elements of $T_\nu{}^{(\mu)}$ obey the Wigner-Eckart theorem[13]:

$$(\phi_{\nu_3}{}^{(\mu_3)},T_{\nu_2}{}^{(\mu_2)}\phi_{\nu_1}{}^{(\mu_1)})=\sum_\gamma\begin{pmatrix}\mu_1 & \mu_2 & \mu_{3\gamma}\\ \nu_1 & \nu_2 & \nu_3\end{pmatrix}(\mu_3\|T^{(\mu_2)}\|\mu_1)_\gamma. \quad (3.3)$$

The symbol

$$\begin{pmatrix}\mu_1 & \mu_2 & \mu_{3\gamma}\\ \nu_1 & \nu_2 & \nu_3\end{pmatrix}$$

is the Clebsch-Gordan coefficient for combining representations μ_1 and μ_2 to form μ_3. The index γ distinguishes possible equivalent representations in the direct product $\mu_1\otimes\mu_2$. The number of terms in the sum on γ equals the multiplicity of μ_3 in $\mu_1\otimes\mu_2$. Thus, in the Clebsch-Gordan series of $8\otimes 8$, two equivalent 8-dimensional representations occur; these are distinguished by the value of γ. Equation (3.3) specifies the dependence of the matrix element on ν_1 and ν_3, and thereby leads to sum rules for the matrix elements of a fixed operator. $(\mu_3\|T^{(\mu_2)}\|\mu_1)_\gamma$ is called the "double-bar matrix element."

If $T_{\nu_2}{}^{(\mu_2)}$ is an irreducible tensor operator, then it may be written in the form

$$T_{\nu_2}{}^{(\mu_2)}=\sum_{\mu_1\mu_3\nu_1\nu_3\gamma}(\mu_3\|T^{(\mu_2)}\|\mu_1)_\gamma\times\phi_{\nu_1}{}^{(\mu_1)}\begin{pmatrix}\mu_1 & \mu_2 & \mu_{3\gamma}\\ \nu_1 & \nu_2 & \nu_3\end{pmatrix}\phi_{\nu_3}{}^{(\mu_3)\dagger}. \quad (3.4)$$

The sums on μ_1 and μ_3 run over all irreducible representations in the representation space $\mathcal{C}$. Of course, some of these representations may be equivalent. There will be separate terms in the sum for any two equivalent representations. The vectors $\phi_\nu{}^{(\mu)}$ form a complete, orthonormal set in $\mathcal{C}$; the $\phi_\nu{}^{(\mu)\dagger}$ are the corresponding vectors of the dual space. Given the completeness of the $\phi_\nu{}^{(\mu)}$, Eq. (3.4) just expresses the Wigner-Eckart theorem. There is a sort of converse of the Wigner-Eckart theorem: viz., that the right-hand side of Eq. (3.4) is a tensor operator of type $T_{\nu_2}{}^{(\mu_2)}$ for arbitrary values of the double-bar matrix elements. By taking all but one of the double-bar matrix elements equal to zero, we conclude that the following expression is a tensor operator of type $T_{\nu_2}{}^{(\mu_2)}$:

$$\sum_{\nu_1\nu_3}\phi_{\nu_1}{}^{(\mu_1)}\begin{pmatrix}\mu_1 & \mu_2 & \mu_{3\gamma}\\ \nu_1 & \nu_2 & \nu_3\end{pmatrix}\phi_{\nu_3}{}^{(\mu_3)\dagger}. \quad (3.5)$$

The converse Wigner-Eckart theorem may be established by Eq. (3.2), Eq. (3.1), and the properties of the D's and the Clebsch-Gordan coefficients.[13]

A matrix element of an entirely arbitrary operator may be expressed as a linear sum of matrix elements of

[16] G. Rajasekaran, Nuovo Cimento **37**, 1004 (1965).
[17] S. L. Glashow, Phys. Rev. **130**, 2132 (1963).

the type (3.3). With A arbitrary and μ_1 and μ_3 fixed, we regard the quantities

$$(\phi_{\nu_3}{}^{(\mu_3)}, A\phi_{\nu_1}{}^{(\mu_1)}) \tag{3.6}$$

as the components of a vector; the components are labeled by index-pairs (ν_3,ν_1). The Clebsch-Gordan coefficients provide a complete set of vectors in terms of which this vector may be expressed. Write the orthogonality condition in the form

$$\sum_{\nu_3\nu_1}\begin{pmatrix}\mu_1 & \bar{\mu}_3 & \bar{\mu}_\gamma\\ \nu_1 & -\nu_3 & -\nu\end{pmatrix}\begin{pmatrix}\mu_1 & \bar{\mu}_3 & \bar{\mu}_{\gamma'}\\ \nu_1 & -\nu_3 & -\nu'\end{pmatrix}=\delta_{\mu\mu'}\delta_{\gamma\gamma'}\delta_{\nu\nu'}, \tag{3.7}$$

where $\bar{\mu}$ is the complex conjugate of the representation μ, and $-\nu$ stands for $(I,-I_3,-Y)$. Next apply the "crossing symmetry" of the coefficients[13]:

$$\begin{pmatrix}\mu_1 & \mu_2 & \mu_{3\gamma}\\ \nu_1 & \nu_2 & \nu_3\end{pmatrix}=\xi_2(-)^{(I_3+\frac{1}{2}Y)_1}\left[\frac{N_3}{N_2}\right]^{1/2}\begin{pmatrix}\mu_1 & \bar{\mu}_3 & \bar{\mu}_{2\gamma'}\\ \nu_1 & -\nu_3 & -\nu_2\end{pmatrix}. \tag{3.8}$$

Here N_i is the dimension of representation μ_i, and $\xi_2=\pm1$ is an arbitrary phase factor. By substituting (3.8) in (3.7), we find

$$\frac{N_\mu}{N_3}\sum_{\nu_3\nu_1}\begin{pmatrix}\mu_1 & \mu & \mu_{3\gamma}\\ \nu_1 & \nu & \nu_3\end{pmatrix}\begin{pmatrix}\mu_1 & \mu' & \mu_{3\gamma'}\\ \nu_1 & \nu' & \nu_3\end{pmatrix}=\delta_{\mu\mu'}\delta_{\gamma\gamma'}\delta_{\nu\nu'}. \tag{3.9}$$

It follows that vectors with components

$$\begin{pmatrix}\mu_1 & \mu & \mu_{3\gamma}\\ \nu_1 & \nu & \nu_3\end{pmatrix}$$

form an orthogonal set with N_1N_3 members, and hence a complete set. Therefore, there exist coefficients $\alpha(\mu,\gamma,\nu)$ such that

$$(\phi_{\nu_3}{}^{(\mu_3)}, A\phi_{\nu_1}{}^{(\mu_1)})=\sum_{\mu\gamma\nu}\begin{pmatrix}\mu_1 & \mu & \mu_{3\gamma}\\ \nu_1 & \nu & \nu_3\end{pmatrix}\alpha(\mu,\gamma,\nu). \tag{3.10}$$

Of course, Eq. (3.10) reduces to Eq. (3.3) if A is an irreducible tensor operator. If A is not a tensor operator, Eq. (3.10) may nevertheless simplify in certain cases. For example, if A commutes with I, I_3, and Y, then only the terms with $\nu=(0,0,0)$ occur in the sum. In this case there is a further reduction due to the absence of $\nu=(0,0,0)$ in certain of the representations μ.

The Gell-Mann–Okubo[2] (GMO) mass formula follows from Eq. (3.3) and the assumption that the mass-perturbation operator is a tensor operator of the type $T_0{}^{(8)}$, where the subscript 0 means $\nu_2=(0,0,0)$. Thus,[18] δM_ν and $\delta m_\nu{}^2$ are proportional to the quantity

$$(\phi_\nu{}^{(\mu)}, T_0{}^{(8)}\phi_\nu{}^{(\mu)})=bY+c[2I(I+1)-\tfrac{1}{2}Y^2 -\tfrac{1}{3}p(p+2)-\tfrac{1}{3}q(q+2)+\tfrac{1}{9}(p-q)^2], \tag{3.11}$$

where the right side is an evaluation of the right side of Eq. (3.3). Here $p(q)$ is the number of upper (lower) indices in the tensor construction of representation μ. One requires that $\nu_2=(0,0,0)$ to ensure that the mass perturbations do not result in nonconservation of I, I_3, or Y.

We now return to our dynamical model in which the B or B^* states are formed as B-P composites. We show that to first order in mass differences the validity of the GMO formula (3.11) for the constituent B and P particles will imply its validity for the composite states. This fact is a trivial consequence of Eqs. (2.34) and (3.3). After restoration of an index λ indicating the I, I_3, and Y values of the composite state, Eq. (2.34) gives the mass shift $w_{1\lambda}$ of the composite state as follows:

$$\begin{aligned} w_{1\lambda}&=-J_w{}^{-1}(\Psi_{0\lambda}{}^\alpha, \Delta\Psi_{0\lambda}{}^\alpha),\\ \Delta&=J_M\delta M+J_{m^2}\delta m^2. \end{aligned} \tag{3.12}$$

If the constituent masses δM_ν and $\delta m_\nu{}^2$ obey the GMO formula (3.11), then the corresponding mass operators δM and δm^2 are tensor operators of type $T_0{}^{(8)}$. This follows if we note that $(\Psi_{0\lambda}{}^\alpha, \delta M\Psi_{0\lambda}{}^\alpha)$ may be regarded as a matrix element of the operator

$$\delta M=1_1\otimes\delta M_2\sim\sum_{\nu_1}\phi_{\nu_1}{}^{(8)}\phi_{\nu_1}{}^{(8)\dagger} \otimes\sum_{\nu_2\gamma}\Psi_{\nu_2}{}^{(8)}\Psi_{\nu_2}{}^{(8)\dagger}\begin{pmatrix}8 & 8 & 8_\gamma\\ \nu_2 & 0 & \nu_2\end{pmatrix}a_\gamma, \tag{3.13}$$

where indices 1 and 2 refer to mesons and baryons, respectively, and the a_γ are arbitrary real numbers. By Eq. (3.5), $1_1\otimes\delta M_2$ is clearly a $T_0{}^{(8)}$ operator. Now since $w_{1\lambda}$ is given as a matrix element of a $T_0{}^{(8)}$ operator by Eq. (3.12), it will obey the GMO formula. More generally the mass-perturbation operators δM and δm^2 might be a linear combination of $T_0{}^{(8)}$ and a singlet operator $T_0{}^{(1)}$. This would only add terms independent of I and Y to the GMO formulas of the B, B^*, and P multiplets.

