Use of Angular-Momentum Tensors*

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The properties of tensor representations are developed for application to angular-momentum problems in elementary-particle reactions.

I. INTRODUCTION

A HIGH-ENERGY resonance X appears, typically, in production-decay reactions such as

$$A+B\to X+C+\cdots$$
$$X\to a+b+c+\cdots.$$

The experiment yields distributions of energies and momenta in an ensemble of these events. If some particles undergo further interactions, e.g.,

$$a\to a'+b'+\cdots,$$

their spin orientations can be identified. One may then ask how to analyze the data, *systematically* and *exhaustively*, to learn the spin and parity of X and fix other quantities of interest. This is a prototype of the problems we wish to study.

Angular-momentum considerations are primary in these problems. One must study *phenomenological* descriptions of reactions, that is, descriptions which express angular-momentum properties (and other symmetry properties, according to the situation) explicitly, but cover our ignorance of dynamical details with undetermined constants and energy-dependent functions. Let it be understood that our descriptions of reactions are always intended to be phenomenological, in this sense.

The tensor calculus discussed in this paper provides an alternative to the conventional specification of angular-momentum states by magnetic quantum numbers and the associated apparatus of Clebsch-Gordan coefficients.[1] We believe the tensor approach is especially suited to problems like the one characterized above. The formalism bears a more intimate relation to the experimental information than in the conventional approach and the mathematics necessary to connect a physical hypothesis with an observed distribution are—in our opinion—more easily learned and more rapidly executed. But this assertion must not be taken as a promise that all calculations become trivial.

A few spin-parity problems have already been examined in this spirit.[2] The present paper gives a general account of tensor properties needed in such problems, plus some extras. The following paper applies the methods to various resonance reactions.

II. ANGULAR-MOMENTUM STATES AND REACTION AMPLITUDES

1. Spin Wave Functions for a Particle at Rest

Consider first a particle X of integral spin j at rest. Its spin state is traditionally described by a wave function $\psi^j{}_m$ where the magnetic quantum number m takes on $2j+1$ integrally spaced values. We prefer to represent the spin state by a tensor of rank j in three-space:

$$T^j{}_{m_1m_2\cdots m_j},\quad m_i=1,2,3.$$

The m_i are (Cartesian) vector indices. The tensor must be symmetrical,

$$T^j{}_{\cdots m\cdots m'\cdots}=T^j{}_{\cdots m'\cdots m\cdots},\tag{2.1}$$

and traceless,

$$\sum_m T^j{}_{\cdots m\cdots m\cdots}=0\tag{2.2}$$

in each pair or indices.

The space of *all* jth-rank tensors is a j-fold product of spaces of spin one; hence it is reducible into a sum of spaces with angular-momentum values ranging from zero to j. The constraints (2.1) and (2.2) assure that the spin-j wave functions cannot be so reduced. One may verify by a simple combinatorial calculation that a symmetric jth-rank tensor has $\frac{1}{2}(j^2+3j+2)$ independent components and that tracelessness imposes $\frac{1}{2}(j^2-j)$ constraints leaving $2j+1$ independent components, as is suitable for spin j.

Now let X have half-integral spin $j+\frac{1}{2}$—so that j still represents an integer. We then use a "mixed" tensor

$$T^{j+1/2}{}_{m_1m_2\cdots m_j\alpha},\quad m_i=1,2,3;\quad \alpha=1,2,$$

with j vector indices and one spinor index α. $T^{j+1/2}$ must still be traceless and symmetric with respect to any pair of vector indices. It must obey the additional constraint

$$\sum_{m,\beta}(\sigma_m)_{\alpha\beta}T^{j+1/2}{}_{mm_2m_3\cdots m_j\beta}=0$$

or, more compactly written,

$$\boldsymbol{\sigma}\cdot T^{j+1/2}=0.\tag{2.3}$$

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[1] The first application of tensors to elementary-particle physics of which we are aware was by W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941). Other applications have been given by C. Fronsdal, Nuovo Cimento Suppl. **9**, 416 (1958) and A. Goldhaber, thesis, Princeton University, 1964 (unpublished).

[2] C. Zemach, Phys. Rev. **133**, B1201 (1964); Nuovo Cimento **32**, 1605 (1964).

Note that a symmetric tensor obeying (2.3) is automatically traceless. A simple combinatorial calculation shows that when (2.1) and (2.3), and hence (2.2) are satisfied, the tensor has $2(j+\frac{1}{2})+1$ components.

Under a rotation, spinors and vectors transform according to well-established laws having the form

$$T^{1/2}{}_{\alpha}{}' = D_{\alpha\beta} T^{1/2}{}_{\beta}, \tag{2.4a}$$

$$T^{1}{}_{m}{}' = R_{mn} T^{1}{}_{n}. \tag{2.4b}$$

The laws implied for tensor transformations are

$$T^{j}{}_{m_1m_2\cdots m_j}{}' = R_{m_1n_1}R_{m_2n_2}\cdots R_{m_jn_j}T^{j}{}_{n_1n_2\cdots n_j}, \tag{2.5a}$$

$$T^{j+1/2}{}_{m_1\cdots m_j\alpha}{}' = R_{m_1n_1}\cdots R_{m_jn_j}D_{\alpha\beta}T^{j+1/2}{}_{n_1\cdots n_j\beta}. \tag{2.5b}$$

Then the spin operator (angular-momentum operator) for spin j is $\mathbf{S}^j$:

$$\mathbf{S}^j = \sum_{a=1}^{j} \mathbf{S}^{(a)}, \tag{2.6}$$

where $\mathbf{S}^{(a)}$ is the spin operator for spin one, operating on the ath vector index. The matrix elements of the components of $\mathbf{S}^{(a)}$ are (with $\hbar=1$)

$$(S^{(a)}{}_k)_{mn} = -i\epsilon^{kmn}, \tag{2.7}$$

as is well known. The spin operator for $j+\frac{1}{2}$ is

$$\mathbf{S}^{j+1/2} = \mathbf{S}^j + \tfrac{1}{2}\boldsymbol{\sigma}. \tag{2.8}$$

The reason for imbedding a $(2j+1)$- or $[2(j+\frac{1}{2})+1]$-component quantity in a space of higher dimension is that its transformation character becomes manifest. The intuition physicists feel in treating vectors and spinors becomes applicable to systems of higher spin. Then, if one needs to combine a multitude of momentum vectors to form states of definite orbital angular momentum, the procedure is more or less obvious by inspection once a few basic rules are established. There is no need to prepare tables of Clebsch-Gordan coefficients.

