PHYSICAL REVIEW VOLUME 141, NUMBER 4 JANUARY 1966

Gravitational Ward Identity and the Principle of Equivalence*

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(Received 26 February 1965)

We have written down the Ward identity for a gravitational field interacting with a scalar field. It is shown that in the limit in which the scalar field is taken on the mass shell, this identity becomes the principle of equivalence.

THE well-known identity, due to Ward,[1] relating vertex to propagator renormalization is a consequence of gauge invariance of the second kind. This identity has been widely generalized by many workers[2] to cover the general case of gauge vector and pseudovector mesons coupled to conserved currents. In the present note, the Ward identity for the theory of gravitation is presented. That such an identity should exist follows from the consideration of the graviton as that gauge boson which ensures the local conservation of energy and momentum in the same sense that the photon ensures the conservation of charge.[3] The underlying group is, in this case, that of coordinate transformations. The Ward identity relates the gravitational vertex function to the self-energy insertion of a particle due to all its interactions. Inquiry into the physical meaning of this identity reveals that on the mass shell it equates gravitational mass (vertex function) to inertial mass (bare mass plus self-energy). The Ward identity serves as the necessary generalization to the principle of equivalence for off-the-mass-shell propagation. One sees in this way the mutual compatibility of general relativity with quantum-field theory.

Before presenting the formal derivation, it is well to set forth our fundamental assumptions:

(1) We adopt Gupta's point of view[4] by considering gravity as a spin-2 field embedded in a flat space. It is minimally coupled to the energy-momentum tensor $T_{\mu\nu}$ so as to ensure the conservation of this quantity. Whether this fictitious flat space is observable or not is a very interesting question concerning the existence of inertial frames; this point is presumably related to Mach's principle. For the present, we simply bypass this question, but hope to return to a detailed discussion of this essential point at a future date.

(2) It is assumed that the vacuum is invariant, not only under special Lorentz transformations, but in general as well.

The proof of the Ward identity is straightforward. Consider the propagator of a scalar Hermitian field $S(x,y)=\langle T\phi(x)\phi(y)\rangle$ which in the absence of external fields is a function of $(x-y)$ alone. Under the infinitesimal coordinate transformation

$$x^\mu \to x^\mu + a^\mu(x)\,,$$
$$y^\mu \to y^\mu + a^\mu(y)\,. \tag{1}$$

We then have

$$\delta S(x,y)=[a^\mu(x)-a^\mu(y)]\partial_\mu S(x-y)\,. \tag{2}$$

[Note that all indices are raised and lowered by the flat metric $g_{\mu\nu}{}^0$, except in Eq. (3) where this is not permissible.] By assumption (2), this difference must be compensated by a change in the propagator due to a change in an external gravitational potential $g_{\mu\nu}$ which is set equal to zero after the increment is calculated. The tensor character of $g_{\mu\nu}$ implies

$$\delta g^{\mu\nu}=-\delta g_{\mu\nu}=(\partial_\mu a_\nu+\partial_\nu a_\mu)\,, \tag{3}$$

so that

$$[a^\mu(x)-a^\mu(y)]\partial_\mu S(x-y)=-\int dx'dy'd\xi$$
$$\times[S(x-x')(\Gamma^{\mu\nu}(x'-\xi,\xi-y')$$
$$+\Gamma^{\nu\mu}(x'-\xi,\xi-y'))S(y'-y)][\partial_\mu a_\nu(\xi)]\,, \tag{4}$$

where $\Gamma^{\mu\nu}+\Gamma^{\nu\mu}$ is the gravitational vertex function, hereafter denoted as $\bar{\Gamma}^{\mu\nu}$. Taking Fourier components and comparing coefficients of $a_\mu(p)$ which are arbitrary, gives

$$p_\mu S(p)-p_\mu' S(p')=-S(p)\bar{\Gamma}_\mu{}^\nu(p,p')S(p')[p_\nu-p_\nu'] \tag{5}$$

or

$$\bar{\Gamma}_\mu{}^\nu(p,p')(p_\nu-p_\nu')=-p_\mu S^{-1}(p')+p_\mu' S^{-1}(p)\,. \tag{6}$$

Equation (6) is the end of the formal development. One may check the validity of the relation for a field without interaction where $S^{-1}=p^2-m_0{}^2$. In this case $\Gamma_{\mu\nu}$ is the p, p' matrix element of

$$T_{\mu\nu}=[p_\mu' p_\nu-(g_{\mu\nu}{}^0/2)(p\cdot p'-m_0{}^2)]\,.$$

We proceed further in the usual manner and go to the limit $p\to p'$ and thence on to the mass shell. Equation (6) becomes

$$\bar{\Gamma}_{\mu\nu}(p,p)=2p_\mu p_\nu(d/dp^2)S^{-1}(p^2)-g_{\mu\nu}{}^0 S^{-1}(p^2)$$
$$=2p_\mu p_\nu[1-d\Pi/dp^2]$$
$$-g_{\mu\nu}{}^0[p^2-m_0{}^2-\Pi(p^2)]\,, \tag{7}$$

* This research was supported in part by the U. S. Air Force under grant No. AF-EOAR 63-51 and monitored by the European Office, Office of Aerospace Research.