The origin of the coupling-constant sum rules is immediately apparent from Eq. (2.35). The vector Ψ_1 of coupling-constant perturbations is expressed as a sum of matrix elements of mass-perturbation operators. If the latter are irreducible tensor operators, their matrix elements may all be expressed in terms of a few parameters by means of the Wigner-Eckart theorem. With an octet mass perturbation we get n coupling constants expressed in terms of m parameters, with $m<n$. Elimination of the m parameters yields $n-m$ sum rules. The sum rules for BBP and B^*BP couplings are worked out explicitly in Sec. 5.

The idea of "octet enhancement" is that equations of the bootstrap type may actually force the mass-

[18] R. J. Oakes and C. N. Yang, Phys. Rev. Letters **11**, 174 (1963).

perturbation operator to be predominantly an octet tensor operator. To see how this goes we take the simplest kind of bootstrap equation for mass perturbations. The equation is Eq. (2.34) for the case of the B octet formed as a B-P composite. It may be written in the form

$$\delta M_\nu=\sum_\lambda A_{\nu\lambda}\delta M_\lambda+D_\nu, \tag{3.14}$$

where the δM_ν are the perturbations of the baryon masses, and D_ν is proportional to the meson squared-mass perturbations $\delta m_\lambda{}^2$. In a more complete model, D_ν would contain many other terms. As far as the following discussion is concerned, D_ν can be considered as an unknown quantity from an "exact" theory. We use the vector notation $\delta\mathbf{M}=A\delta\mathbf{M}+\mathbf{D}$, where A is a 4×4 matrix. Mass-perturbation vectors $\delta\mathbf{M}$ with components proportional to diagonal matrix elements of $T_0{}^{(\mu)}$ form a complete set of eigenvectors of A. In fact, a complete set of four eigenvectors of A may be constructed from the following quantities:

$$\Psi_\nu{}^{(27)}=\left(\frac{27}{8}\right)^{1/2}\begin{pmatrix}8 & 27 & 8\\ \nu & 0 & \nu\end{pmatrix},\quad \Psi_\nu{}^{(8_a)}=\begin{pmatrix}8 & 8 & 8_a\\ \nu & 0 & \nu\end{pmatrix},$$
$$\Psi_\nu{}^{(8_s)}=\begin{pmatrix}8 & 8 & 8_s\\ \nu & 0 & \nu\end{pmatrix},\qquad \Psi_\nu{}^{(1)}=\left(\frac{1}{8}\right)^{1/2}\begin{pmatrix}8 & 1 & 8\\ \nu & 0 & \nu\end{pmatrix}. \tag{3.15}$$

$\Psi_\nu{}^{(\mu)}$ is proportional to $(\phi_\nu{}^{(8)},T_0{}^{(\mu)}\phi_\nu{}^{(8)})$ for $\mu=27$ or 1, since 8 is involved only once in $8\otimes27$ or $8\otimes1$ [i.e., the sum over γ in Eq. (3.3) contains only one term]. On the other hand, $(\phi_\nu{}^{(8)},T_0{}^{(8)}\phi_\nu{}^{(8)})$ is proportional to a linear combination of $\Psi_\nu{}^{(8_a)}$ and $\Psi_\nu{}^{(8_s)}$, since 8 is included twice in $8\otimes8$. Only the representations $\mu=27$, 8, 1 are allowed for mass-perturbation operators of the octet baryons, since these are the only representations in $8\otimes8$ with $I=I_3=Y=0$. The quantity $\Psi_\nu{}^{(\mu)}$ is independent of I_3 for fixed I and Y. Thus, the eight values of $\nu=(I,I_3,Y)$ yield only four distinct values of $\Psi_\nu{}^{(\mu)}$; these make up the four-component vectors on which A acts. Orthogonality of the eight-component vectors with components (3.15) follows from Eq. (3.9) if it is recalled that $\nu_1=\nu_3$ when $\nu=0$. To arrange for orthogonality of the four-component vectors we introduce a metric

$$g_\nu=2I+1. \tag{3.16}$$

We use a square bracket notation for scalar products based on this metric:

$$[\Psi^{(\mu\gamma)},\Psi^{(\mu'\gamma')}]=\sum_{I,Y}\Psi^{(\mu\gamma)}g_\nu\Psi^{(\mu'\gamma')}=\delta_{\mu\mu'}\delta_{\gamma\gamma'}. \tag{3.17}$$

Now consider the equation defining A:

$$(A\delta\mathbf{M})_\lambda=-(J_M/J_w)(\Psi_{0\lambda}{}^{(8)},\delta M\Psi_{0\lambda}{}^{(8)}). \tag{3.18}$$

If in (3.18) we take δM_ν proportional to one of the four quantities $\Psi_\nu{}^{(\mu)}$ of (3.15), then δM is a tensor operator of type $T_0{}^{(\mu)}$, $\mu=27$, 8, or 1. If $\mu=27$ or 1, then Eq. (3.18) and the Wigner-Eckart theorem show that $(A\delta\mathbf{M})_\nu$ is also proportional to $\Psi_\nu{}^{(\mu)}$, so $\Psi_\nu{}^{(\mu)}$ is an eigenvector of A. On the other hand, if δM_ν is proportional to $\Psi_\nu{}^{(8_a)}$ or $\Psi_\nu{}^{(8_s)}$, then $(A\delta\mathbf{M})_\nu$ will be a linear combination of $\Psi_\nu{}^{(8_a)}$ and $\Psi_\nu{}^{(8_s)}$. One can find two orthogonal combinations of $\Psi^{(8_a)}$ and $\Psi^{(8_s)}$ which are eigenvectors. If the eigenvectors are written $\Psi=c_a\Psi^{(8_a)}+c_s\Psi^{(8_s)}$, the eigenvalue equation is

$$\sum_{\gamma'}[\Psi^{(8\gamma)},A\Psi^{(8\gamma')}]c_{\gamma'}=\lambda c_\gamma. \tag{3.19}$$

The matrix of (3.19) is real and symmetric, and its eigenvalues are distinct (cf. Sec. 5). The two orthogonal solutions of (3.16) lead to eigenvectors $\Psi^{(8\alpha)}$, $\Psi^{(8\beta)}$ which are orthogonal in the sense of (3.17). The A matrix may now be written in spectral form as follows:

$$A_{\nu\lambda}=\sum_{\mu=27,8_\alpha,8_\beta,1}\Psi_\nu{}^{(\mu)}A^\mu\Psi_\lambda{}^{(\mu)}g_\lambda,$$
$$A^\mu=[\Psi^{(\mu)},A\Psi^{(\mu)}]. \tag{3.20}$$

By using a similar spectral form for $(1-A)^{-1}$, Eq. (3.14) may be transformed to

$$\delta M_\nu=\sum_\mu(1-A^\mu)^{-1}[\Psi^{(\mu)},\mathbf{D}]\Psi_\nu{}^{(\mu)}. \tag{3.21}$$

The viewpoint of "enhancement" is that one value of A^μ may be much closer to 1 than all the others, and that the corresponding term of Eq. (3.21) will dominate. Of course, it is assumed that largeness of $(1-A^\mu)^{-1}$ will not be nullified by smallness of $[\Psi^{(\mu)},\mathbf{D}]$. Very little is known about D in general, so this assumption can only be tested within the context of some particular model. In Sec. 5 we show that one of the $A^{(8)}$'s is indeed much closer to 1 than $A^{(27)}$, so the conjecture of octet enhancement is verified for our simple model. In the model the ratio $A^{(8_\alpha,\beta)}/A^{(27)}$ depends only on group theory. To the extent that the model is realistic, we have something like a group-theoretical explanation of octet enhancement.

The reader familiar with the literature will be aware that the material of this section is not very original.[17,4,6,7] We have presented this review in order to bring the notation into accord with that commonly used in the theory of angular momentum, and to show more clearly the role of the Wigner-Eckart theorem.

4. TEST OF THE ACCURACY OF MASS-DIFFERENCE EXPANSIONS

In the case of the decuplet model we compare precise numerical evaluations of the dispersion integral J with its first-order expansion in mass differences [cf. Eq. (2.15)]. The model was described in Sec. 2. It is the same as the model of Ref. 1, except that the values of the unperturbed masses have been changed slightly. In the present work we assume that the mass perturbation operator is purely of octet type [as in Eq. (3.11)] while

in Ref. 1 it was taken to be octet plus singlet. Thus, the unperturbed masses are $M=\frac{1}{2}(M_\Lambda+M_\Sigma)=8.255m_\pi$ and $m^2=\frac{1}{2}(m_\eta^2+m_\pi^2)=(2.88m_\pi)^2$ instead of the values $M=M_\Lambda$, $m=m_\pi$ of Ref. 1. The exchanged mass $\bar{M}$ is taken to be $11m_\pi$. This relatively large value is necessary to avoid a spurious overlap of left and right singularities, an overlap due to the approximation of including mass differences only in the centrifugal barrier factor ρ. The subtraction point $\hat{w}$ has the value $3m_\pi$, in agreement with Ref. 1.

Table I shows the comparison of exact and first-order integrals for the various channels over a range of energies. We tabulate $J-J_0$ and J_1-J_0, where J, J_1, and J_0 are the exact integral, its first-order approximation, and the unperturbed integral, respectively. Since J_0 is large with respect to the perturbations, this way of tabulating gives a clearer idea of the accuracy than a simple comparison of J_1 with J. The table shows that the accuracy is rather poor. In fact, the sum of the second- and higher order perturbations is often not much smaller than the first-order perturbation. The ratio of these quantities, $(J-J_1)/(J_1-J_0)$, is shown in the table.