These basic rules would be even simpler if we used tensors built from spinors rather than from vectors. But then we would lose the simple relation to experimental data which is, after all, expressed in terms of momentum vectors.

2. Covariant Spin Wave Functions

The description of massless particles is taken up in the next section. For the present, we assume the particle has a nonzero mass m and hence a rest frame. Let its four-momentum be k_μ in a general frame and $(m,0,0,0)$ in the rest frame.

A relativistically covariant description of an integral spin j is given by a symmetric tensor of rank j in four-space,

$$T^{j}{}_{\mu_1\mu_2\cdots\mu_j}, \quad \mu_i = 0, 1, 2, 3$$

which is traceless in the Lorentz metric, and transverse to the particle's four-momentum:

$$k_\mu T^{j}{}_{\mu\cdots} = 0 \tag{2.9}$$

In the rest frame, tensor components with any timelike indices are identically zero, by (2.9). The remaining components, bearing only spacelike indices constitute the three-space tensor already used for the rest-frame description. One verifies that this tensor has $2j+1$ independent components in an arbitrary frame either by a combinatorial calculation or by noting the one-to-one correspondence to the rest frame where the calculation has already been done.

To treat spin $\frac{1}{2}$ relativistically, we may adopt either the four-component (Dirac) scheme or the two-component scheme. In the Dirac approach, the four-component spinor ψ and its adjoint $\bar{\psi}$ can be decomposed into two-component spinors u and v,

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \bar{\psi} = \begin{pmatrix} u^* \\ -v^* \end{pmatrix}, \tag{2.10a}$$

related by

$$v = [\boldsymbol{\sigma}\cdot\mathbf{k}/(k_0+m)]u. \tag{2.10b}$$

The significance of u is that, apart from normalization, it is precisely the spin-$\frac{1}{2}$ wave function for the particle in its rest frame, called $T^{1/2}{}_\alpha$ in part 1. The familiar reduction of a covariant spin matrix element to a matrix element between "upper components" via (2.10b) can be regarded as the re-expression of the covariant form into a form which refers to the rest-frame spin components of the particles.

In the two-component scheme for spin $\frac{1}{2}$, the wave function for a particle in any frame is, of course, a two-component quantity φ_α. Its relation to the rest-frame spinor u_α is

$$\varphi = (k_\mu\sigma_\mu/m)^{-1/2}u, \tag{2.11}$$

where $\sigma_u \equiv (1,\boldsymbol{\sigma})$ is the familiar set of 2×2 matrices. If matrix elements are formed with φ on the right and φ^* on the left, one Lorentz covariant can be made from matrices in the spin space, namely the four-vector $\varphi^*\sigma_\mu\varphi$.

Now for spin $j+\frac{1}{2}$: If we follow the two-scheme, we use the tensor

$$T^{j+1/2}{}_{\mu_1\mu_2\cdots\mu_j\alpha}; \quad \mu_i = 0, 1, 2, 3; \quad \alpha = 1, 2.$$

The vector indices are subject to the constraints already enumerated. In addition, we have the covariant version of (2.3),

$$\sigma_\mu T^{j+1/2}{}_{\mu\cdots} = 0. \tag{2.12}$$

If we follow the four-scheme, the tensor is written the same way, but α ranges from 1 to 4. This method was used by Rarita and Schwinger.[1] The generalization of (2.3) is

$$\gamma_\mu T^{j+1/2}{}_{\mu\cdots} = 0. \tag{2.13}$$

We also need the Dirac equation

$$(\gamma k - m)T^{j+1/2} = 0 \tag{2.14}$$

to distinguish positive energy states in the four-scheme.

3. Spin Wave Functions for Particles of Zero Mass

For orientation purposes, let us review the situation for vector particles of zero mass, i.e., photons. We begin to describe the spin by a four-vector T_μ, orthogonal to the momentum

$$k_\mu T_\mu = 0. \tag{2.15}$$

The space of the T_μ's satisfying (2.15) is three dimensional, as it is for a massive vector particle. But this space is not irreducible under Lorentz transformations. In a given Lorentz frame, let $\mathcal{E}$ be the two-dimensional part of three-space consisting of the plane perpendicular to $\mathbf{k}$. T_μ is expressible in components as $T_\mu = (T_0, T_3, \mathbf{e})$, where T_3 is the component in the direction $\mathbf{k}$ and $\mathbf{e}$ is in $\mathcal{E}$. In this representation, $k_\mu = (k_0, k_3, 0, 0)$ and $k_0 = k_3$. Equation (2.15) implies $T_0 = T_3$. Hence

$$T^2_\mu = -\mathbf{e}\cdot\mathbf{e}. \tag{2.16}$$

Thus $\mathbf{e}\cdot\mathbf{e}$ is itself an invariant under a Lorentz transformation, regardless of what happens to T_0, T_3. The true spin states of a free photon are vectors in $\mathcal{E}$. Under a Lorentz transformation, two things happen: The spin state is rotated in $\mathcal{E}$ by some angle α and the relation of $\mathcal{E}$ to actual physical space, which depends on $\mathbf{k}$, also changes. The reason for imbedding $\mathcal{E}$ in the higher dimensional space of the T_μ's is as before: The transformation properties are made manifest and the combination of photon spin with momentum vectors to make invariants is simplified.

As is well known, $\mathcal{E}$ is also reducible. If $\mathbf{e}_1$, $\mathbf{e}_2$ are a basis in $\mathcal{E}$ (states of linear polarization) then the helicity states $\mathbf{e}_+ = -(\mathbf{e}_1 + i\mathbf{e}_2)/\sqrt{2}$, $\mathbf{e}_- = (\mathbf{e}_1 - i\mathbf{e}_2)/\sqrt{2}$ (states of circular polarization) transform according to

$$\mathbf{e}_+ \to e^{+i\alpha}\mathbf{e}_+, \quad \mathbf{e}_- \to e^{-i\alpha}\mathbf{e}_- \tag{2.17}$$

under a Lorentz transformation.

