Complete Analytic Extension of the Symmetry Axis of Kerr's Solution of Einstein's Equations

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The 2-dimensional metric on the symmetry axis of the Kerr solution is examined and it is shown that in the form usually given it is incomplete when $a^2 \leqslant m^2$. The method developed by Kruskal for completing the Schwarzschild solution is adapted to the distinct cases $a^2 < m^2$ and $a^2 = m^2$. In each case a singularity-free metric is obtained which is periodic with respect to a timelike coordinate, and which is shown to be a complete analytic extension. The generalization to the full 4-dimensional Kerr solution is discussed, and finally the questions of uniqueness and causality are considered.

1. INTRODUCTION

It is remarkable that although the Schwarzschild metric was discovered nearly 50 years ago, no really determined effort to understand its full topology was made until quite recently. In 1960, after a partial solution of the problem by Finkelstein[1] in 1958, Kruskal[2] finally published the full solution which he had obtained some years earlier, while about the same time Fronsdal[3] solved the problem by a different method. One reason for the long delay was that many people thought that the region of space-time within the Schwarzschild horizon could have no physical meaning. As a result of the interest which has developed in problems of gravitational collapse it is now generally realized that this view was mistaken.

Most large stars have angular momentum per unit mass which is comparable in gravitational units with their mass, and this may well be true for the other types of body whose gravitational collapse is under discussion. The only presently known exact solution of the field equations which might be relevant to such a situation is that of Kerr[4] (1963). Consequently it seems that the clarification of the complete topology of the Kerr solution is worthwhile for the same reasons as applied to the Schwarzschild case.

In this paper the complete solution for the axis of symmetry is obtained, and the probable form of the full 4-dimensional solution is sketched. The methods of Finkelstein and Kruskal are used since these seem much more straightforward than the embedding technique of Fronsdal.

2. THE NECESSITY OF EXTENSION

The Kerr solution may be expressed in quasispheroidal coordinates as

$$ds^2 = -\left(1-\frac{2mr}{r^2+a^2\cos^2\theta}\right)dt^2 + 2\frac{2amr\sin^2\theta}{r^2+a^2\cos^2\theta}dt\,d\varphi + \sin^2\theta\left(r^2+a^2+\frac{2a^2mr\sin^2\theta}{r^2+a^2\cos^2\theta}\right)d\varphi^2 + (r^2+a^2\cos^2\theta)d\theta^2$$
$$+2\frac{2mr}{r^2+a^2}dr\,dt + 2\frac{2amr\sin^2\theta}{r^2+a^2}dr\,d\varphi + \frac{r^2+a^2\cos^2\theta}{r^2+a^2}\left(1+\frac{2mr}{r^2+a^2}\right)dr^2,$$

where, as was stated by Kerr[4] and demonstrated at length by Boyer and Price,[5] m is the mass and a is the angular momentum per unit mass as observed at large r. When $a=0$ this is the Schwarzschild solution, and when $m=0$, the two regions, r positive and r negative each represent a complete flat space in spheroidal-type coordinates, the two flat spaces being connected through the interior of a ring of radius a.

On the axis this reduces to the 2-dimensional form

$$ds^2 = -\left(1-\frac{2mr}{r^2+a^2}\right)dt^2 + 2\frac{2mr}{r^2+a^2}dr\,dt + \left(1+\frac{2mr}{r^2+a^2}\right)dr^2.$$

This may be equivalently expressed in terms of a null coordinate

$u=t+r$ as $ds^2 = -[1-2mr/(r^2+a^2)]du^2 + 2du\,dr$. (1)

This metric is analytic and nonsingular over the manifold $\mathfrak{M}(u,r)$ defined as the whole of the (u,r) plane.

The equations for a geodesic are

$$-\left(1-\frac{2mr}{r^2+a^2}\right)\frac{du}{d\lambda}+\frac{dr}{d\lambda}=-E,$$

$$-\left(1-\frac{2mr}{r^2+a^2}\right)\left(\frac{du}{d\lambda}\right)^2+2\frac{du}{d\lambda}\frac{dr}{d\lambda}=\epsilon,$$

[1] David Finkelstein, Phys. Rev. **110**, 965 (1958).
[2] M. D. Kruskal, Phys. Rev. **119**, 1743 (1960).
[3] C. Fronsdal, Phys. Rev. **116**, 778 (1959).
[4] Roy P. Kerr, Phys. Rev. Letters **11**, 238 (1963).
[5] R. H. Boyer and T. G. Price, Proc. Cambridge Phil. Soc. **61**, 531 (1965).

where ϵ is an indicator with value $+1$, 0, -1 in the spacelike, null, timelike cases, respectively, λ is an affine parameter—the metric parameter in the non-null cases —and E is a constant representing the energy in the timelike case.