The poor accuracy of the first-order expansion brings out some interesting questions. In particular, there is

TABLE I. Numerical values of the dispersion integrals: J_0 is the integral evaluated with the degenerate masses, J the integral evaluated with the actual masses, and J_1 the integral evaluated using the first-order mass expansion.

W	J_0	$J-J_0$	J_1-J_0	$(J-J_1)/(J_1-J_0)$	$J-J_0$	J_1-J_0	$(J-J_1)/(J_1-J_0)$
			$N-\pi$			$\Sigma-K$	
8.5	−0.01442	−0.00597	−0.00315	0.90	0.00076	0.00084	−0.09
9.0	−0.01625	−0.00700	−0.00380	0.85	0.00092	0.00103	−0.12
9.5	−0.01822	−0.00774	−0.00463	0.67	0.00110	0.00126	−0.13
10.0	−0.02040	−0.00808	−0.00574	0.41	0.00135	0.00159	−0.15
10.5	−0.02283	−0.00797	−0.00744	0.07	0.00168	0.00208	−0.19
11.0	−0.02572	−0.00721	−0.01126	−0.36	0.00225	0.00326	−0.31
11.5	−0.02947	−0.00540	−0.01280	−0.58	0.00336	0.00364	−0.08
12.0	−0.03300	−0.00361	−0.01069	−0.66	0.00360	0.00289	+0.25
12.5	−0.03621	−0.00195	−0.00868	−0.78	0.00288	0.00222	+0.30
			$N-K$			$\Xi-K$	
		−0.00202	−0.00181	0.12	0.00188	0.00229	−0.18
		−0.00244	−0.00214	0.14	0.00223	0.00276	−0.19
		−0.00301	−0.00257	0.17	0.00263	0.00334	−0.21
		−0.00385	−0.00309	0.25	0.00315	0.00414	−0.24
		−0.00547	−0.00387	0.20	0.00379	0.00532	−0.29
		−0.00633	−0.00542	0.17	0.00477	0.00797	−0.40
		−0.00586	−0.00642	−0.09	0.00644	0.00912	−0.29
		−0.00649	−0.00747	−0.13	0.00766	0.00776	−0.01
		−0.00449	−0.00557	−0.20	0.00826	0.00645	+0.28
			$\Sigma-\pi$			$\Lambda-\pi$	
		−0.00054	−0.00050	0.08	−0.00168	−0.00131	0.28
		−0.00074	−0.00062	0.19	−0.00228	−0.00159	0.43
		−0.00112	−0.00081	0.38	−0.00321	−0.00197	0.63
		−0.00177	−0.00105	0.70	−0.00394	−0.00248	0.59
		−0.00223	−0.00149	0.50	−0.00435	−0.00330	0.32
		−0.00218	−0.00259	−0.16	−0.00417	−0.00523	−0.20
		−0.00117	−0.00274	−0.57	−0.00296	−0.00580	−0.49
		−0.00023	−0.00174	−0.87	−0.00180	−0.00446	−0.60
		+0.00056	−0.00090	−1.62	−0.00077	−0.00327	−0.77
			$\Sigma-\eta$			$\Xi-\pi$	
		0.00109	0.00129	−0.15	0.00103	0.00095	0.08
		0.00131	0.00158	−0.17	0.00118	0.00111	0.06
		0.00157	0.00195	−0.20	0.00132	0.00128	0.03
		0.00192	0.00247	−0.22	0.00150	0.00150	0
		0.00237	0.00327	−0.27	0.00161	0.00175	−0.08
		0.00311	0.00520	−0.40	0.00175	0.00213	−0.18
		0.00447	0.00576	−0.22	0.00268	0.00274	−0.02
		0.00528	0.00443	+0.19	0.00339	0.00313	+0.08
		0.00510	0.00325	+0.57	0.00387	0.00334	+0.16
			$\Lambda-K$			$\Xi-\eta$	
		0.00003	0.00004	−0.25	0.00213	0.00273	−0.22
		0.00006	0.00007	−0.14	0.00252	0.00331	−0.24
		0.00008	0.00010	−0.20	0.00298	0.00403	−0.26
		0.00013	0.00016	−0.19	0.00355	0.00502	−0.29
		0.00020	0.00027	−0.41	0.00428	0.00650	−0.34
		0.00038	0.00062	−0.39	0.00534	0.00992	−0.46
		0.00068	0.00058	+0.17	0.00713	0.01125	−0.36
		0.00027	0.00016	+0.69	0.00851	0.00930	−0.08
		−0.00003	−0.00015	−0.80	0.00935	0.00749	+0.19

the question of how to explain the "propagation" of the GMO rule that was obtained in Ref. 1. In that work the GMO equal-spacing rule for the decuplet held approximately, in spite of the presence of perturbations of all orders in the constituent-particle mass differences. Deviations of the level spacings from equality were of the order of 10% of the spacings themselves. Table I indicates that this situation cannot be understood merely as a manifestation of the first-order result on propagation of the GMO rule (i.e., that to first order in mass differences the composite states will obey the GMO rule if their constituents do). Higher order effects are certainly substantial, but they somehow conspire to give smaller deviations from the GMO rule than would be expected from a casual look at Table I. Something of the sort was already apparent in the work of Ref. 1, where the energies of the decuplet states were plotted as a function of the symmetry perturbation strength x (Fig. 2 of Ref. 1). Since the octet mass-perturbation operators of the constituent particles are proportional to x, any curvature in the graphs of decuplet energies versus x is due to contributions nonlinear in mass differences. There is appreciable curvature in the graphs of Ref. 1, except for the Ω graph. The Ξ^*, Y^*, and N^* graphs all bend in the same direction. The amount of bending increases with isospin in such a way as to maintain approximately equal spacing. It is hoped that a close study of the second-order formula Eq. (2.36) will throw light on this interesting behavior.

5. SUM RULES IN EXPLICIT FORM AND NUMERICAL EVALUATIONS

To illustrate the general discussion of Secs. 2 and 3, we consider specific details of our two models: (a) the $J^P=\frac{3}{2}^+$ states B^* formed as B-P composites through B exchange, and (b) the $J^P=\frac{1}{2}^+$ states B formed as B-P composites through B^* exchange. In both models symmetry-breaking effects arise as a consequence of mass splittings within the component baryon and meson octets. For the component masses we take GMO formulas. The mass splittings are treated according to the formalism of Sec. 2.

(a) The B^* Decuplet

Reference 1 contains a detailed treatment of the B^* model. In that work the mass-difference expansion was avoided, but there was a consequent lack of transparency in the calculations. The remarks of this and the previous section are intended to elucidate the purely numerical results of Ref. 1.

As in the previous section, the mass-perturbation operator is assumed to be of purely octet type, so that the degenerate B, P, and B^* masses are given by $M=\frac{1}{2}(M_\Lambda+M_\Sigma)$, $m^2=\frac{1}{2}(m_\eta{}^2+m_\pi{}^2)$, $M^*=M_{Y_1{}^*}$. According to Eq. (3.8) the various mass shifts may be expressed in terms of matrix elements of irreducible tensor operators $T_\nu{}^{(\mu)}$, with $\nu=(I,I_3,Y)=(0,0,0)$. For example, for the baryon octet,

$$\delta M_\nu=(\phi_\nu{}^{(8)},\delta M\phi_\nu{}^{(8)})=\sum_{\mu\gamma}\delta M^{(\mu\gamma)}\begin{pmatrix}8 & \mu & 8\gamma\\ \nu & 0 & \nu\end{pmatrix}. \quad (5.1)$$

We consult the tables of de Swart for the coefficients of Eq. (5.1), and invert the equation to obtain the following expressions:

$$\begin{aligned}
&\delta M^{(1)}=\tfrac{1}{8}[2\delta M_N+\delta M_\Lambda+3\delta M_\Sigma+2\delta M_\Xi],\\
&\delta M^{(8_s)}=(1/\sqrt{5})[-\delta M_N-\delta M_\Lambda+3\delta M_\Sigma-\delta M_\Xi],\\
&\delta M^{(8_a)}=\delta M_N-\delta M_\Xi,\\
&\delta M^{(27)}=(9/8\sqrt{5})[2\delta M_N-3\delta M_\Lambda-\delta M_\Sigma+2\delta M_\Xi], \quad (5.2)\\
&\delta m^{2(1)}=\tfrac{1}{8}[4\delta m_K{}^2+\delta m_\eta{}^2+3\delta m_\pi{}^2],\\
&\delta m^{2(8_s)}=(1/\sqrt{5})[-2\delta m_K{}^2-\delta m_\eta{}^2+3\delta m_\pi{}^2],\\
&\delta m^{2(27)}=(9/8\sqrt{5})[4\delta m_K{}^2-3\delta m_\eta{}^2-\delta m_\pi{}^2].
\end{aligned}$$

The subscripts s and a refer, respectively, to the symmetric and antisymmetric octet representations. The perturbations of the 27-type are small, and by defining the degenerate masses as we have, the singlet-type perturbations are also made small.