In building invariants to describe a reaction, one must take care to combine T_μ with momenta in such a way that the irrelevant components T_0, T_3 do not enter. This means one must use, not T_μ directly,[3] but rather the electromagnetic field tensor

$$F_{\mu\nu} = k_\nu T_\mu - k_\mu T_\nu. \tag{2.18}$$

In a three-space description, $F_{\mu\nu}$ breaks down into 2 three-vectors:

$$\mathbf{E} \equiv (F_{01}, F_{02}, F_{03}) = k_0\mathbf{e}, \quad \mathbf{E}\cdot\mathbf{k} = 0; \tag{2.19a}$$

$$\mathbf{H} \equiv (F_{23}, F_{31}, F_{12}) = \mathbf{k}\times\mathbf{e}, \quad \mathbf{H}\cdot\mathbf{k} = 0. \tag{2.19b}$$

The most general way photon spin mixes with photon momentum in a three-space description is then expressible as a sum over two kinds of multipole terms, the electric multipoles $k_0 T^j(\mathbf{kk}\cdots\mathbf{ke})$ and the magnetic multipoles $T^j(\mathbf{k}\cdots\mathbf{kk}\times\mathbf{e})$, with parities $(-1)^j$ and $(-1)^{j+1}$, respectively. (For notation, see Sec. III.)

There is nothing particularly novel in our conclusions on how to describe photon interactions. However, we do not know of any comparable discussion in the elementary-particle literature which makes these points without going into the complexities of field theory, or other dynamical questions.

The generalization for massless particles of arbitrary spin, should the need for it arise, is now simple. One uses, as in the massive case, a tensor $T^j{}_{\mu_1\cdots\mu_j}$ (or $T^{j+1/2}{}_{\mu_1\cdots\mu_j\alpha}$) with the aforementioned constraints. But only the components with all μ_i referring to one of the directions in the space $\mathcal{E}$ are relevant to the spin representation. With this added constraint, there are only two independent tensor components for any spin. In fact, if spacelike indices are expressed in spherical coordinates $+$, $-$, 0 instead of 1, 2, 3 (see Sec. VI for details), it is easily seen that only the positive and negative helicity components $T^j{}_{++\cdots+}$ and $T^j{}_{--\cdots-}$ (or $T^{j+1/2}{}_{++\cdots+,+1/2}$ and $T^{j+1/2}{}_{--\cdots-,-1/2}$) are nonzero. Under a Lorentz transformation, these are multiplied by $e^{\pm ij\alpha}$ (or $e^{\pm i(j+1/2)\alpha}$).

To build a covariant description of a reaction in which the massless particle participates, one uses the generalization of (2.18), introducing a new set of indices $\nu_1\cdots\nu_j$. In a three-space description, there are various types of multipoles. For spin 2, the covariant tensor would be written $T_{\mu_1\mu_2}$. There is a corresponding symmetric traceless three-space dyadic $\mathsf{T} \equiv T_{m_1m_2}$ with constraints $\mathbf{k}\cdot\mathsf{T} = \mathsf{T}\cdot\mathbf{k} = 0$. There are three species of multipole, built out of $k_0^2\mathsf{T}$, $k_0(\mathbf{k}\times\mathsf{T})$, and $\mathbf{k}\times\mathsf{T}\times\mathbf{k}$.

4. Relativity Is Not an Essential Complication

We stress a point already made for spin-$\frac{1}{2}$ but valid for any spin (for massive particles): There is a simple linear relation expressing a covariant tensor in an arbitrary frame in terms of the corresponding rest-frame tensor. To construct the relation, one needs to know only the Lorentz transformation law for vectors and, for half-integral spins, either (2.10) or (2.11).

Let us imagine a reaction amplitude M expressed, first of all, in covariant form; it will be a Lorentz-invariant function of the four-momenta of the reaction and the covariant spin tensors. We are then at liberty to refer the momenta to the CMF (center-of-mass frame) of the reaction and to refer the tensor specifying the spin of each particle to that particle's RF (rest frame). The spin tensors for massless particles can be broken up into multipole contributions referred to the CMF. We shall say the amplitude is being described in terms of "proper" variables. Alternatively, in a phenomenological description, one may as well start out with a representation of the amplitude in terms of center-of-mass momenta and rest-frame spin tensors. One of the advantages (to be spelled out later) is that the phase space for a cascade of reactions becomes the product of

[3] One can also use as an invariant $j_\mu T_\mu$, where j_μ is a conserved current, i.e., $j_\mu k_\mu = 0$. In a phenomenological description, such a j_μ can be written $j_\mu = \lambda k_\mu + C_{\mu\nu}k_\nu$, where $C_{\mu\nu}$ is antisymmetric, whence $j_\mu T_\mu = \frac{1}{2}C_{\mu\nu}F_{\mu\nu}$.

the phase spaces for the component reactions, each referred to its own center of mass.

The description of a reaction is then in terms of vectors and tensors in three-space and has a nonrelativistic appearance, although remaining relativistically correct. No information regarding parity or angular momentum is lost. On the other hand, nonrelativistic analyses already in the literature—the nucleon-nucleon polarization studies, for example—do not become incorrect at high energies, but merely need a proper interpretation. If the mention of spins and momenta is interpreted according to the above rule, the work will generally be valid relativistically without alteration. In this sense we may say that relativity is not an essential complication.

One pitfall in the application of this procedure experimentally is worth emphasizing. The relationship between the coordinate axes in the CMF and the axes in the RF of the particle whose spin is of interest—call it Y—should be that of the pure Lorentz transformation (velocity transformation) defined by the Y velocity in the CMF. All momentum vectors will be measured first in a laboratory frame. They should then be transformed to the CMF and from there to the YRF. If one short-cuts this process by transforming directly from lab to YRF, the rest frame achieved differs from the desired one by a rotation. This rotation depends on the relation of the lab frame to the other frames and is not relevant to the physics of the problem.

The advantages of using what we call proper variables were first recognized by Stapp.[4] They have also been exploited by R. Gatto and others.

III. BASIC TENSOR PROPERTIES

1. Conventions

From now on, "tensor" will mean three-space tensor rather than covariant tensor. For neatness, we may write $T^j{}_{123\cdots}$ rather than $T^j{}_{m_1m_2m_3\cdots}$. Alternatively, a single symbol m can represent a set of indices $\{m_1m_2\cdots m_j\}$. The spinor index will ordinarily be suppressed. The indices j and m will often be suppressed if no ambiguity results. A vector $T^1{}_m$ can also be indicated by bold face: $\mathbf{T}$.

These conventions—or, rather, habits—concerning indices and their suppression will be freely applied to operators $O^j{}_{mm'}$ on the tensors.

The tensor product of two tensors of equal rank is expressed by a colon:

$$T^j:A^j\equiv T^j{}_{123\cdots j}A^j{}_{123\cdots j}\equiv T^j{}_{m_1m_2\cdots m_j}A^j{}_{m_1m_2\cdots m_j}. \quad (3.1)$$

A tensor that does not satisfy one or more of the required constraints will be called "raw." In contrast, a T^j that does satisfy them is "pure." A tensor is presumed to be pure if no adjective is supplied.