They may be rewritten

$$dr/d\lambda = \pm\Delta(r)\,,$$

$$du/d\lambda = [1-2mr/(r^2+a^2)]^{-1}[E\pm\Delta(r)]\,, \qquad (2)$$

where

$$\Delta(r) = \{E^2+\epsilon[1-2mr/(r^2+a^2)]\}^{1/2}.$$

It can be seen that $dr/d\lambda$ is bounded. When $a^2>m^2$ the factor $[1-2mr/(r^2+a^2)]$ has no real zeros and so $du/d\lambda$ is bounded also. Hence each geodesic may be continued to infinite values of the affine parameter. Thus when $a^2>m^2$ the manifold $\mathfrak{M}(u,r)$ with metric (1) is geodesically complete.

From now on we shall consider the more interesting case when $0<a^2<m^2$. The factor $[1-2mr/(r^2+a^2)]$ now has two (positive) real zeros $r_\pm = m\pm(m^2-a^2)^{1/2}$.

In the null case, (2) can be solved explicitly. We may put $E=1$ without loss of generality by renormalizing the affine parameter, and we obtain two classes of solution:

$$u=C\,,\quad r=-\lambda\,,$$

and

$$u=C+F(r)\,,\quad r=\lambda\,,$$

where C is a constant and

$$F(r) = 2r + A^{-1}\ln|r-r_+| - B^{-1}\ln|r-r_-|\,, \qquad (3)$$

with

$$A^{-1} = 2m[(1-a^2/m^2)^{-1/2}+1]\,,$$

$$B^{-1} = 2m[(1-a^2/m^2)^{1/2}-1]$$

so that

$$dF(r)/dr = 2[1-2mr/(r^2+a^2)]^{-1}.$$

A and B are both positive. $F(r)$ is monotonic in each of the regions

$$\text{I:}\quad r>r_+>r_-\,,$$

$$\text{II:}\quad r_+>r>r_-\,,$$

$$\text{III:}\quad r_+>r_->r\,.$$

It diverges to $\mp\infty$ as $r\to r_\pm$.

Thus for one class of null geodesics u is unbounded as $r\to r_\pm$ even though λ tends to a finite limit. It can be seen that there will be classes of non-null geodesics behaving in a similar manner.

We conclude that the metric is geodesically incomplete as $r\to r_+$ and $u\to-\infty$ and as $r\to r_-$ and $u\to+\infty$.

3. SYMMETRIC DOUBLE-NULL COORDINATES

In order to proceed further we introduce a second null coordinate $w=F(r)-u$ and obtain the metric in the symmetric double-null form

$$ds^2 = [1-2mr/(r^2+a^2)]du\,dw\,, \qquad (4)$$

where r is uniquely determined in terms of u, w by

$$F(r) = u+w \qquad (5)$$

provided that one of the regions I, II, III is specified.

Thus the lines $r=r_+$, $r=r_-$ (i.e., $w=\pm\infty$) divide the manifold $\mathfrak{M}(u,r)$ into three open manifolds $\mathfrak{M}_{\rm I}$, $\mathfrak{M}_{\rm II}$, $\mathfrak{M}_{\rm III}$ each of which consists of a whole (u,w) plane. In each one, r is determined by (5) with the restriction to the region I, II, III, respectively, and the metric is given by (4).

Conversely, $\mathfrak{M}(u,r)$ can be regarded as an analytic extension of one of $\mathfrak{M}_{\rm I}$, $\mathfrak{M}_{\rm II}$, $\mathfrak{M}_{\rm III}$.

We may take advantage of the symmetry of (4) and (5) to eliminate u instead of w, obtaining the metric

$$ds^2 = -[1-2mr/(r^2+a^2)]dw^2 + 2dw\,dr\,,$$

which is analytic and nonsingular over $\mathfrak{M}(w,r)$—the whole (w,r) plane.

$\mathfrak{M}(w,r)$ is also subdivided into manifolds isomorphic to $\mathfrak{M}_{\rm I}$, $\mathfrak{M}_{\rm II}$, $\mathfrak{M}_{\rm III}$, and so can be regarded as an alternative analytic extension of any one of these.

We shall show in the next section how it is possible to obtain an analytic extension which is geodesically complete by repeated use of these two ways of extending one of $\mathfrak{M}_{\rm I}$, $\mathfrak{M}_{\rm II}$, $\mathfrak{M}_{\rm III}$.

4. THE EXTENSION WHEN $0<a^2<m^2$

We introduce a new manifold $\mathfrak{M}^*$ with coordinates ψ,ξ running from $-\infty$ to ∞ (see Fig. 1 (a)).