Similar mass-shifts characteristic of the representations **1**, **8**, **27**, and **64** may be introduced to describe the decuplet mass shifts (these are the representations in $10\otimes\bar{1}0$).

$$\begin{aligned}
\delta M_{N^*}&=\delta M_*{}^{(1)}+\frac{1}{2\sqrt{2}}\delta M_*{}^{(8)}+\frac{1}{3\sqrt{7}}\delta M_*{}^{(27)}+\frac{1}{8\sqrt{14}}\delta M_*{}^{(64)},\\
\delta M_{Y_1{}^*}&=\delta M_*{}^{(1)}\qquad\qquad -\frac{5}{9\sqrt{7}}\delta M_*{}^{(27)}-\frac{1}{2\sqrt{14}}\delta M_*{}^{(64)},\\
\delta M_{\Xi^*}&=\delta M_*{}^{(1)}-\frac{1}{2\sqrt{2}}\delta M_*{}^{(8)}-\frac{1}{3\sqrt{7}}\delta M_*{}^{(27)}+\frac{3}{4\sqrt{14}}\delta M_*{}^{(64)},\\
\delta M_\Omega&=\delta M_*{}^{(1)}-\frac{1}{\sqrt{2}}\delta M_*{}^{(8)}+\frac{1}{\sqrt{7}}\delta M_*{}^{(27)}-\frac{1}{2\sqrt{14}}\delta M_*{}^{(64)}.
\end{aligned} \quad (5.3)$$

From Eq. (2.34) it follows that

$$\begin{aligned} -J_w(M^*)\delta M_*{}^{(1)} &= \Delta^{(1)}, \\ -J_w(M^*)\delta M_*{}^{(8)} &= (1/\sqrt{10})\Delta^{(8_s)}+(1/\sqrt{2})\Delta^{(8_a)}, \\ -J_w(M^*)\delta M_*{}^{(27)} &= (\sqrt{35}/15)\Delta^{(27)}, \\ \delta M_*{}^{(64)} &= 0 \end{aligned} \tag{5.4}$$

where

$$\Delta^{(\mu\gamma)} = J_M(M^*)\delta M^{(\mu\gamma)}+J_{m^2}(M^*)(\delta m^2)^{(\mu\gamma)}. \tag{5.5}$$

If $\Delta^{(27)}=0$, as is nearly the case experimentally, then the GMO formula holds for the B octet. In that case, Eqs. (5.4) show that the GMO formula will also hold for the model of the decuplet B^*, since only $\delta M_*{}^{(1)}$ and $\delta M_*{}^{(8)}$ are nonzero.

We now turn to the sum rules for the B^*BP coupling constants of the model. In the lowest order of mass differences the coupling of $B_{I,Y}{}^*$ to B_iP_i is the ith component of the vector $\gamma^{1/2}\Psi$, where γ is given by Eq. (2.53), and $\Psi=\Psi_0+\Psi_1$ is defined by Eqs. (2.30) and (2.35). As usual, we assume that the mass-perturbation operator Δ is a $T_0{}^{(8)}$ operator, in which case Ψ_1 becomes

$$\Psi_1 = \sum_{\beta=27,8_s,8_a} \Psi_0{}^{\beta}\frac{(\Psi_0{}^{\beta},\Delta\Psi_0{}^{10})}{\lambda_{10}{}^{-1}-\lambda_{\beta}{}^{-1}}. \tag{5.6}$$

The sum over β includes only **27** and **8**, since these are the only representations in $8\otimes 8$ which are also in $8\otimes 10$ ($\Delta\Psi_0{}^{10}$ contains only the representations in $8\otimes 10$). By the Wigner-Eckart theorem [Eq. (3.3)] we have

$$(\Psi_0{}^{\beta},\Delta\Psi_0{}^{10}) = \begin{pmatrix} 10 & 8 \\ IY & 00 \end{pmatrix}\Big|\begin{matrix}\beta \\ IY\end{matrix}\Big)(\beta\|\Delta\|10). \tag{5.7}$$

Also, the components of the Ψ_0's are just isoscalar factors:

$$\Psi_{0i}{}^{\beta} = \begin{pmatrix} 8 & 8 \\ (I_1Y_1)_i & (I_2Y_2)_i \end{pmatrix}\Big|\begin{matrix}\beta \\ IY\end{matrix}\Big). \tag{5.8}$$

We express the Ψ_1's in terms of the parameters

$$\mathcal{P}_{\beta} = \frac{(\beta\|\Delta\|10)}{\lambda_{10}{}^{-1}-\lambda_{\beta}{}^{-1}}. \tag{5.9}$$

The Wigner-Eckart theorem also allows us to give the I, Y dependence of $\gamma^{1/2}$ in terms of the single parameter

$$\mathfrak{D} = -\frac{1}{2J_0'}\left[\left(\frac{h'}{h}-\frac{J_0''}{J_0'}\right)(10\|\Delta\|10)+(10\|\Delta'\|10)\right]. \tag{5.10}$$

We state the sum rules in terms of the quantity

$$X_i = \Psi_i\gamma^{1/2}/\Psi_{0i}\gamma_0{}^{1/2}, \tag{5.11}$$

i.e., the ratio of the coupling constant to its degenerate limit. To make correspondence with the work of Gupta and Singh[19] we define some additional parameters:

$$\begin{aligned} p &= \tfrac{3}{5}\mathcal{P}_{27}, & q &= \tfrac{1}{5}\mathcal{P}_{8_s}, \\ r &= -(1/\sqrt{5})\mathcal{P}_{8_a}, & s &= -(1/2\sqrt{2})\mathfrak{D}. \end{aligned} \tag{5.12}$$

The X's are expressed linearly in terms of these parameters:

$$\begin{aligned} X(N^*N\pi) &= 1+(5/4)p-s, \\ X(N^*\Sigma K) &= 1-(5/4)p-s, \\ X(Y_1{}^*\Lambda\pi) &= 1+p+2q, \\ X(Y_1{}^*\Sigma\pi) &= 1-2r, \\ X(Y_1{}^*\Sigma K) &= 1+p-3q+r, \\ X(Y_1{}^*\Sigma\eta) &= 1-p-2q, \\ X(Y_1{}^*\Xi K) &= 1-p+3q+r, \\ X(\Xi^*\Xi\pi) &= 1+\tfrac{1}{4}p+3q-r+s, \\ X(\Xi^*\Lambda\bar{K}) &= 1+\tfrac{3}{4}p-q+r+s, \\ X(\Xi^*\Sigma\bar{K}) &= 1-\tfrac{1}{4}p-3q-r+s, \\ X(\Xi^*\Xi\eta) &= 1-\tfrac{3}{4}p+q+r+s, \\ X(\Omega^-\Xi\bar{K}) &= 1+2s. \end{aligned} \tag{5.13}$$

These equations agree with the equations of Gupta and Singh if their parameter a is put equal to 1; a is merely an over-all scale factor which does not affect sum rules. The sum rules stated by Gupta and Singh are obtained by elimination of p, q, r, and s in Eq. (5.13). The Gupta-Singh X's are not precisely the same quantities as ours, but they satisfy the same sum rules.

There is a rough consistency of the sum rules from (5.13) with the coupling constants computed in Ref. 1, for values of the symmetry perturbation strength x at or near its maximum value of 1. If the sum rules are written as $\sum\lambda_iX_i=0$, then $|\sum\lambda_iX_i/\delta\bar{X}|$ is about 0.25 at worst, where $\delta\bar{X}$ is an average magnitude of perturbation of the X's. This degree of failure of the sum rules seems consistent with the magnitude of higher order mass perturbations reported in the previous section. As in the case of the mass sum rule, the failure is perhaps less than one would have guessed from Table I.

To make a comparison with the nonperturbative calculation of Ref. 1 we compute the first-order coupling-constant perturbations with exactly[20] the dynamical model used in Ref. 1. The double-bar matrix elements involved are as follows:

$$\begin{aligned} (27\|\Delta\|10) &= -(1/\sqrt{5})\Delta^{(8_s)}-+\tfrac{1}{3}\Delta^{(8_a)}, \\ (8_s\|\Delta\|10) &= -(1/\sqrt{5})\Delta^{(8_s)}-\tfrac{1}{2}\Delta^{(8_a)}, \\ (8_a\|\Delta\|10) &= \tfrac{1}{2}\Delta^{(8_s)}, \\ (10\|\Delta\|10) &= (1/\sqrt{10})\Delta^{(8_s)}+(1/\sqrt{2})\Delta^{(8_a)}. \end{aligned} \tag{5.14}$$

[19] V. Gupta and V. Singh, Phys. Rev. **135**, B1442 (1964). E. Johnson and E. R. McCliment, *ibid.* **139**, B951 (1965), have derived the Gupta-Singh sum rules in a dynamical model based on third-order perturbation theory.

[20] As we remarked in the last section, the model of Ref. 1 uses $M=M_\Lambda$ and $m=m_\eta$ as the unperturbed masses, instead of the values stated above, which are appropriate to a pure octet mass shift operator.

The quantities $\Delta^{(8_s)}$ and $\Delta^{(8_a)}$ are defined in (5.5); $\Delta^{(8_s)-}$ differs from $\Delta^{(8_s)}$ by a minus sign in the meson term:

$$\Delta^{(8_s)-}=J_M(M^*)\delta M^{(8_s)}-J_{m^2}(M^*)(\delta m^2)^{(8_s)}. \quad (5.15)$$

With numerically computed values of (5.9), we use (5.6) to obtain the first-order coupling-constant vectors.

$$\begin{aligned}\Psi(N^*)&=0.98\Psi^{10}+0.162\Psi^{27},\\ \Psi(Y^*)&=0.99\Psi^{10}+0.123\Psi^{27}\\ &\quad -0.014\Psi^{8_s}+0.011\Psi^{8_a}, \quad (5.16)\\ \Psi(\Xi^*)&=0.997\Psi^{10}+0.075\Psi^{27}-0.014\Psi^{8_s}+0.011\Psi^{8_a}.\end{aligned}$$

We give the corresponding nonperturbative expressions of Ref. 1 for comparison.

$$\begin{aligned}\Psi(N^*)&=0.98\Psi^{10}+0.18\Psi^{27},\\ \Psi(Y^*)&=0.99\Psi^{10}+0.13\Psi^{27}+0.0007\Psi^{8_s}\\ &\quad +0.008\Psi^{8_a}+0.003\Psi^{\bar{1}\bar{0}}, \quad (5.17)\\ \Psi(\Xi^*)&=0.997\Psi^{10}+0.07\Psi^{27}+0.001\Psi^{8_s}+0.007\Psi^{8_a}.\end{aligned}$$

The first-order theory does rather well in predicting the amplitude of the strong **27** component, but its account of the weaker components is poor. In the amplitudes for the weaker components there are near cancellations between several large terms. This may account for the difference between the exact and first-order results. The large magnitude of the **27** component can be understood easily in the first-order theory. In the N^* channel the amplitude of the **27** term is

$$\frac{(\Psi_0{}^{27},\Delta\Psi_0{}^{10})}{\lambda_{10}{}^{-1}-\lambda_{27}{}^{-1}}=\frac{\frac{1}{2}(\Delta^{N\pi}-\Delta^{\Sigma K})}{\lambda_{10}{}^{-1}-\lambda_{27}{}^{-1}}, \quad (5.18)$$

where

$$\Delta^{BP}=J_M\delta M^B+J_{m^2}\delta m^{2P}. \quad (5.19)$$

The N and π masses differ greatly from the unperturbed masses M_Λ and m_η, while Σ and K masses are close to the unperturbed values. Thus, $\Delta^{N\pi}-\Delta^{\Sigma K}$ is relatively large. The factor $(\lambda_{10}{}^{-1}-\lambda_{27}{}^{-1})^{-1}$ is also larger than corresponding factors for other representations. This means that of the various nonresonant representations, the **27** is the closest to resonance. For an F/D ratio corresponding to $f=0.326$ (notation of Martin and Wali[21]) we have $(\lambda_{27}{}^{-1}-\lambda_{10}{}^{-1})^{-1}=1.58g^2/4\pi$, $(\lambda_{8_s}{}^{-1}-\lambda_{10}{}^{-1})^{-1}=-1.014g^2/4\pi$, $(\lambda_{8_a}{}^{-1}-\lambda_{10}{}^{-1})^{-1}=-0.675g^2/4\pi$. Given the **27** amplitude in the N^* state, the **27** amplitudes in the Y^* and Ξ^* states of Eq. (5.16) are determined by Clebsch-Gordan coefficients alone.