If a 3×3 matrix O_{ab} operates on the nth index of $T^j{}_m$, we can write, without subscripts, $O^{(n)}T^j$. Similarly $\mathbf{a}^{(n)}\cdot T^j$ denotes the contraction of a vector $\mathbf{a}$ with the nth index of T^j. Of course, the superscript "(n)" is unnecessary here if T^j is symmetric.

The sign $\sum_P$ preceding an indexed quantity signifies a summation over all essentially different permutations. Thus

$$\sum_P A^2{}_{12}T^2{}_{34}=A^2{}_{12}T^2{}_{34}+A^2{}_{13}T^2{}_{24}+A^2{}_{14}T^2{}_{23}+A^2{}_{23}T^2{}_{14}+A^2{}_{24}T^2{}_{13}+A^2{}_{34}T^2{}_{12}. \quad (3.2)$$

A contraction of, say, the first two indices of a raw tensor T_{1234} could be expressed as T_{nn34} or as $T_{\cdot\cdot34}$.

2. Projection Operators

To make a raw tensor T^j symmetric, we apply the projection operator $\mathcal{P}_s{}^j$:

$$\mathcal{P}_s{}^jT^j{}_{12\cdots j}=(j!)^{-1}\sum_P T^j{}_{12\cdots j}. \quad (3.3)$$

The factorial ensures that $\mathcal{P}_s$ has the normalization of a projection; $\mathcal{P}_s{}^2=\mathcal{P}_s$. Moreover $\mathcal{P}_sT^j=T^j$ if T^j is already symmetric.

Let $\mathcal{P}_t{}^j$ be the operator that makes a raw tensor traceless. For small j, the effect of $\mathcal{P}_t$ on tensors which are already symmetric is quite simple:

$$\mathcal{P}_tT_{12}=T_{12}-\tfrac{1}{3}\delta_{12}T_{\cdot\cdot}, \quad (3.4)$$

$$\mathcal{P}_tT_{123}=T_{123}-\tfrac{1}{5}(\delta_{12}T_{\cdot\cdot3}+\delta_{13}T_{\cdot\cdot2}+\delta_{23}T_{\cdot\cdot1}) \quad (3.5)$$

$$\mathcal{P}_tT_{1234}=T_{1234}-(1/7)\sum_P \delta_{12}T_{nn34}+(1/35)(\delta_{12}\delta_{34}+\delta_{13}\delta_{24}+\delta_{14}\delta_{23})T_{nnkk}. \quad (3.6)$$

The general formula (for symmetric T^j) is

$$\mathcal{P}_t{}^jT^j{}_{12\cdots j}=T^j{}_{12\cdots j}-(2j-1)^{-1}\sum_P \delta_{12}T_{nn3\cdots j}+(2j-1)^{-1}(2j-3)^{-1}\sum_P \delta_{12}\delta_{34}T_{nnkk5\cdots j}-\text{etc.} \quad (3.7)$$

as is verified by taking a trace of (3.7) and counting carefully. These equations are, of course, consistent with $\mathcal{P}_t{}^2=\mathcal{P}_t$.

Let $\mathcal{P}_\sigma{}^j$ be the projection that secures (2.3). If $T^{j+1/2}$ is already traceless and symmetric, it represents at most an angular-momentum mixture of $j+\frac{1}{2}$ and $j-\frac{1}{2}$. Then we have the familiar expression

$$\mathcal{P}_\sigma{}^jT^{j+1/2}=\frac{j+1+\boldsymbol{\sigma}\cdot\mathbf{S}^j}{2j+1}T^{j+1/2}. \quad (3.8)$$

In view of (2.7), this can be written out

$$\mathcal{P}_\sigma T^{j+1/2}=\frac{j+1}{2j+1}\left[T^{j+1/2}+\frac{i}{j+1}(\boldsymbol{\sigma}^{(1)}\times T^{j+1/2}+\boldsymbol{\sigma}^{(2)}\times T^{j+1/2}+\cdots)\right]. \quad (3.9)$$

[4] H. P. Stapp, Lawrence Radiation Laboratory Report No. UCRL 8096, 1957 (unpublished).

For the simplest case of spin $\frac{3}{2}$, we have, if the often unnecessary normalization factor is dropped,

$$\mathcal{P}_\sigma \mathbf{T} = \mathbf{T} + \tfrac{1}{2} i\boldsymbol{\sigma} \times \mathbf{T} \tag{3.10}$$

which clearly obeys $\boldsymbol{\sigma}\cdot(\mathcal{P}_\sigma \mathbf{T})=0$.

If the operator $\mathcal{Q}_\sigma{}^j$

$$\mathcal{Q}_\sigma{}^j = 1 - \mathcal{P}_\sigma{}^j = (j - \boldsymbol{\sigma}\cdot\mathbf{S}^j)/(2j+1) \tag{3.11}$$

is applied to this $T^{j+1/2}$, the result has j vector indices but represents pure spin $j-\frac{1}{2}$.

There is yet another useful way to generate a pure half-integral spin. Notice that with $T^{j+1/2}$ traceless and symmetric, but not transverse to $\boldsymbol{\sigma}$, the tensor

$$A^{j-1/2} = \boldsymbol{\sigma}\cdot T^{j+1/2} \tag{3.12}$$

is pure $j-\frac{1}{2}$ as it obeys all constraints without the application of $\mathcal{P}_\sigma$.

These projections commute with one another. Putting them together, we get total projections

$$\mathcal{P}^j = \mathcal{P}_s{}^j \mathcal{P}_t{}^j \quad \text{and} \quad \mathcal{P}^{j+1/2} = \mathcal{P}_s{}^j \mathcal{P}_t{}^j \mathcal{P}_\sigma{}^j \tag{3.13}$$

which reduce arbitrary raw tensors to pure tensors. Recall that the trace of a projection equals the dimension of the subspace onto which it projects:

$$\mathrm{Tr}\mathcal{P}^j = \sum_m (\mathcal{P}^j)_{mm} = 2j+1 \tag{3.14a}$$

$$\mathrm{Tr}\mathcal{P}^{j+1/2} = \sum_{m,\alpha} (\mathcal{P}^j)_{m\alpha m\alpha} = 2(j+\tfrac{1}{2})+1\,. \tag{3.14b}$$

Finally, consider the product of pure tensors A^j, B^j derived from raw tensors $\bar{A}^j$, $\bar{B}^j$

$$\begin{aligned} A^*\!:\!B &= \bar{A}^*\!:\!\mathcal{P}_s\mathcal{P}_t\bar{B} \quad (\text{or } \bar{A}^*\!:\!\mathcal{P}_s\mathcal{P}_t\mathcal{P}_\sigma B) \\ &= \bar{A}^*\!:\!B = A^*\!:\!\bar{B}\,. \end{aligned} \tag{3.15}$$

We see that a constraint need not be imposed on both tensors as a projection need only be used once. If B is pure, only the raw version of A is required. Or, if A is symmetric and B is traceless, one need not enforce symmetry on B or tracelessness on A, and so on. This principle allows useful simplifications in calculations.