Let the lines $\cot(\psi-\xi)=0$, $[\cot(\psi+\xi)=0]$ subdivide it into open regions labelled $\cdots(n,-)^*$, $(n+1,-)^*,\cdots$; $[\cdots(-,n)^*, (-,n+1)^*,\cdots]$, etc., according as they contain the points with $\xi=0$ and $\cdots$, $\psi=n\pi$, $\psi=(n+1)\pi$,-$\cdots$, (for integer n). We let (n,p) be the subregion $(n,-)^*\cap(-,p)^*$, and define the submanifold $\mathfrak{M}_c$ to consist of those subregions with $|n-p|\leqslant 1$ together with the lines separating any two such subregions. Let $(n,-)$ stand for $\mathfrak{M}_c\cap(n,-)^*$ and similarly define $(-,n)$. It follows at once that $(n,-)$ consists of the union of $(n,n-1)$, (n,n), $(n,n+1)$, and the two lines that separate them. Similarly for $(-,n)$ with respect to $(n-1,n)$, (n,n), $(n+1,n)$.

Now define u as a function of ξ,ψ by $g(u)=\tan(\psi+\xi)$, where $g(u)$ is a function which is to be unbounded with continuous nonvanishing derivative, and whose precise definition may be given independently in each of the regions $(-,n)$. We take $g(u)=\pm h(u)$ according as n is odd/even, where $h(u)$ is a fixed monotonic increasing function to be specified later.

Similarly, we define w as a function of ξ, ψ by $f(w)=\tan(\psi-\xi)$ and fixing $f(w)$ in each of the regions $(n,-)$ by taking $f(w)=\pm h(w)$ according as n is odd/even.

We now have a continuous nonsingular mapping of the (u,w) plane onto each of the regions (n,p). We fix r as a function of ξ,ψ by specifying that (n,p) be a map of $\mathfrak{M}_{\rm I}$ if $n\neq p$ and $\max\{n,p\}$ is odd; of $\mathfrak{M}_{\rm II}$ if $n=p$, and

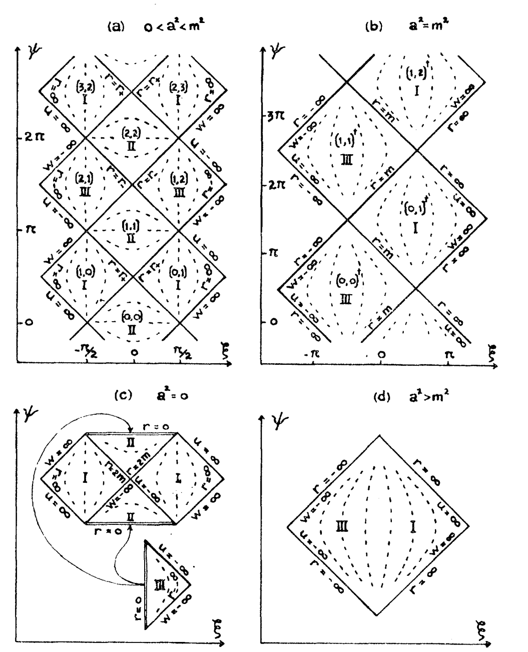

FIG. 1. The manifolds $\mathfrak{M}_c$ and $\mathfrak{M}_c{}^\dagger$ are illustrated in Figs. 1(a) and 1(b), respectively. Broken lines are used to sketch some of the Killing vector paths (r constant) and solid lines are used to show the horizons (where the Killing vectors become null), and the boundaries at infinite r. The null lines are everywhere at 45° to the axes. The manner in which the distant flat-space parts of the $\mathfrak{M}_{\mathrm{I}}$ and $\mathfrak{M}_{\mathrm{III}}$ regions are compressed into finite parts of the (ξ,ψ) plane by the coordinate transformations recalls the conformal diagrams used by R. Penrose, in *Relativity Groups and Topology*, (Gordon and Breach Science Publishers, New York, 1964), p. 565.

For purposes of comparison we show the effect of similar coordinate transformations on the Kruskal diagram—the case $a^2=0$ [Fig. 1(c)] and on the corresponding diagram for the case $a^2>m^2$ [Fig. 1(d)] using exactly the same conventions. Since the precise transformations do not need to be specified the diagrams do not have any scale on them. In the Kruskal case we have included the manifold obtained by continuing through the singularity at $r=0$ to the region of negative r. For analytical continuity two negative r regions should be fitted to the positive r region to complete the square as shown by arrows, but in order to retain the convention that ξ is spacelike and ψ timelike the regions have had to be separated and rotated relative to each other by 90°. This emphasizes that analytical continuation through $r=0$ is purely formal when $a^2=0$ since in this case an irremovable singularity (represented by a double line) develops at $r=0$ thus breaking off the physical connection between the regions of positive and negative r which existed for nonzero values of a^2. In the limit when $m=0$, the negative r part of Fig. 1(c) becomes the appropriate diagram for flat space. In the case when $a^2=m^2$ the regions of large positive and negative r are marked I and III, respectively, although the precise boundary between these regions has ceased to exist.