The dominant **10–27** mixing helps account for the small experimental branching ratio $R=(Y_1^*\to\Sigma+\pi)/(Y_1^*\to\Lambda+\pi)$. If the pure $SU(3)$ prediction for this ratio is corrected only by centrifugal barrier factors, $R=0.16$ is obtained. The experimental upper limit for R is about 0.05. Equations (5.17) lead to $R=0.07$ (footnote 20 of Ref. 1). This decrease is mainly due to the fact that Ψ^{27} contains no Σ-π component, while the $\Lambda\pi$ components of Ψ^{10} and Ψ^{27} add.

In view of the disagreement between Eqs. (5.16) and Eqs. (5.17), it is desirable to find a simple check of Eqs. (5.17), since the latter were obtained from a relatively complicated computer program. Such a check is provided by first-order perturbation theory, but with a different choice of the perturbation operator. We write Eq. (2.27) in the form

$$[\Lambda^{-1}+J_0(w)+U^T\Delta J(w)U]\Psi=0. \quad (5.20)$$

The energy w at which this equation is satisfied is already known from the complete calculation. We solve for Ψ by first-order perturbation theory with $U^T\Delta J(w)U$ as the perturbation. This amounts to replacing Δ by $\Delta J(w)$ in Eq. (2.35). The results are closer to the exact equation (5.17) in the sense that 8_s is suppressed relative to 8_a:

$$\begin{aligned}\Psi(N^*)&=0.98\Psi^{10}+0.188\Psi^{27},\\ \Psi(Y^*)&=0.99\Psi^{10}+0.128\Psi^{27}+0.0004\Psi^{8_s}\\ &\quad +0.013\Psi^{8_a}-0.002\Psi^{\bar{1}\bar{0}}, \quad (5.21)\\ \Psi(\Xi^*)&=0.997\Psi^{10}+0.072\Psi^{27}-0.0016\Psi^{8_s}+0.011\Psi^{8_a}.\end{aligned}$$

In Ref. 1 it was noted that the calculated coupling constants did not agree with the sum rules of Sudarshan.[22] The reason for the disagreement is merely that the coupling constants of Ref. 1 were not calculated with sufficient numerical precision. The Sudarshan sum rules contain a large number of terms, so that the accumulated roundoff error becomes troublesome. These sum rules, although stated in terms of squared coupling constants, are equivalent in first order to linear combinations of the Gupta-Singh[19] sum rules. The connection between linear and squared sum rules is trivial. The linear rule has the form $\sum\lambda_iX_i=0$, where the X_i are all equal to 1 in the degenerate limit. Thus, if $X_i=1+\delta X_i$ we have $\sum\lambda_i\delta X_i=0$ and $\sum\lambda_i=0$. It follows that $\sum\lambda_iX_i{}^2=0$ is correct to first order.

The coupling constant G_{B^*BP} defined via a Lagrangian in Ref. 1 is related to the $\gamma^{1/2}\Psi$ that we have been discussing by the following equation:

$$\frac{G_{B^*(BP)_i}{}^2}{4\pi}=\frac{12M_{B^*}{}^2}{M_B{}^*+M_B-m_P}\gamma\Psi_i{}^2\beta_i. \quad (5.22)$$

Here β_i is an isotopic spin factor, which may be looked up in Eq. (7) of Martin and Wali.[13] One may show that the G's (divided by their symmetric limits) satisfy the sum rules as well as the X's, provided that the factor multiplying $\gamma\Psi_i{}^2\beta_i$ in (5.22) is expanded to first order in mass differences.

[21] A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963).

[22] E. C. G. Sudarshan, *Proceedings of the Topical Conference on Recently Discovered Resonance Particles* (Ohio University, Athens, Ohio, 1963); this paper contains a revision of the work of C. Dullemond, A. J. MacFarlane, and E. C. G. Sudarshan, Phys. Rev. Letters **10**, 423 (1963).

(b) The B Octet

Calculations for the $\frac{1}{2}^+$ octet reported in the following are all to first order in the mass differences. At present there are no exact evaluations with which to compare the first-order results.

The constituent B and P masses are given by GMO formulas, with parameters chosen to fit the observed masses as well as possible. As in Sec. 4, the unperturbed masses are $M=\frac{1}{2}(M_\Lambda+M_\Sigma)=8.255m_\pi$ and $m^2=\frac{1}{2}(m_\eta{}^2+m_\pi{}^2)=(2.88m_\pi)^2$. The exchanged decuplet mass is $M=11.0m_\pi$ which is somewhat larger than the degenerate decuplet mass $M=M_{Y_1{}^*}=9.92m_\pi$ appropriate to a pure octet mass-perturbation operator. The M value was chosen to avoid an overlap of left and right singularities, as explained in Sec. 4. The matrix N_0 depends on a single coupling constant $G_{B^*BP}{}^2$. In addition, we have as free parameters the subtraction point $\hat{w}$ and the cutoff energy a of Eq. (2.11). The B^*BP coupling constant was given the value obtained in the pure symmetry limit of the model of Ref. 1; viz., $G_{B^*BP}{}^2/4\pi=0.09$. As in the decuplet case, the average level spacing depends strongly on $\hat{w}$. On the other hand, the average mass of the multiplet depends most strongly on a. By fitting one of the observed level spacings the subtraction point is essentially fixed. With $\hat{w}=3.0m_\pi$ and $a=7.25m_\pi$ we obtain the following bound state masses:

$$M_N=6.97m_\pi,\quad M_\Lambda=7.81m_\pi,$$
$$M_\Sigma=8.67m_\pi,\quad M_\Xi=9.08m_\pi,$$
$$M=\tfrac{1}{2}(M_\Lambda+M_\Sigma)=8.25.$$

The corresponding experimental values are

$$M_N=6.72m_\pi,\quad M_\Lambda=7.98m_\pi,$$
$$M_\Sigma=8.53m_\pi,\quad M_\Xi=9.51m_\pi.$$

The quoted bound state masses obey the GMO rule, since they were computed in first order.

We now turn to the coupling-constant perturbations. As we remarked in Sec. 2, the N_0 matrix for the octet problem is diagonalized in a basis which involves a definite linear combination of the 8_a and 8_s states. The state labeled 8_1 in Eq. (2.21) corresponds to the B-P bound state; i.e., $\lambda_{8_1}{}^{-1}+J_0(M)=0$, where M is the unperturbed bound-state energy. The unperturbed coupling-constant vector is

$$\begin{aligned}&\Psi_0{}^{8_1}=(\cos\theta)\Psi_0{}^{8_s}-(\sin\theta)\Psi_0{}^{8_a},\\&\tan\theta=(\sqrt{5})/(\sqrt{6}+1),\\&\theta=33^\circ,\end{aligned}\tag{5.23}$$

which corresponds to a D/F mixing parameter[21] $f=0.326$ for the coupling of the composite baryon to its constituents. Of course, this particular mixture is specific to the decuplet exchange model, but it is consistent with values obtained from other considerations [limited empirical evidence and $U(6)$ theory]. As Martin and Wali showed,[21] the choice $f=0.326$ guarantees that only the **10** representation resonates in the octet-exchange model of the decuplet. As far as the BBP coupling-constant sum rules are concerned, the value of θ may be arbitrary. The first-order perturbation of the coupling-constant vector is

$$\Psi_1=\sum_{\beta=27,10\ \bar{1}\bar{0},8_2,1}\Psi_0{}^\beta\frac{(\Psi_0{}^\beta,\Delta\Psi_0{}^{8_1})}{\lambda_{8_1}{}^{-1}-\lambda_\beta{}^{-1}}.\tag{5.24}$$

We define the vector X with components X_i in terms of $\Psi=\Psi_0+\Psi_1$:

$$X_i=\left[1-\frac{1}{2J_0'}\left(\frac{h'}{h}-\frac{J_0''}{J_0'}\right)(\Psi_0{}^{8_1},\Delta\Psi_0{}^{8_1})-\frac{1}{2J_0'}(\Psi_0{}^{8_1},\Delta'\Psi_0{}^{8_1})\right]\frac{\Psi_i}{\Psi_{0i}{}^{8_s}}.\tag{5.25}$$

For $X(\Sigma\Sigma\pi)$ this expression makes no sense, because $\Psi_{0i}{}^{8_s}$ vanishes. In this case $\Psi_{0i}{}^{8_s}$ is replaced by $\sqrt{(6/5)}$. It should be noted that the X_i do not reduce to unity in the pure-symmetry limit. As in the decuplet case, we introduce parameters to describe the off-diagonal perturbations. In the notation of Eq. (3.3),

$$\mathcal{P}_{\beta^{(\gamma)}}=\frac{(\beta\|\Delta\|8_1)_\gamma}{\lambda_{8_1}{}^{-1}-\lambda_\beta{}^{-1}},\qquad\beta=27,\ 10,\ \bar{1}\bar{0},\ 8_2,\ 1.\tag{5.26}$$