[1] J. Ward, Phys. Rev. **77**, 293 (1950).

[2] See, for example, J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento **17**, 757 (1960); and Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

[3] See, for example, M. Gell-Mann, *Proceedings of the 1961 Solvay Conference* (Interscience Publishers, Inc., New York, 1961), p. 131.

[4] S. Gupta, Rev. Mod. Phys. **29**, 334 (1957).

where Π is the self-energy. Near the mass shell we have

$$\Pi = m^2 - m_0^2 + (p^2 - m^2)(d\Pi/dp^2)$$

whereupon Eq. (7) becomes

$$\lim_{p^2\to m^2} \bar{\Gamma}_{\lambda\mu} = [-(p^2-m^2)g_{\lambda\mu}{}^0 + 2p_\lambda p_\mu][1-d\Pi/dp^2]. \quad (8)$$

Thus in the long-wavelength limit the coupling is the same as with a noninteracting field of mass m, multiplied by the vertex renormalization factor $[1-d\Pi/dp^2] = Z_1^{-1}$. In usual fashion this factor is taken out by the renormalization factors of $S(p)$ [cf. Eq. (5)], so that the Ward identity simply reduces to $Z^{-1}Z_2 = 1$ when one uses the renormalized mass. This result implies the universality of the gravitational coupling to the energy momentum tensor calculated with the true inertial mass (for particles on the mass shell) whatever the origin of the mass and independent of the Lagrangian of the gravitational field. This is the expression of the principle of equivalence. Off the mass shell, Eq. (6) is the appropriate generalization of the principle of equivalence.

We close with two remarks: (1) On the basis of a previous paper,[5] it is possible that theories which break Lorentz invariance[6] lead to a graviton mass. (2) Because of the Ward identity, it is probable that gravitation is a renormalizable theory. We hope to return to these problems as well as to the question of the meaning of a flat space in a filled universe in subsequent work.

We should like to express our gratitude to Professor D. Speiser of the University of Louvain who alerted us to the link between our Ward identity and the principle of equivalence. One of us (R.B.) wishes to acknowledge a stimulating discussion with Professor H. Bondi of King's College on the subject matter of this paper.

[5] F. Englert and R. Brout, Phys. Rev. Letters **13**, 321 (1964).
[6] For example J. Bjorken, Ann. Phys. (N. Y.) **24**, 174 (1963); I. Bialynicki-Birula, Phys. Rev. **130**, 465 (1963).

PHYSICAL REVIEW VOLUME 141, NUMBER 4 JANUARY 1966

Hydrodynamic Calculations of General-Relativistic Collapse*

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(Received 19 August 1965)

The general-relativistic equations governing the motion of a large mass under the influence of its own gravitational field and its own pressure have been approximated by finite-difference equations. A spherically symmetric, co-moving frame of reference was used. The pressure was assumed to be zero at the outer boundary. Rest mass was assumed to be conserved and heat transfer by neutrinos, radiation, etc., was not taken into account. Numerical solutions were obtained on a computer for several simplified equations of state, chosen to bracket the behavior of stellar material in late stages of collapse, and several masses. The maximum stable masses obtained were of the same order of magnitude, but somewhat larger than the maximum stable masses calculated statically. The behavior of light signals, of the metric coefficients, and of the hydrodynamic quantities as functions of time is described for collapse past the Schwarzschild radius. Such collapse leads to regions where the surface area of concentric spheres decreases as the rest mass contained by the spheres increases.

I. INTRODUCTION

THE gravitational collapse of spherically symmetric masses under conditions where the general theory of relativity is expected to apply has been calculated by solving the field equations in finite-difference approximation on digital computers. This paper presents results of this calculation for materials with simple equations of state and for simple initial and boundary conditions. The purpose is to provide a description of the collapse in the presence of nonzero pressures, and to verify current estimates of maximum stable mass. The calculation can be extended to take into account pair production, heat transfer, and other mechanisms of interest for the description of astrophysical processes.

II. EQUATIONS

We consider an ideal fluid and neglect all heat transfer except that due to the motion of the fluid itself. The specific entropy of the fluid at a given mass point is then constant except when the mass point goes through a shock. We neglect pair production and annihilation, and the interaction of the fluid with external fields, so that rest mass is conserved. Assuming spherical symmetry leads to the metric[1]

$$ds^2 = a^2(\mu,t)c^2dt^2 - b^2(\mu,t)d\mu^2 - R^2(\mu,t)d\Omega^2, \quad (1)$$

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1962), 2nd ed., pp. 331–332.