For practical computations, the original form of the metric in terms of r and u or w is much more convenient than the new form in terms of ξ and ψ. When $0<a^2<m^2$ the new form has the essential function of providing coordinate patches to cover the special points left out by the original analytic patches, but when $a^2=m^2$ the new form is no more than a framework for exhibiting the complete manifold $\mathfrak{M}_c{}^\dagger$.

of $\mathfrak{M}_{\mathrm{III}}$ if $n\neq p$ and $\min\{n,p\}$ is odd, and by taking $r=r_+$ on the boundary separating $\mathfrak{M}_{\mathrm{II}}$ from $\mathfrak{M}_{\mathrm{I}}$ and $r=r_-$ on the boundary separating $\mathfrak{M}_{\mathrm{II}}$ from $\mathfrak{M}_{\mathrm{III}}$ so that r is a continuous function of ξ, ψ.

It follows that each of the regions $(n,-)$ [$(-,n)$] is a continuous nonsingular map of $\mathfrak{M}(w,r)$ [$\mathfrak{M}(u,r)$]. These regions together cover the whole of $\mathfrak{M}_c$ except for the isolated points P_n with coordinates $\xi=0, \psi=(n+\frac{1}{2})\pi$.

The metric on $\mathfrak{M}_c$ which results from the mapping we have described is

$$ds^2=Q(\xi,\psi)[-d\psi^2+d\xi^2], \tag{6}$$

where Q is defined implicitly by

$$Q(\xi,\psi)=-\left(1-\frac{2mr}{r^2+a^2}\right)\frac{\sec^2(\psi+\xi)\sec^2(\psi-\xi)}{g'(u)f'(w)},$$

and thus depends on our choice of h.

By choosing $h(u)=e^{Bu}-e^{-Au}$ we ensure, as can easily be verified, that Q is positive definite and C_1 everywhere in $\mathfrak{M}_c$ which implies that at all points of $\mathfrak{M}_c$ the metric is nonsingular and the curvature finite. Q is in fact analytic within the regions $\mathfrak{M}_{\mathrm{I}}$, $\mathfrak{M}_{\mathrm{II}}$, $\mathfrak{M}_{\mathrm{III}}$, but not on the boundaries between them.

The manifold $\mathfrak{M}_c$ with metric (6) and this definition of h is completely covered by subregions [isomorphic to $\mathfrak{M}(u,r)$] which are analytic manifolds—except for the isolated points P_n. Therefore, $\mathfrak{M}_c$ is an analytic manifold except possibly at these points even though its metric is not explicitly analytic everywhere in the coordinate system we have chosen.

By taking a slightly different coordinate system we can see that $\mathfrak{M}_c$ is analytic at the points P_n also. We let the modified coordinates be $\tilde{\xi}$, $\tilde{\psi}$ introduced in exactly the same way as ξ, ψ except that we replace h chosen as above by $\tilde{h}$ chosen by letting

$$\tilde{h}(u)=e^{Bu} \quad \text{for} \quad u>k_1,$$
$$\tilde{h}(u)=-e^{-Au} \quad \text{for} \quad u<k_2,$$

where k_1 and k_2 are two constants with $k_1>k_2$. The values of these constants and the definition of $\tilde{h}(u)$ in the intermediate range $k_2\leqslant u\leqslant k_1$ are of no importance because the only purpose of bringing in the new system is to investigate the points P_n where u and w become infinite simultaneously. ($\tilde{h}$ could not be analytic throughout the intermediate region although it could if desired be made infinitely differentiable, and so the metric can no longer be analytic everywhere within $\mathfrak{M}_{\mathrm{I}}$, $\mathfrak{M}_{\mathrm{I}}$, $\mathfrak{M}_{\mathrm{III}}$ although it could be infinitely differentiable.) The modified metric is

$$ds^2=\tilde{Q}(\tilde{\xi},\tilde{\psi})[-d\tilde{\psi}^2+d\tilde{\xi}^2],$$

where, near the points P_n,

$$\tilde{Q}(\tilde{\xi},\tilde{\psi})=\csc^2(\tilde{\psi}+\tilde{\xi})\csc^2(\tilde{\psi}-\tilde{\xi})G(r).$$