3. Sums over Spin

Let T^j be the wave function for a spin state of particle X. One naturally chooses for it a normalization of unity

$$T^{j*}\!:\!T^j \equiv T^j{}_m{}^* T^j{}_m = 1\,. \tag{3.16}$$

The corresponding formula for the outer product summed over all spin states, $2j+1$ in number, is

$$\sum_{X\text{ spin}} T^j{}_m T^j{}_{m'}{}^* = (\mathcal{P}^j)_{mm'}\,. \tag{3.17}$$

Note that (3.17) is consistent with (3.16) and (3.14). If $M = T^*{}_m\!:\!A_m$ is the amplitude for a reaction in which X appears as a product, then the counting rate, summed over X-spin states is proportional to

$$\sum_{X\text{ spin}} |M|^2 = A^*\!:\!\mathcal{P}A \qquad (=A^*\!:\!A \text{ if } A \text{ is pure})\,. \tag{3.18}$$

4. Tensors from Vectors

Given a family of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}, \cdots$, one can construct a raw tensor R^j of integral spin by stringing the vectors along with repetitions

$$R^j{}_{1234\cdots j}(\mathbf{abbc}\cdots\mathbf{d}) = a_1 b_2 b_3 c_4 \cdots d_j\,, \tag{3.19}$$

and then a pure tensor by applying the projection

$$T^j(\mathbf{abbc}\cdots\mathbf{d}) = \mathcal{P}^j R^j(\mathbf{abbc}\cdots\mathbf{d})\,. \tag{3.20}$$

For example,

$$T^2{}_{12}(\mathbf{ab}) = \tfrac{1}{2}(a_1b_2 + a_2b_1) - \tfrac{1}{3}\delta_{12}\mathbf{a}\cdot\mathbf{b}\,. \tag{3.21}$$

The total number of different tensors of rank j obtainable from a set of vectors in this way is easily counted. If the vectors represent momenta of particles, the tensors represent different orbital-angular-momentum configurations. To obtain all possible orbital configurations, one must add to the list of momenta the various cross products among them.

5. Tensors from One Vector

If the tensors are built from a single vector $\mathbf{p}$, we write $R^j(\mathbf{p})$ and $T^j(\mathbf{p})$ rather than $R^j(\mathbf{pp}\cdots\mathbf{p})$, $T^j(\mathbf{pp}\cdots\mathbf{p})$.

Now let $\mathbf{p}$, $\mathbf{n}$ be unit vectors and $d\Omega_p = \sin\theta d\theta d\varphi$ be the element of solid angle for $\mathbf{p}$. The Legendre polynomial can be defined by

$$\begin{aligned} T^j(\mathbf{n})\!:\!T^j(\mathbf{p}) &= R^j(\mathbf{n})\!:\!\mathcal{P}_t{}^j R^j(\mathbf{p}) = c_j P_j(x)\,, \\ x &= \mathbf{n}\cdot\mathbf{p}\,, \end{aligned} \tag{3.22}$$

with the constant c_j fixed by the condition $P_j(1)=1$. If we admit that we have seen the Legendre polynomials before, the simplest way of obtaining c_j is to compare the leading term in x on the left of (3.22), namely x^j, with the known leading term of $P_j(x)$; this yields

$$c_j = 2^j(j!)^2/(2j)! = j!/(2j-1)!!\,. \tag{3.23}$$

Alternatively, c_j can be derived directly (see Sec. VI.4) and all the terms of $P_j(x)$ can be inferred from the properties of $\mathcal{P}_t{}^j$ whose explicit form is already known.

As analogs of (3.16), (3.17), we have

$$T^j(p)\!:\!T^j(p) = c_j \tag{3.24}$$

$$\frac{1}{4\pi}\int T^j{}_m(p) T^j{}_{m'}(p)\, d\Omega_p = \frac{c_j}{2j+1}(\mathcal{P}^j)_{mm'}\,. \tag{3.25}$$

The coefficient in (3.25) is chosen so that if we set $m=m'$ and sum over m, we have consistency with (3.24) and (3.14a). If (3.25) is multiplied by $T^j{}_m(n)T^j{}_{m'}(n)$ and summed over m,m' the orthonormality formula for Legendre polynomials is obtained.

These formulas are sometimes useful:

$$T^j(\mathbf{p}):T^k(\mathbf{p})=(c_j/c_{j-k})T^{j-k}(\mathbf{p}) \quad (\text{for } j\geqq k), \quad (3.26)$$

$$T^j(\mathbf{p}\cdots\mathbf{pn}\cdots\mathbf{n}):T^j(\mathbf{p})=c_jP_l(x) \quad (l \text{ is the number of } \mathbf{n}\text{'s}), \quad (3.27)$$

$$(2j+1)(\mathbf{p}\cdot\mathbf{n})T^j(\mathbf{p})=jT^j(\mathbf{pp}\cdots\mathbf{pn})+(2j+1)\mathbf{n}\cdot T^{j+1}(\mathbf{p}). \quad (3.28)$$

The last of these can be proved by (a) setting $\mathbf{p}=\mathbf{n}$ to get one relation on the coefficients and (b) multiplying by $T^j(\mathbf{p})$ to get another. The Legendre recursion relation follows if (3.28) is multiplied by $T^j(\mathbf{n})$.