In the case of $\beta=8_2$, γ takes on two values, 1 and 2. There are two more parameters corresponding to the diagonal perturbations:

$$\mathfrak{D}(\gamma)=-\frac{1}{2J_0'}\left[\left(\frac{h'}{h}-\frac{J_0''}{J_0'}\right)(8_1\|\Delta\|8_1)_\gamma+(8_1\|\Delta'\|8_1)_\gamma\right],\qquad\gamma=1,2.\tag{5.27}$$

Finally, there is the 8_a–8_s mixing angle θ, so there are nine parameters in all. We introduce the following convenient constants:

$$\begin{aligned}&a=\cos\theta;\quad a'=-(\sqrt{5}/3)\sin\theta,\\&b=(1/\sqrt{5})(\cos\theta)\mathfrak{D}^{(1)}+\tfrac{1}{3}(\sin\theta)\mathcal{P}_{8_2}{}^{(1)},\\&b'=(-1/\sqrt{5})(\sin\theta)\mathfrak{D}^{(1)}+\tfrac{1}{3}(\cos\theta)\mathcal{P}_{8_2}{}^{(1)},\\&c=\tfrac{1}{2}(\cos\theta)\mathfrak{D}^{(2)}+(\sqrt{5}/6)(\sin\theta)\mathcal{P}_{8_2}{}^{(2)},\\&c'=-\tfrac{1}{2}(\sin\theta)\mathfrak{D}^{(2)}+(\sqrt{5}/6)(\cos\theta)\mathcal{P}_{8_2}{}^{(2)},\\&p=(\sqrt{5}/10)\mathcal{P}_{27};\quad q=(\sqrt{5}/12)\mathcal{P}_{\bar{1}\bar{0}},\\&r=(\sqrt{5}/12)\mathcal{P}_{10};\quad s=(\sqrt{5}/8)\mathcal{P}_1.\end{aligned}\tag{5.28}$$

The X's come out as follows:

$$
\begin{aligned}
X(NN\pi)&=(a-\tfrac{1}{2}b+c)-(a'-\tfrac{1}{2}'b'+c')-p-2q,\\
X(NN\eta)&=(a-\tfrac{1}{2}b+c)+3(a'-\tfrac{1}{2}b'+c')+9p-6q,\\
X(N\Lambda K)&=(a-\tfrac{1}{2}b+c)-3(a'-\tfrac{1}{2}b'+c')+9p+6q,\\
X(N\Sigma K)&=(a-\tfrac{1}{2}b+c)+(a'-\tfrac{1}{2}b'+c')-p+2q,\\
X(\Sigma\Lambda\pi)&=(a+b)-3p-3q-3r,\\
X(\Sigma\Sigma\pi)&=-(a'+b')-q+r,\\
X(\Sigma N\bar{K})&=(a+b)+(a'+b')+2p-2q+2r,\\
X(\Sigma\Sigma\eta)&=(a+b)-3p+3q+3r,\\
X(\Sigma\Xi K)&=(a+b)-(a'+b')+2p+2q-2r,\\
X(\Lambda\Sigma\pi)&=(a-b)-\tfrac{3}{4}p-s,\\
X(\Lambda NK)&=(a-b)-3(a'-b')-\tfrac{9}{2}p+2s,\\
X(\Lambda\Lambda\eta)&=(a-b)+(27/4)p+s,\\
X(\Lambda\Xi K)&=(a-b)+3(a'-b')-\tfrac{9}{2}p+2s,\\
X(\Xi\Xi\pi)&=(a-\tfrac{1}{2}b-c)+(a'-\tfrac{1}{2}b'-c')-p-2r,\\
X(\Xi\Lambda K)&=(a-\tfrac{1}{2}b-c)+3(a'-\tfrac{1}{2}b'-c')+9p+6r,\\
X(\Xi\Sigma\bar{K})&=(a-\tfrac{1}{2}b-c)-(a'-\tfrac{1}{2}b'-c')-p+2r,\\
X(\Xi\Xi\eta)&=(a-\tfrac{1}{2}b-c)-3(a'-\tfrac{1}{2}b'-c')+9p-6r.
\end{aligned}
\tag{5.29}
$$

The numerical evaluation of the coupling-constant perturbations has been carried out on the basis of the model described above. The integrals involved in the $\mathcal{P}$'s and $\mathfrak{D}$'s have the following values (all quantities with the dimensions of mass are expressed in terms of $m_\pi=139.63$ MeV):

$$
\begin{aligned}
&J_0=-3.232, \quad J_M=0.5668, \quad J_{m^2}=\ 0.08536,\\
&J_0'=-0.8225, \quad J_M'=0.1795, \quad J_{m^2}'=\ 0.02743,\\
&J_0''=-0.122, \quad h=0.698, \quad h'=-0.0995.
\end{aligned}
\tag{5.30}
$$

The empirical values of the baryon and meson masses are given closely by the following octet-type perturbations:

$$
\delta M^{(8_s)}=0.615;\ \delta M^{(8_a)}=-2.79;\ (\delta m^2)^{(8_s)}=-16.2. \tag{5.31}
$$

TABLE II. Numerical values of $X(B',BP)$. The Lagrangian coupling constant $g_{B'BP}$ is given by Eq. (5.36). $X_0(B',BP)$ is the unperturbed value of X.

$B'BP$	X_0	X	$B'BP$	X_0	X
$NN\pi$	1.24	1.24	$\Lambda\Sigma\pi$	0.84	0.84
$NN\eta$	−0.38	−0.29	$\Lambda N\bar{K}$	2.06	2.06
$N\Lambda K$	2.06	1.89	$\Lambda\Lambda\eta$	0.84	0.81
$N\Sigma K$	0.43	0.42	$\Lambda\Xi K$	−0.38	−0.36
$\Sigma\Lambda\pi$	0.84	0.81	$\Xi\Xi\pi$	0.43	0.36
$\Sigma\Sigma\pi$	0.41	0.45	$\Xi\Lambda\bar{K}$	−0.38	−0.29
$\Sigma N\bar{K}$	0.43	0.51	$\Xi\Sigma\bar{K}$	1.24	1.32
$\Sigma\Sigma\eta$	0.84	0.91	$\Xi\Xi\eta$	2.06	1.92
$\Sigma\Xi K$	1.24	1.16			

From these figures we obtain

$$
\begin{aligned}
\Delta^{(8_s)}&=J_M\delta M^{(8_s)}+J_{m^2}(\delta m^2)^{(8_s)}=-1.03,\\
\Delta^{(8_s)-}&=J_M\delta M^{(8_s)}-J_{m^2}(\delta m^2)^{(8_s)}=1.73,\\
\Delta^{(8_a)}&=J_M\delta M^{(8_a)}=-1.58,\\
\Delta'^{(8_s)}&=-0.335,\\
\Delta'^{(8_s)-}&=+0.555,\\
\Delta'^{(8_a)}&=-0.501.
\end{aligned}
\tag{5.32}
$$

The necessary double-bar matrix elements are the following:

$$
\begin{aligned}
(27\|\Delta\|8_1)&=-\tfrac{1}{5}(\cos\theta)\Delta^{(8_s)}-\tfrac{1}{3}(\sin\theta)\Delta^{(8_a)}=0.461,\\
(10\|\Delta\|8_1)&=\tfrac{2}{5}(\cos\theta)\Delta^{(8_s)}-(1/\sqrt{5})\\
&\quad\times[(\cos\theta)\Delta^{(8_a)}+(\sin\theta)\Delta^{(8_s)}]=0.503,\\
(10\|\Delta\|8_1)&=\tfrac{2}{5}(\cos\theta)\Delta^{(8_s)}+(1/\sqrt{5})\\
&\quad\times[(\cos\theta)\Delta^{(8_a)}+(\sin\theta)\Delta^{(8_s)}]=-1.19,\\
(1\|\Delta\|8_1)&=-(\cos\theta)\Delta^{(8_s)}+(\sin\theta)\Delta^{(8_a)}=0,\\
(8_2\|\Delta\|8_1)_1&=\tfrac{1}{2}(\cos 2\theta)\Delta^{(8_a)}-\tfrac{2}{5}(\sin 2\theta)\Delta^{(8_s)}=0.056,\\
(8_2\|\Delta\|8_1)_2&=\tfrac{1}{2}(\cos 2\theta)\Delta^{(8_s)-}=0.353,\\
(8_1\|\Delta\|8_1)_1&=-\tfrac{1}{10}(4\cos 2\theta-1)\Delta^{(8_s)}\\
&\quad-\tfrac{1}{2}(\sin 2\theta)\Delta^{(8_a)}=+0.787,\\
(8_1\|\Delta\|8_1)_2&=\tfrac{1}{2}\Delta^{(8_a)}-\tfrac{1}{2}(\sin 2\theta)\Delta^{(8_s)-}=-1.58,\\
(8_1\|\Delta'\|8_1)_1&=+0.249,\\
(8_1\|\Delta'\|8_1)_2&=-0.504.
\end{aligned}
\tag{5.33}
$$

The other quantities needed are the perturbation denominators and the combination of integrals that appears in the diagonal perturbation:

$$
\begin{aligned}
&(\lambda_{27}^{-1}-\lambda_{8_1}^{-1})^{-1}=0.0332,\\
&(\lambda_{\overline{10}}^{-1}-\lambda_{8_1}^{-1})^{-1}=(\lambda_{10}^{-1}-\lambda_{8_1}^{-1})=0.127,\\
&(\lambda_1^{-1}-\lambda_{8_1}^{-1})^{-1}=-0.184,\\
&(\lambda_{8_2}^{-1}-\lambda_{8_1}^{-1})^{-1}=-0.0920,\\
&\frac{1}{2}\left(\frac{h'}{h}-\frac{J_0''}{J_0'}\right)=-0.1423.
\end{aligned}
\tag{5.34}
$$