The function $G(r)$ is defined only near $r=r_+$ and $r=r_-$ where it is positive definite and analytic:

$$G(r)=\frac{e^{-2Ar}}{A^2(r^2+a^2)}(r-r_-)^{(A+B)/B} \quad \text{near} \quad r=r_+,$$

$$G(r)=\frac{e^{2Br}}{B^2(r^2+a^2)}(r_+-r)^{(A+B)/A} \quad \text{near} \quad r=r_-.$$

Near the points P_{2n} where r is near r_+ we have

$$-1[-2mr/(r^2+a^2)]=\cot(\tilde{\psi}+\tilde{\xi})\cot(\tilde{\psi}-\tilde{\xi})A^2G(r)\,,$$

and near the points P_{2n+1} where r is near r_- we have

$$-[1-2mr/(r^2+a^2)]=\cot(\tilde{\psi}+\tilde{\xi})\cot(\tilde{\psi}-\tilde{\xi})B^2G(r)\,.$$

It follows that r and $G(r)$ are analytic functions of $\tilde{\xi}$, $\tilde{\psi}$ near the points P_n and consequently so is $\tilde{Q}$.

It has been established that the manifold $\mathfrak{M}_c$ is analytic and well behaved everywhere. It is also geodesically complete since every geodesic may be continued to unlimited values of the affine parameter within $\mathfrak{M}_{\mathrm{I}}$, $\mathfrak{M}_{\mathrm{II}}$, or $\mathfrak{M}_{\mathrm{III}}$ unless it reaches one of the boundaries $r=r_\pm$, and within $\mathfrak{M}_c$ it is possible to cross every such boundary and continue on the other side.

5. THE EXTENSION WHEN $a^2=m^2$

So far we have ignored the two interesting special cases $a^2=0$ and $a^2=m^2$ (see Fig. 1). The former is the Schwarzschild case which has been adequately dealt with by Kruskal, so we shall not consider it further.

We shall now adapt our previous method to the latter case. We introduce a second null coordinate w as before, except that we must replace $F(r)$ given by (3) by $F^\dagger(r)$ given by

$$F^\dagger(r)=2r+4m\ln|(r-m)|-4m^2(r-m)^{-1}\,. \quad (7)$$

Since we now have $r_+=r_-=m$, the manifold $\mathfrak{M}_{\mathrm{II}}$ no longer exists—$\mathfrak{M}(u,r)$ is divided into only two open manifolds $\mathfrak{M}_{\mathrm{I}}$ and $\mathfrak{M}_{\mathrm{III}}$ by the line $r=m$. As before $\mathfrak{M}_{\mathrm{I}}$ and $\mathfrak{M}_{\mathrm{III}}$ can be analytically extended in two distinct ways—to $\mathfrak{M}(u,r)$ or to $\mathfrak{M}(w,r)$.

Again we introduce $\mathfrak{M}^*$, the (ξ,ψ) plane, and define the mapping

$$u=\tan\tfrac{1}{2}(\psi+\xi)\,,\quad w=\cot\tfrac{1}{2}(\psi-\xi)\,.$$

The lines on which u and w become infinite divide $\mathfrak{M}^*$ into regions which we label $(-,n)^{*\dagger}$ or $(n,-)^{*\dagger}$ according as they contain the points $\xi=0$, $\psi=2n\pi$ or $(2n+1)\pi$. As before we define $(n,p)^\dagger$ to be the subregion $(n,-)^{*\dagger}\cap(-,p)^{*\dagger}$ and we take $\mathfrak{M}_c^\dagger$ to consist of the subregions with $n=p$ or $n=p-1$ together with the lines separating two such subregions. We let $(n,-)^\dagger$ stand for $\mathfrak{M}_c^\dagger\cap(n,-)^{*\dagger}$ and similarly define $(-,n)^\dagger$.

We fix r as a continuous function of (ξ,ψ) by specifying that $(n,p)^\dagger$ be a map of $\mathfrak{M}_{\mathrm{I}}$ if $n\neq p$ and $\mathfrak{M}_{\mathrm{III}}$ if $n=p$ and by setting $r=m$ on the boundaries. It can be shown that r is in fact C_1.

The resulting metric on $\mathfrak{M}_c^\dagger$ is given by

$$ds^2=Q^\dagger(\xi,\psi)[-d\psi^2+d\xi^2]\,, \quad (8)$$

where $Q^\dagger(\xi,\psi)$ is defined implicitly by

$$Q^\dagger(\xi,\psi)=[(r-m)^2/(r^2+m^2)]\times\tfrac{1}{4}\sec^2[\tfrac{1}{2}(\psi+\xi)]\csc^2[\tfrac{1}{2}(\psi-\xi)]\,.$$

It is clearly analytic and positive definite within the subregions $\mathfrak{M}_{\mathrm{I}}$ and $\mathfrak{M}_{\mathrm{III}}$ and it can be shown to be continuous (C_0) everywhere.