To express the combination of orbital angular momentum with a two-component spinor u, we write

$$T^{j+1/2}(\mathbf{abc}\cdots,u)$$

and, simply, $T^{j+1/2}(\mathbf{a},u)$ if $\mathbf{a}=\mathbf{b}=\mathbf{c}=\cdots$. Then

$$T^{j+1/2}(\mathbf{p},u)=\mathcal{P}_\sigma T^j(\mathbf{p})u =(2j+1)^{-1}\{(j+1)T^j(\mathbf{p})+jiT^j(\mathbf{p}\cdots\mathbf{p}\,\boldsymbol{\sigma}\times\mathbf{p})\}u. \quad (3.29)$$

6. Coupling of Angular Momenta

If the components of two tensors T^j, U^k of integral spin are multiplied together, we have a raw tensor of rank $j+k$

$$T^j{}_{m_1m_2\cdots m_j}U^k{}_{n_1n_2\cdots n_k}.$$

Suppose we contract over zero, one, two, $\cdots$ pairs of indices, one of T^j, the other of U^k so that a total of l indices remain, and then apply the projection $\mathcal{P}^l$ to these. This produces a state of spin l from T^j and U^k, where l has one of the values $j+k$, $j+k-2$, $j+k-4,\cdots,|j-k|$. Alternatively, we can apply the antisymmetric symbol $\epsilon^{pm_1n_1}$, then contract over some number of the remaining indices and apply the projection; this produces states of $l=j+k-1$, $j+k-3$, etc. These rules can be expressed in terms of appropriate symbols for the projection operators, and these symbols would then be the coupling coefficients (Clebsch-Gordan coefficients) for integral angular momentum in the tensor notation. Our rules do not give the over-all normalization of the coefficients. The calculation of the normalizations is complicated and we avoid it here.

If we combine two half-integral spins

$$T^{j+1/2}{}_{m_1m_2\cdots m_j\alpha}U^{k+1/2}{}_{n_1\cdots n_k\beta},$$

we can contract over α, β and treat the vector indices as before; this gives all compound states except those of the highest spin $j+k+1$. To obtain this highest state, we combine α, β into a vector by applying $(C\sigma_p)_{\alpha\beta}$ where

$$C=\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$$

and then apply the projection without any contractions.

Finally, there is the mixed case

$$T^{j+1/2}{}_{m_1\cdots m_j\alpha}U^k{}_{n_1\cdots n_k}.$$

By treating the vector indices as before and applying the suitable projection, including $\mathcal{P}_\sigma$, we get all compound spin states if $j\geqq k$. But if $j<k$, we miss the lowest one, of spin $k-j-\frac{1}{2}$. The latter is obtained by making j contractions and then taking the scalar product of $\boldsymbol{\sigma}$ with one of the remaining indices.

These rules are generally quite simple to apply in dealing with systems of low spin and often serve as a painless substitute for tables of Clebsch-Gordan coefficients.

IV. TENSOR OPERATORS

1. Definition of a Tensor Operator

Let $\mathbf{S}$ be the spin operator for a system of spin J (integral or half-integral) so that $\mathbf{S}^2=J(J+1)$. Let $T^k{}_m(\mathbf{S})$ be the traceless symmetric tensor operator built from $\mathbf{S}$, that is,

$$T^k{}_{m_1\cdots m_k}(\mathbf{S})=\mathcal{P}^k(S_{m_1}S_{m_2}\cdots S_{m_k}). \quad (4.1)$$

A matrix element $(T^k{}_m(\mathbf{S}))_{m'm''}$ of a component of $T^k(\mathbf{S})$ connects the three spin states (J,m'), (J,m''), and (k,m) and so is merely a special notation for a coupling coefficient. This shows that $T^k(\mathbf{S})$ is identically zero for $k>2J$.

Moreover, for $k\neq k'$, we have

$$\mathrm{Tr}\{T^k(\mathbf{S})T^{k'}(\mathbf{S})\}=0. \quad (4.2)$$

The corresponding operator trace for $k=k'$ defines a normalization constant,

$$\mathrm{Tr}\{T^k{}_m(\mathbf{S})T^k{}_{m'}(\mathbf{S})\}=\mathcal{P}^{(k)}{}_{mm'}d_{Jk}. \quad (4.3)$$

Now d_{Jk} is a polynomial of degree k in $\mathbf{S}^2$, and vanishes for spin quantum numbers $J'<\frac{1}{2}k$. In the classical limit ($\mathbf{S}^2\to\infty$ or $\hbar\to 0$), Eq. (4.3) must be consistent with $T^k(\mathbf{p}):T^k(\mathbf{p})=c_k$. Hence

$$d_{Jk}=\frac{2J+1}{2k+1}\mathbf{S}^2(\mathbf{S}^2-\tfrac{1}{2}\times\tfrac{3}{2})\times(\mathbf{S}^2-1\times 2)\cdots\left[\mathbf{S}^2-\left(\frac{k-1}{2}\right)\left(\frac{k+1}{2}\right)\right]c_k =\frac{c_k}{2k+1}2^{-2k}\frac{(2J+k+1)!}{(2J-k)!}. \quad (4.4)$$

One may verify that the family of operators $T^k(\mathbf{S})$, $k=0, 1, 2,\cdots, 2J$ have, in toto, $(2J+1)^2$ independent components and are therefore suitable as a basis for matrices operating on a spin-J system. They are, of course, the analogs of the well-known "irreducible tensorial operators."

The application we have in mind is the following. Let $M^J{}_m$ be the amplitude for a reaction in which particle

X of spin J participates. Then the density matrix of X is

$$\lambda_{mm'}=M^J{}_m M^J{}_{m'}^*. \tag{4.5}$$

Let the density matrix be expanded in tensor operators

$$\lambda_{mm'}=\sum_{k=0}^{2J} d_{Jk}{}^{-1}\Lambda^k:(T^k(\mathbf{S}))_{mm'}. \tag{4.6}$$

Then, by (4.3)

$$\Lambda^k=\mathrm{Tr}\{\lambda T^k(\mathbf{S})\}$$
$$=M^{J*}:T^k(\mathbf{S})M^J. \tag{4.7}$$

The Λ^k are the "tensor moments" of the density matrix λ.