From Eqs. (5.33), (5.34), (5.26), and (5.27) we obtain the parameters of Eq. (5.28):

$$
\begin{aligned}
a&=\ 0.839, \quad a'=-0.406,\\
b&=+0.0068, \quad b'=-0.0023,\\
c&=-0.0075, \quad c'=+0.0193,\\
p&=-0.00342, \quad q=-0.0119,\\
r&=+0.0282, \quad s=0.
\end{aligned}
\tag{5.35}
$$

Substitution in Eqs. (5.29) leads to the X's listed in Table II. In Table II we use the notation $X(B'BP)$ for the constant related to the B' pole residue in the B-P scattering amplitude. The vertex symmetry $X(B'BP)$

$=X(BB'P)$ should hold in a correct theory. It does not hold in our model, according to Table II.[22a] The reason for lack of vertex symmetry is our omission of B^* baryons as external lines in the ND^{-1} procedure. To illustrate the point we consider a Lagrangian density involving only the interactions $\Lambda\Sigma\pi$, $Y^*\Sigma\pi$, $Y^*\Lambda\pi$. The third-order graphs that contribute to $X(\Lambda'\Sigma\pi)$ are shown in Fig. 1. Our computation of $X(\Lambda'\Sigma\pi)$ amounts to a calculation of the third-order Y^* exchange graph by a dispersion method. The Λ exchange graph is not included in our work; it would come in if the Y^* state were allowed as one of the channels in the ND^{-1} method. However, both of the graphs of Fig. 1 are necessary for symmetry of the third-order vertex, since one of the graphs goes into the other on crossing of the Σ and Λ lines. Cutkosky and collaborators[11,23] have stressed the importance of vertex symmetry, and have discussed ways of incorporating it in dynamical models. Lin and Cutkosky[11] have remarked that a method based on an exact solution of the ND^{-1} integral equations (with the left-cut term given by single-particle exchanges) certainly spoils the vertex symmetry. That follows because the method effectively attaches an infinite string of ladder graphs running in the s-channel direction to the third-order graph (more precisely, it attaches an infinite string of fragments of ladder graphs; the method does not correspond exactly to any sum of Feynman graphs). Thus, a basic asymmetry between the s and u channels is introduced, with a consequent failure of vertex symmetry. A method like that of the present paper, where N is set equal to the Born matrix, will presumably give vertex symmetry if (i) both B and B^* baryons are included as external and exchanged particles; (ii) all mass perturbations are allowed (i.e., not only mass perturbations in the centrifugal barrier factor ρ); (iii) the cutoffs of the integrals corresponding to the graphs of Fig. 1 are handled appropriately. Whether items (i) through (iii) actually guarantee the symmetry is perhaps not completely clear without more investigation. An argument to the effect that they do guarantee it has been given by Lin and Cutkosky[11] in a static-model context. Unfortunately, one thing *is* clear; if N is set equal to the Born matrix with both B and B^* states as external lines, the resulting T-matrix will not be symmetrical. We were able to avoid this well-known difficulty in the present paper only because of the factorization of the N matrix expressed by Eq. (2.8). Of course, the factorization fails if there are two or more types of Born terms, or if arbitrary mass perturbations are allowed. The usual remedy for the T-matrix asymmetry is to use the exact solution of the ND^{-1} integral equation, but that leads to vertex asymmetry. It seems that

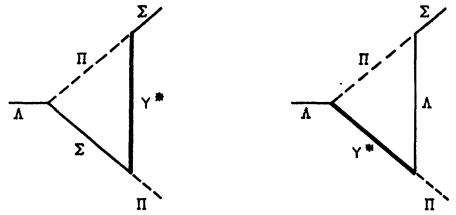

FIG. 1. Third-order vertex graphs which contribute to $g_{\Lambda\Sigma\pi}$.

no simple scheme based on single-particle exchanges and ND^{-1} methods can guarantee both T matrix and vertex symmetries. Furthermore, such schemes involve difficulties in meeting threshold conditions,[24] trouble with overlapping left and right singularities, and failure of crossing symmetry. Cutkosky and co-workers have taken the Bethe-Salpeter equation as a starting point for dynamical investigations of symmetries. Their method has the advantage of ensuring vertex symmetry. However, the bootstrap equations for the coupling constants that they finally work with are exactly the equations obtained from the ND^{-1} method (in the form $N=$Born matrix), ignoring the problem of T-matrix symmetry.

The connection of the X's with the usual coupling constants defined by a Lagrangian density is given by the formula

$$\frac{g_{B'BP}^2}{4\pi}=\frac{4\beta_{B'BP}M_B^2}{M_{B'}-M_B+m_P}\left(\frac{-h(M)}{J_0'(M)}\right)\times[\Psi_{0(B'BP)}{}^{8s}X_{(B'BP)}]^2, \quad (5.36)$$

where $g^2/4\pi$ is equal to the g^2 of Appendix I of Ref. 21. The isotopic spin factor $\beta_{B'BP}$ has the following values:

$$\begin{aligned}
\beta_{NN\pi}&=\tfrac{1}{3}, & \beta_{NN\eta}&=1,\\
\beta_{N\Lambda K}&=1, & \beta_{N\Sigma K}&=\tfrac{1}{3},\\
\beta_{\Sigma\Lambda\pi}&=1, & \beta_{\Sigma\Sigma\pi}&=\tfrac{1}{2},\\
\beta_{\Sigma N\bar{K}}&=\tfrac{1}{2}, & \beta_{\Sigma\Sigma\eta}&=1,\\
\beta_{\Sigma\Xi K}&=\tfrac{1}{2}, & \beta_{\Lambda\Sigma\pi}&=\tfrac{1}{3},\\
\beta_{\Lambda NK}&=\tfrac{1}{2}, & \beta_{\Lambda\Lambda\eta}&=1,\\
\beta_{\Lambda\Xi K}&=\tfrac{1}{2}, & \beta_{\Xi\Xi\pi}&=\tfrac{1}{3},\\
\beta_{\Xi\Lambda K}&=1, & \beta_{\Xi\Sigma K}&=\tfrac{1}{3},\\
&& \beta_{\Xi\Xi\eta}&=1.
\end{aligned}$$

With the numbers given by our model the $NN\pi$ coupling constant has the value

$$G_{NN\pi}{}^2/4\pi=36,$$

while the experimental value is about 15. The number obtained from the model naturally depends to some

[22a] *Note added in proof.* The failure of vertex symmetry does not appear to be very great in Table II. However, the discrepancy is much more marked if one looks at the coupling constants g^2 that appear in the Lagrangian, rather than the X's. [The relation between g^2 and X is given in Eq. (5.36).]

[23] R. E. Cutkosky and M. Leon, Phys. Rev. **135**, B1445 (1964).

[24] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963); A. W. Martin and J. L. Uretsky, *ibid.* **135**, B803 (1964).

extent on how the cutoff is treated. If we replace the cutoff function $\phi(w)$ of Eq. (2.11) by

$$\phi(w)=\left[\frac{(M+m)^2+a^2}{w^2+a^2}\right]^n,$$

where n is an integer greater than 1, the $NN\pi$ coupling constant is reduced. For example, if $n=2$ we find $a=25$ and $G_{NN\pi}{}^2/4\pi=26$, while with $n=3$ we have $a=35$ and $G_{NN\pi}{}^2/4\pi=23$. In both of these cases the choice $\hat{w}=3.5m_\pi$ for the subtraction point yields a baryon mass spectrum nearly the same as that reported above for $n=1$. In each of the cases $n=1$, 2, and 3, the poles of the cutoff function at $w=\pm ia$ are reasonably remote from the physical region and the baryon pole. In choosing the form of the cutoff function, our intention was to put its singularities far away from the physical region. In some preliminary work we tried the Abers-Zemach[25] cutoff function

$$\phi(w)=\left[1+\frac{w^2-(M+m)^2}{z^2}\right]^{-1},$$

which was used in Ref. 13. This turned out to be unsatisfactory, since the necessary value of z leads to poles of ϕ at $w=\pm 5.5$. The pole at $w=5.5$ is much too close to the nucleon bound-state pole. It results in an unreasonably large residue for the latter pole and a value of 150 for $G_{NN\pi}{}^2/4\pi$.

If we regard the composite baryon octet as being identical with the constituent baryon octet, then the model provides a way of trying out the idea of octet enhancement which was discussed in Sec. 3. The work of Dashen and Frautschi[4,7] on this question is already more complete than ours, in a sense, since their model is more elaborate. On the other hand, they use the pole approximation and the static limit, both of which we avoid. Their model has both B and B^* states as exchanged particles, and they allow mass perturbations for both external and exchanged particles. Since the B exchange term in the $J^P=\frac{1}{2}^+$ B-P scattering is quite small, the main difference between their treatment and ours has to do with perturbations of the exchanged particle masses.

The results of Dashen and Frautschi suggest that these perturbations are not entirely negligible, but that they are dominated by perturbations of the external particle masses, which are included in our work. Indeed, our neglect of exchanged particle mass perturbations in both Ref. 1 and the present paper was based on a knowledge of their secondary importance.

Although the Dashen-Frautschi model of the baryon octet is, in effect, similar to ours, we have serious doubts about the suitability of either model for investigation of the octet enhancement conjecture. A principal doubt arises from the necessity of including B^* states in external lines, if vertex symmetry is to be retained. Since some of the B^*-P thresholds are quite comparable to B-P thresholds, the undoubtedly important effects of mass perturbations in the external B^*-P lines could change the entire picture.

Our model results in the following expressions for the eigenvalues A^1 and A^{27} defined in Eq. (3.20).

$$\begin{aligned} A^1&=c\,,\\ A^{27}&=c(1/15)(4\cos 2\theta-1)\,,\\ c&=-J_M(M)/J_0'(M)\,. \end{aligned} \tag{5.37}$$

For the 2×2 matrix of Eq. (3.19) we have

$$A^{(8)}=c\begin{bmatrix} -\frac{1}{10}(4\cos 2\theta-1) & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \frac{1}{2} \end{bmatrix}, \tag{5.38}$$

where the first row and first column refer to 8_s. After determination of the eigenvalues of $A_\alpha{}^8$ and $A_\beta{}^8$ of the matrix $A^{(8)}$, a numerical evaluation yields $A^1=0.69$, $A_\alpha{}^8=0.52$, $A_\beta{}^8=-0.22$, $A^{27}=0.03$. Indeed, $A_\alpha{}^8$ is much greater than A^{27}, so the conjecture of octet enhancement is verified in this simple case.