The manifold $\mathfrak{M}_c^\dagger$ with metric (8) is an analytic extension of $\mathfrak{M}_{\mathrm{I}}$ and $\mathfrak{M}_{\mathrm{III}}$ since it is completely covered by the regions $(n,-)^\dagger$ and $(-,n)^\dagger$ which are isomorphic to $\mathfrak{M}(w,r)$ and $\mathfrak{M}(u,r)$, respectively. There are no exceptional points this time.

We deduce that this manifold is geodesically complete by similar arguments to those used in the previous case.

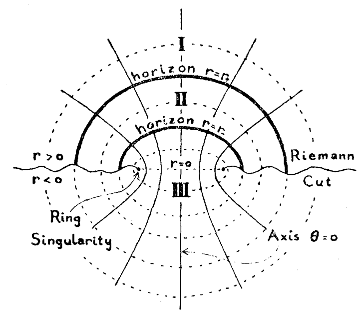

FIG. 2. A cross section $u=$const, $\varphi=$const of the full 4-dimensional Kerr metric (in the case $0<a^2<m^2$) is represented in order to show the relative positions of the features which have been referred to in Sec. 6. The horizons $r=r_\pm$ are marked with heavy lines. The broken lines mark other lines $r=$const and the faint continuous lines mark lines $\theta=$const. The regions of positive and negative r are separated outside the ring by a Riemann cut.

As a^2 increases the horizons $r=r_\pm$ approach each other, meeting when $a^2=m^2$ (at which stage $r_+=r_-=m$) and disappearing beyond this point. As a^2 decreases to zero, the outer horizon expands until in the Schwarzschild limit $r_+=2m$, while the inner horizon contracts to a point and the ring singularity contracts to a point singularity within it so that the regions of positive and negative r become completely cut off from one another.

6. IMPLICATIONS FOR THE 4-DIMENSIONAL CASE

We have confined our attention to the axis of symmetry because it is easy to see exactly what is going on there. Nevertheless it seems probable that the basic topological properties of the 4-dimensional manifold are essentially the same.

When the Kerr solution was first discovered, the feature which attracted immediate attention was the irremovable ring singularity at $r=0$, $\cos\theta=0$ in the spheroidal type coordinate system we are using. It connects the regions of positive and negative r, and has the property of a quadratic branch point in the complex plane, i.e., by passing through, looping round, and then passing through a second time in the same direction one returns to the space from which one started. It is possible for a timelike line to pass through in either direction. The negative r part of the manifold is identical in structure with what the positive r part would become if the parameter m (which we have taken to be positive) were replaced by its negative.

Another salient feature is the existence of stationary null hypersurfaces at $r=r_\pm$ when $0<a^2<m^2$ and at $r=m$ when $a^2=m^2$, to which attention has been called

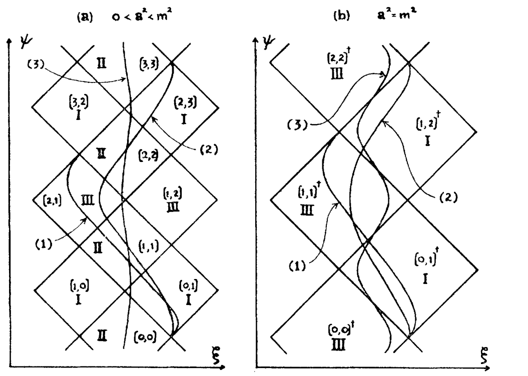

FIG. 3. The diagrams show the forms of typical trajectories followed by particles falling through the $r=r_+$ horizon from a type I region with different energies. The orbits are marked (1), (2), or (3) according as the energy is high, medium, or low (in the sense defined in Sec. 7). Figure 3(a) shows the case when $0<a^2<m^2$ [cf. Fig. 1(a)], the initial type I region having been taken to be (0,1). Figure 3(b) shows the case when $a^2=m^2$ [cf. Fig. 1(b)], the initial type I region being $(0,1)^\dagger$.

by Boyer and Price.[5] They are always on the positive r side of the ring (see Fig. 2). They have exactly the same horizon property as the stationary null lines $r=r_\pm$ or $r=m$ on the axis, namely that timelike lines can cross them in one direction only. They can be regarded as dividing the 4-space into 4-dimensional submanifolds:

$$\mathfrak{M}_{\mathrm{I}}(4)\,,\quad \text{for which}\quad r>r_+;$$
$$\mathfrak{M}_{\mathrm{II}}(4)\,,\quad \text{for which}\quad r_+>r>r_-;$$
$$\mathfrak{M}_{\mathrm{III}}(4)\,,\quad \text{for which}\quad r_->r.$$

These regions are generalizations of the $\mathfrak{M}_{\mathrm{I}}$, $\mathfrak{M}_{\mathrm{II}}$, $\mathfrak{M}_{\mathrm{III}}$ which we have been considering on the axis and as before $\mathfrak{M}_{\mathrm{II}}(4)$ is empty when $a^2 \doteq m^2$.