Each $T^k{}_m(\mathbf{S})$ is Hermitian as an operator. The lowest order tensors are

$$T^0(\mathbf{S})=1,\quad T^1(\mathbf{S})=\mathbf{S}, \tag{4.8a}$$

$$T^2{}_{12}(\mathbf{S})=\tfrac{1}{2}(S_1S_2+S_2S_1)-\tfrac{1}{3}\delta_{12}\mathbf{S}^2, \tag{4.8b}$$

$$T^3{}_{123}(\mathbf{S})=\frac{1}{3!}\sum\nolimits_P S_1S_2S_3 -\tfrac{1}{5}(\mathbf{S}^2-\tfrac{1}{3})(\delta_{12}S_3+\delta_{13}S_2+\delta_{23}S_1), \tag{4.8c}$$

$$T^4{}_{1234}(\mathbf{S})=\frac{1}{4!}\sum\nolimits_P S_1S_2S_3S_4-(1/7)(\mathbf{S}^2-\tfrac{5}{6})\sum\nolimits_P \delta_{12}S_3S_4 +(1/35)\mathbf{S}^2(\mathbf{S}^2-2)(\delta_{12}\delta_{34}+\delta_{13}\delta_{24}+\delta_{14}\delta_{23}). \tag{4.8d}$$

Notice that because of the noncommutivity of different components of $\mathbf{S}$, $T^k(\mathbf{S})$ is *not* the same function of $\mathbf{S}$ that $T^k(\mathbf{p})$ is of $\mathbf{p}$.

The explicit forms of $T^k(\mathbf{S})$ will not ordinarily be needed. It will be sufficient to consider matrix elements of $(\mathbf{n}_1\cdot\mathbf{S})(\mathbf{n}_2\cdot\mathbf{S})(\mathbf{n}_3\cdot\mathbf{S})\cdots$ between M^{J*} and M^J and then perform the projections on the resulting functions of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3\cdots$.

2. Operator Legendre Functions

The previous section contains all that is needed of the theory of tensor operators for our contemplated analyses of reactions. But having defined them, we should explain how they can be constructed.

Tensor operator formulas can often be deduced by making appropriate comparisons with ordinary tensor formulas. For example $\mathbf{S}\cdot T^n(\mathbf{S})$ must be proportional to $T^{n-1}(\mathbf{S})$. The exact formula is

$$\mathbf{S}\cdot T^n(\mathbf{S})=(n/(2n-1))(\mathbf{S}^2-\tfrac{1}{4}(n^2-1))T^{n-1}(\mathbf{S}) \tag{4.9}$$

because (4.9) must be consistent with (3.28) when $\hbar\to 0$, and because $T^n(\mathbf{S})$ vanishes for $n=2J+1$ [and hence for $J(J+1)=\frac{1}{4}(n^2-1)$] while $T^{n-1}(\mathbf{S})$ does not.

Operator Legendre functions $\bar{P}_n(x)$ may be defined by

$$T^n(\mathbf{p}):T^n(\mathbf{S})=c_n\bar{P}_n(x), \tag{4.10}$$

where $\mathbf{p}$ is a unit vector and $x=\mathbf{p}\cdot\mathbf{S}$. For $\hbar\to 0$, they are substantially the same as ordinary Legendre functions, the precise relation being $\bar{P}_n(x\sqrt{S^2})=(\sqrt{S^{2n}})\times P_n(x)$. $\bar{P}_n(x)$ vanishes for $n>2J$ when x takes on one of the eigenvalues $J, J-1,\cdots,-J$.

The standard Legendre recursion relation is

$$(2n+1)xP_n(x)=(n+1)P_{n+1}(x)+nP_{n-1}(x). \tag{4.11}$$

Then, by the same logic that led to (4.9), the operator version is

$$(2n+1)x\bar{P}_n(x)=(n+1)\bar{P}_{n+1}(x) +n(\mathbf{S}^2-\tfrac{1}{4}(n^2-1))\bar{P}_{n-1}(x). \tag{4.12}$$

The $\bar{P}_n$'s agree with the P_n's for $n=0, 1, 2$, but the noncommutivity of components of $\mathbf{S}$ makes a difference for $n\geqq 3$.

Let $R^{(n)}$ represent the symmetrized product of n components of $\mathbf{S}$, i.e.,

$$R^{(n)}{}_{12\cdots n}=(n!)^{-1}\sum\nolimits_P S_1S_2\cdots S_n. \tag{4.13}$$

To construct $T^{(n)}(\mathbf{S})$ explicitly, we must set

$$T^{(n)}{}_{123\cdots n}(\mathbf{S})=R^{(n)}{}_{12\cdots n}-a_1\sum\nolimits_P \delta_{12}R^{(n-2)}{}_{34\cdots n} +a_2\sum\nolimits_P \delta_{12}\delta_{34}R^{(n-4)}{}_{56\cdots n}-\text{etc.} \tag{4.14}$$

and then determine the coefficients a_i as functions of $\mathbf{S}^2$. The scalar product of (4.14) with $(p)_1(p)_2(p)_3\cdots(p)_n$ is

$$c_n\bar{P}_n(x)=x^n-\tfrac{1}{2}n(n-1)a_1(x^{n-2}/1!) +\tfrac{1}{2}n(n-1)\tfrac{1}{2}(n-2)(n-3)a_2(x^{n-4}/2!)-\text{etc.} \tag{4.15}$$

For a given n, the series expansion of $\bar{P}_n$ can be deduced from the recursion relation (4.12). The coefficients a_i necessary for (4.14) are then obtained by comparison with (4.15). We obtained (4.8) in this way.

3. An Expansion for the Rotation Operator

The operator which rotates a system of spin J about an axis $\mathbf{p}$ through an angle θ is $\exp(i\theta\mathbf{p}\cdot\mathbf{S})$. For $J=\frac{1}{2}$, this simplifies to

$$e^{i\theta(\mathbf{p}\cdot\mathbf{S})}=\cos\tfrac{1}{2}\theta+i(\boldsymbol{\sigma}\cdot\mathbf{p})\sin\tfrac{1}{2}\theta. \tag{4.16}$$

More generally, there will be a finite expansion of the form

$$e^{i\theta\mathbf{p}\cdot\mathbf{S}}=\sum_{n=0}^{2J}(i)^n(2n+1)\bar{j}_n(\theta)\bar{P}_n(\mathbf{p}\cdot\mathbf{S}) \tag{4.17}$$

with the functions $\bar{j}_n(\theta)$ to be determined. We have

$$\bar{j}_0(\theta)=\frac{1}{2J+1}\mathrm{Tr}\{e^{i\theta\mathbf{p}\cdot\mathbf{S}}\}=\frac{1}{2J+1}\sum_{m=-J}^{+J}e^{im\theta} =\frac{1}{2J+1}\frac{\sin(J+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}. \tag{4.18}$$

Differentiating (4.17) with respect to θ and using (4.12),

TABLE I. Reduction of tensor products to Legendre functions and ordinary scalar products for tensors of integral spin.