6. CONCLUSIONS AND COMMENTS

We have studied the first-order mass difference expansions in some detail. In the decuplet model, the exact and first-order treatments of mass differences have been compared. The linear theory gave a good account of the dominant **27–10** mixing, but with regard to finer details it was not satisfactory. The linear theory seems to be useful only for a gross sketch of mass difference effects. This is made clear by the comparison of exact and first-order dispersion integrals in Sec. 4; the sum of higher order terms is often nearly as large as the first-order term. In particular, the linear theory does not explain the approximate propagation of the GMO rule obtained in the numerical calculations of Ref. 1. We hope that some work with the second-order theory will lead to a qualitative explanation of this phenomenon.

It would be highly desirable to improve the formulation of the dynamics in work of this type. We have seen that ND^{-1} methods based on single-particle exchanges do not allow both T-matrix symmetry and vertex symmetry. This is unfortunate, since the vertex symmetry seems especially important. In order to ensure it, one has to include channels which seem important *a priori*; e.g., the B^*-P channels in the model of the B octet.

The studies of octet enhancement carried out so far have involved a failure of vertex symmetry, and neglect of mass difference effects beyond the first order. We think that both of these shortcomings are rather serious. The octet enhancement work has also omitted coupling-constant perturbations up to now, and these could well be important.

[25] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963).

The lack of vertex symmetry is a feature of almost all N/D calculations published up to now. It is not usually noticed, since a coupling constant is usually computed in only one way. For instance, many people have computed the $N^*N\pi$ coupling constant as the residue of the N^* pole in the $N\pi$ amplitude, but calculations of the N-pole residue in the $N^*\pi$ amplitude are rare.

ACKNOWLEDGMENTS

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APPENDIX

Here we prove formulas for the derivatives with respect to M (or m^2) of the following principal-value integral, which was encountered in Sec. 2:

$$P\int_{M+m}^{\infty}\frac{\rho(w';M,m)h(w')dw'}{(w'-\hat{w})(w'-w)}. \tag{A1}$$

We consider the slightly more general problem of an integral of the type

$$K=P\int_{M+m}^{\infty}\frac{(w'-M-m)^{1/2}\alpha(w';M,m)dw'}{w'(w'-w)}. \tag{A2}$$

When M and m are near the points M_0 and m_0 at which the derivative is to be calculated, α is assumed to be analytic in w' in a neighborhood of the cut (M_0+m_0,∞). For $w'\geq M_0+m_0$, α is also required to be analytic in M and m^2 in neighborhoods of M_0 and m_0^2. The integral (A1) is a special case of (A2), as is almost any other partial-wave dispersion integral that one is likely to encounter. We first prove the formula for the first derivative of K:

$$\frac{\partial K}{\partial M}=P\int_{M+m}^{\infty}\frac{dw'}{w'(w'-w)}\frac{\partial}{\partial M}\times[(w'-M-m)^{1/2}\alpha(w';M,m)]. \tag{A3}$$

The proof is easiest for orbital angular momentum $l\geq 1$, in which case α has a zero at $w=M+m$. In that case all we have to do is turn the principal value integral of (A2) into an ordinary integral by subtraction of the pole:

$$K=\int_{M+m}^{\infty}\frac{(w'-M-m)^{1/2}\alpha(w';M,m)-(w-M-m)^{1/2}\alpha(w;M,m)}{w'(w'-w)}dw' +(w-M-m)^{1/2}\alpha(w;M,m)P\int_{M+m}^{\infty}\frac{dw'}{w'(w'-w)}. \tag{A4}$$

The second term of (A4) has the value

$$(w-M-m)^{1/2}\alpha(w;M,m)\frac{1}{w}\ln\frac{w-M-m}{M+m}. \tag{A5}$$

The derivative with respect to M of the integrand of the first term of (A4) is continuous as a function of w' and M, so the elementary rule for differentiating the integral with respect to M applies.[26] The derivative of the second term of (A4) is obtained from the formula (A5). Combining the results, we find Eq. (A3) to be correct. For s waves, the derivative of the integrand is not continuous in w' at $w'=M+m$, since $(w'-M-m)^{-1/2}\alpha(w';M,m)$ is infinite at that point. This case can be reduced to the one just treated by writing K in the form

$$K=P\int_{M+m}^{\infty}\frac{(w'-M-m)^{1/2}}{w'(w'-w)}[\alpha(w';M,m)-\alpha(M+m;M,m)]dw'+\alpha(M+m;M,m)P\int_{M+m}^{\infty}\frac{(w'-M-m)^{1/2}dw'}{w'(w'-w)}. \tag{A6}$$

The second term of (A6) has the value

$$\pi\alpha(M+m;M,m)(M+m)^{1/2}/w, \tag{A7}$$

so its derivative with respect to M exists. The integrand of the first term of (A6) satisfies the conditions of the theorem already proved, so we have

$$\frac{\partial K}{\partial M}=P\int_{M+m}^{\infty}\frac{dw'}{w'(w'-w)}\frac{\partial}{\partial M}\{(w'-M-m)^{1/2}[\alpha(w';M,m)-\alpha(M+m;M,m)]\} +\frac{\pi}{w}\frac{\partial}{\partial M}[\alpha(M+m;M,m)(M+m)^{1/2}]. \tag{A8}$$

[26] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1963); cf. Sec. 4.2.

The last term (A8) is cancelled by the second term in the integral, which exists by itself. Hence, we once again obtain (A3).

We can make successive use of (A3) to obtain higher derivatives of K up to an order which depends on l. In fact

$$\frac{\partial^{(n)}K}{\partial M^{(n)}}=P\int_{M+m}^{\infty}\frac{dw'}{w'(w'-w)}\frac{\partial^{(n)}}{\partial M^{(n)}}[(w'-M-m)^{1/2}\alpha(w';M,m)], \tag{A9}$$

provided $n\leq l+1$. For $n>l+1$ the formula fails, but in that case we can save the day by the trick used in (A6). For $n=l+1$, Eq. (A9) is just like the derivative of an s-wave dispersion integral; i.e., it has the form

$$\frac{\partial^{(l+1)}K}{\partial M^{(l+1)}}=P\int_{M+m}^{\infty}\frac{(w'-M-m)^{-1/2}\beta(w';M,m)}{w'(w'-w)}dw', \tag{A10}$$

where β satisfies the same analyticity conditions as α, and does not vanish at $w'=M+m$. We subtract and add a term to obtain

$$\frac{\partial^{(l+1)}K}{\partial M^{(l+1)}}=P\int_{M+m}\frac{(w'-M-m)^{-1/2}[\beta(w';M,m)-\beta(M+m;M,m)]}{w'(w'-w)}dw'-\pi\beta(M+m;M,m)(M+m)^{-1/2}/w. \tag{A11}$$

The integral of (A11) may be differentiated under the integral sign, as in (A3). To get the derivative of order $l+3$, we note that the derivative of the integral in (A11) has the form (A10) with some new function $\gamma(w';M,m)$ replacing β. Thus, the process may be repeated any number of times; derivatives of all orders exist. Of course, derivatives with respect to m^2, or cross derivatives between M and m^2, may be treated similarly. With numerical integration it would be best to use Eq. (A9) only up to $n=l$. One should compute the $(l+1)$th derivative from the lth only after subtracting and adding a term in the manner of Eq. (A6). This avoids numerical integration over the singularity of $(w'-M-m)^{-1/2}$. Numerical evaluation of principal-value integrals is conveniently done by means of an identity like (A4).

For the study of mass difference effects one also needs derivatives of K with respect to w. These can be obtained to any order by differentiating under the integral sign in the formula

$$K=\int_{M+m}^{\infty}\frac{(w'-M-m)^{1/2}[\alpha(w';M,m)-\alpha(w;M,m)]}{w'(w'-w)}dw'+\pi\alpha(w;M,m)(M+m)^{1/2}/w. \tag{A12}$$

For example, the integrand of the derivative of the first term of (A12) is

$$\frac{(w'-M-m)^{1/2}}{w'}\left[\frac{\alpha(w')-\alpha(w)}{w'-w}-\frac{\partial\alpha}{\partial w}(w)\right]\frac{1}{w'-w}, \tag{A13}$$

which is continuous in w' and w. Higher derivatives of this expression are also continuous in w' and w, so differentiation to any order under the integral sign is permitted. However, if derivatives of α can be obtained conveniently only by numerical methods, it may be simpler to do a direct numerical differentiation of K.

It is educational to look at a simple model of an s-wave dispersion integral. In the nonrelativistic approximation and with a single pole as the only left singularity of the amplitude we have

$$K=P\int_{M+m}^{\infty}(w'-M-m)^{1/2}\frac{dw'}{(w'+a)(w'-w)}=\frac{\pi(M+m+a)^{1/2}}{w+a},\quad a<M+m. \tag{A14}$$

If we write $M=M_0+\alpha x$, $m=m_0+\beta x$, where x is a mass perturbation strength, we see that K is analytic in x except for a branch point at $-(a+M_0+m_0)/(\alpha+\beta)$. Suppose $x=1$ corresponds to the physical values of the masses, and $x=0$ to the unperturbed masses. The power series in x about $x=0$ converges for

$$|x|<\frac{|a+M_0+m_0|}{|\alpha+\beta|}. \tag{A15}$$

The larger the distance from the unperturbed threshold to the left singularity $w=-a$, the larger the radius of convergence of the series. The radius of convergence can be large even when α and β are large, provided that they have opposite signs. Since the pole at $w=-a$ moves to the left as the range of the force between the particles is decreased, the radius of convergence goes to infinity in the limit of a zero-range force.