It seems reasonably likely that the analytic extension of the full Kerr manifold can be obtained by fitting together these manifolds in exactly the same way as we have done on the axis. If this is the case then the (ξ,ψ) diagrams which have been obtained can be thought of as illustrating the basic topology of the full 4-dimensional manifold except that the ring singularity which exists within $\mathfrak{M}_{\mathrm{III}}(4)$ cannot be represented in a two-dimensional picture.[6]

7. TYPICAL ORBITS

To illustrate the physical significance of our results we consider some of the possible motions of a test particle on the axis of symmetry (see Fig. 3). One may obtain many of the properties of free-particle orbits by considering the geodesic equation for r which is

$$dr/d\tau=\pm[E^2-1+2mr/(r^2+a^2)]^{1/2},$$

where τ is the proper time. In the limit $E^2\to\infty$ we have just the null geodesics $u=$const, and $w=$const.

The expression $[1-2mr/(r^2+a^2)]$ plays the part of a squared potential. At large positive r it has the effect of an attractive force which in the limit is the same as that due to a nonrotating mass m. Closer in, the force reverses its direction—there is a potential barrier blocking the ring in the region about $r=0$. The barrier has maximum height $1+m/|a|$ at the point $r=-|a|$. Beyond this point the force changes direction again and becomes repulsive so that in the limit of large negative r its effect is the same as that of a nonrotating negative mass $-m$. Thus a free particle can only pass right through the ring between the regions of large positive and large negative r in either direction if it has $E^2>1+m/|a|$. The energy corresponding to escape velocity is $E^2=1$. Unless it has $E^2>1$, a particle cannot exist at indefinitely large distance in either the I spaces or the III spaces. Therefore orbits fall into three classes:

(1) high energy, $E^2\geqslant 1+m/|a|$,
(2) medium energy, $1\leqslant E^2<1+m/|a|$,
(3) low energy, $E^2<1$.

In order to calculate the form of the orbits one must use a second geodesic equation, e.g., the u, r equation

$$\frac{du}{dr}=\left(1-\frac{2mr}{r^2+a^2}\right)^{-1}\left\{1\pm E\left[E^2-\left(1-\frac{2mr}{r^2+a^2}\right)\right]^{-1/2}\right\}. \quad (9)$$

Although exact treatment is awkward it is easy to see the general characteristics of the trajectories. The following remarks refer primarily to the case $0<m^2<a^2$. Where necessary the appropriate modification for the case $m^2=a^2$ is inserted in brackets.

Class (1): The particle travels directly from $r=\infty$ to $r=-\infty$, e.g., through

$$(0,1)\to(1,1)\to(2,1)\;[(0,1)^\dagger\to(1,1)^\dagger],$$

or from $r=-\infty$ to $r=\infty$, e.g., through

$$(2,1)\to(2,2)\to(2,3)\;[(1,1)^\dagger\to(1,2)^\dagger],$$

except in the limiting case $E^2=1+m/|a|$ when it will never pass the position $r=-|a|$ but will spend an infinite proper time approaching it.

Class (2): A particle in the distant part of a III region will always remain there; after penetrating a certain distance it will be repelled. A particle initially in the distant part of a I region will at first fall inwards, but the potential barrier across the ring will cause it to bounce out again into a subsequent I region after entering the III region. In the limiting case $E^2=1$ it will go no further than the mouth of the ring, $r=0$. With higher energy it will penetrate farther towards $r=-|a|$. If it

[5] R. H. Boyer and T. G. Price, Proc. Cambridge Phil. Soc. **61**, 531 (1965).

[6] See the work of R. H. Boyer and R. W. Lindquist (to be published). *Note added in proof.* See the work of R. H. Boyer and R. W. Lindquist [reported at the London Conference on Relativistic Theories of Gravitation, 1965 (to be published)]. These authors have examined the full 4-dimensional manifold when $0<a^2<m^2$, and their results confirm the above conjecture in this case.

starts in (0,1) it will progress through

$$(0,1) \to (1,1) \to (2,1) \to (2,2) \to (2,3) \quad [(0,1)^\dagger \to (1,1)^\dagger \to (1,2)^\dagger].$$

Class (3): Such a particle cannot exist in the outer parts of III. It must follow the same sequence as has just been described for class (2) particles, but instead of following it once it will continue to fall in and bounce out indefinitely. The oscillations are identical in the sense that if in one period the motion is given by $u=f(r)$, in the next period it will be given by $u=f(r)+C$, where C is a constant. Since u is an ignorable coordinate this constant has no local significance at all—it is only relevant for large-scale analytic considerations. Its effect will be to prevent the orbit from appearing as a periodic function of ψ in the ξ,ψ diagram. C may be evaluated by contour integration of the right-hand side of Eq. (9) in the complex r plane giving the result

$$C=2\pi mE(3-2E^2)(1-E^2)^{-3/2}.$$

Since $C\to 0$ as $E\to 0$ the motion will become periodic in ψ in the zero-energy limit. This result could also have been obtained by observing that one of the zero-energy solutions is $\xi=0$, the others being obtained by dragging this orbit along by the Killing vector field. (In the case $m^2=a^2$ a zero-energy solution does not exist, but as the energy approaches zero the orbit approaches the lines $r=m$.) As was to be expected, $C\to\infty$ in the opposite limit $E\to 1$, at which the motion ceases to be oscillating.

8. QUESTIONS OF UNIQUENESS AND CAUSALITY

Finally we come to the question of uniqueness. The two extensions we have described are clearly not unique since there are several ways in which different parts may be identified. Since they are both periodic in ψ one could obtain a universe which was cyclic in time by identifying parts for which the value of ψ differed by $2n\pi$ for some integer n. Causality would not of course be preserved in such a universe.

A different type of identification is the kind discussed by Fuller and Wheeler with reference to the Schwarzschild solution. In this case this would mean that the large r regions of different $\mathfrak{M}_{\mathrm{I}}(4)$ and $\mathfrak{M}_{\mathrm{III}}(4)$ submanifolds would be regarded as lying in distant parts of the same flat space background rather than in different ones as we have assumed until now. In the Schwarzschild case there were only two such regions to be identified, leading to the idea of a wormhole. In our case there are infinitely many regions, which may be identified in any combinations we may choose. We might obtain a picture bearing a closer relation to a honeycomb or, in a more disordered situation, to an ant's nest, in which a large, possibly infinite number of holes of assorted positive and negative masses are connected by a single subterraneous tunnel consisting of the region $\mathfrak{M}_{\mathrm{II}}(4)$ if they exist and the parts of the $\mathfrak{M}_{\mathrm{III}}(4)$ in the neighborhood of the ring singularities. Fuller and Wheeler[7] pointed out that causality could not be violated in the Schwarzschild case. However, there is nothing to prevent its violation here. We have seen that free particles of any energy may enter or emerge from the positive mass holes and that particles with sufficiently high energy to pass through the rings, and light rays in particular, may enter or emerge from negative mass holes as well. Therefore, identifications of this type will, in general, imply the possibility of causality violation. It is true that if the identified regions were sufficiently far apart in the (ψ,ξ) plane free particles would not pass between them. For example, in the case $a^2<m^2$ a free particle entering from (0,1) would emerge only in (2.3) or, if it could pass through the ring, in (2,1). If it entered through the ring in (2,1) it would emerge only in (2.3). Similarly, for the case $a^2=m^2$, a particle entering from $(0,1)^\dagger$ would emerge either in $(1,2)^\dagger$ or through the ring in $(1,1)^\dagger$ and one entering through the ring in $(1,1)^\dagger$ would emerge only in $(1,2)^\dagger$. Nevertheless, a nongeodesic particle could get to any region higher in the (ψ,ξ) plane than the one at which it entered, e.g., a rocket ship with sufficient fuel reserves to bring it to near-zero energy, $E^2\approx 0$ after its arrival in the tunnel and to restore it to a sufficiently large value of E^2 when it approached the region in which it wished to emerge. This would violate causality if the initial and final regions had the same flat-space background and were sufficiently far apart in it.

Apart from these internal identifications the manifolds $\mathfrak{M}_c$ and $\mathfrak{M}_c^\dagger$ appear to be unique in that they are the only complete analytic extensions of the submanifolds. One should be cautious about making a claim of this kind, since Misner and Taub[8] have shown that the Taub-space part of Newman, Unti, and Tamburino's space can be continued analytically in two essentially distinct ways. However this possibility arises from the curious pathological nature of the space concerned, and neither continuation is able to remove the incompleteness. The Kerr space is perfectly normal in this respect on the axis and appears to be so everywhere so that, because our method of continuation has been essentially straightforward, there seems to be no reason to doubt its uniqueness.

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[7] R. W. Fuller and J. A. Wheeler, Phys. Rev. **128**, 919 (1962).
[8] C. W. Misner and A. H. Taub, (unpublished report), and also C. W. Misner, J. Math. Phys. 4, 924 (1963) (note added in proof).