1. $T^j(\mathbf{p}):T^j(\mathbf{q})=c_jP_j(x);\quad x=\mathbf{p}\cdot\mathbf{q},\ c_j=j!/(2j-1)!!$
2. $T^j(\mathbf{p}):T^j(\mathbf{q}\cdots\mathbf{qa})=c_jj^{-1}\{(\mathbf{p}\cdot\mathbf{a})P_j'(x)-(\mathbf{q}\cdot\mathbf{a})P_{j-1}'(x)\}$
3. $T^j(\mathbf{p}):T^j(\mathbf{q}\cdots\mathbf{qab})=c_jj^{-1}(j-1)^{-1}\{(\mathbf{p}\cdot\mathbf{a})(\mathbf{p}\cdot\mathbf{b})P_j'' -[(\mathbf{p}\cdot\mathbf{a})(\mathbf{q}\cdot\mathbf{b})+(\mathbf{p}\cdot\mathbf{b})(\mathbf{q}\cdot\mathbf{a})]P_{j-1}'' +(\mathbf{q}\cdot\mathbf{a})(\mathbf{q}\cdot\mathbf{b})P_{j-2}''-(\mathbf{a}\cdot\mathbf{b})P_{j-1}'\}$
4. $T^j(\mathbf{p}\cdots\mathbf{pb}):T^j(\mathbf{q}\cdots\mathbf{qa})=c_jj^{-2}\{[(\mathbf{p}\cdot\mathbf{a})(\mathbf{q}\cdot\mathbf{b})+(\mathbf{p}\cdot\mathbf{b})(\mathbf{q}\cdot\mathbf{a})]P_j'' -[(\mathbf{q}\cdot\mathbf{a})(\mathbf{q}\cdot\mathbf{b})+(\mathbf{p}\cdot\mathbf{a})(\mathbf{p}\cdot\mathbf{b})]P_{j-1}''+(\mathbf{a}\cdot\mathbf{b})P_j' -(2j+1)(\mathbf{q}\cdot\mathbf{a})(\mathbf{p}\cdot\mathbf{b})P_{j-1}'\}$
5. $T^j(\mathbf{p}):T^j(\mathbf{p})=c_j$
6. $T^j(\mathbf{p}):T^j(\mathbf{p}\cdots\mathbf{pa})=c_j(\mathbf{a}\cdot\mathbf{p})$
7. $T^j(\mathbf{p}\cdots\mathbf{pb}):T^j(\mathbf{p}\cdots\mathbf{pa}) =c_j(2j)^{-1}\{(j+1)(\mathbf{a}\cdot\mathbf{b})+(j-1)(\mathbf{p}\cdot\mathbf{b})(\mathbf{p}\cdot\mathbf{a})\}$
8. $T^j(\mathbf{p}\cdots\mathbf{pb}):T^j(\mathbf{p}\cdots\mathbf{pa}_1\mathbf{a}_2)=c_j(2j)^{-1}\{(j-2)(\mathbf{p}\cdot\mathbf{a}_1)(\mathbf{p}\cdot\mathbf{a}_2)(\mathbf{p}\cdot\mathbf{b}) +(j+1)[(\mathbf{p}\cdot\mathbf{a}_2)(\mathbf{b}\cdot\mathbf{a}_1)+(\mathbf{p}\cdot\mathbf{a}_1)(\mathbf{b}\cdot\mathbf{a}_2)]-j(\mathbf{p}\cdot\mathbf{b})(\mathbf{a}_1\cdot\mathbf{a}_2)\}$

we get

$$\sum(i)^n(2n+1)(d/d\theta)\bar{j}_n(\theta)\bar{P}_n=i\mathbf{p}\cdot\mathbf{S}e^{i\theta(\mathbf{p}\cdot\mathbf{S})}$$
$$=\sum_{n=0}^{2J}i^{n+1}\bar{j}_n(\theta)[(n+1)\bar{P}_{n+1}+(n-1)[\mathbf{S}^2-\tfrac{1}{4}(n^2-1)]\bar{P}_{n-1}] \tag{4.19}$$

whence

$$(2n+1)(d/d\theta)\bar{j}_n(\theta)=n\bar{j}_{n-1}(\theta)-(n-1)(\mathbf{S}^2-\tfrac{1}{4}(n^2-1))\bar{j}_{n+1}(\theta). \tag{4.20}$$

This allows the $\bar{j}_n$'s to be calculated from $\bar{j}_0$.

Finally, set $(\mathbf{p}\cdot\mathbf{S})\to Jz$, $\theta\to k/J$, $\bar{j}_n(\theta)\to J^{-n}j_n(k)$ and let $J\to\infty$ through integral values. Then

$$\bar{j}_0\equiv j_0(k)=k^{-1}\sin k \tag{4.21}$$

which is the zeroth-order spherical Bessel function. Also, (4.20) becomes

$$(2n+1)j_n'(k)=nj_n(k)-(n+1)j_{n+1}(k) \tag{4.22}$$

which is the Bessel recursion formula. Equation (4.17) becomes the familiar relation

$$e^{ikz}=\sum_{n=0}^{\infty}(i)^n(2n+1)j_n(k)P_n(z). \tag{4.23}$$

It is amusing that (4.16) and (4.23) are both special cases of a single operator equation.

V. CALCULATION OF TENSOR PRODUCTS

1. Integral Spin. (Table I)

When angular-momentum configurations are expressed in terms of tensors, amplitudes and probabilities will be given by scalar products of tensors. For spins which are not too large, the tensors can be written out explicitly and the products directly calculated. This procedure represents a simple extension of the ordinary methods of vector analysis.

More systematic methods are appropriate for certain classes of matrix elements that occur frequently. We have already noted the relation

$$T^j(\mathbf{p}):T^j(\mathbf{q})=c_jP_j(x)|p|^j|q|^j \tag{5.1}$$

written here without the assumption that $\mathbf{p},\mathbf{q}$ are unit vectors. Other formulas follow from this one by a differential technique. Set $\mathbf{q}\to\mathbf{q}+\mathbf{a}$ in (5.1) and expand in powers of $\mathbf{a}$. Then

$$T^j(\mathbf{p}):[T^j(\mathbf{q})+jT^j(\mathbf{q}\cdots\mathbf{qa})+\cdots]=c_j\{P_j+[\mathbf{a}\cdot\hat{p}-x(\mathbf{a}\cdot\hat{p})]P_j'+\cdots\}\times\{1+j(\hat{q}\cdot\mathbf{a})/q+\cdots\}p^jq^j. \tag{5.2}$$

Now equate coefficients of $\mathbf{a}$ and let $\mathbf{p},\mathbf{q}$ again be unit vectors. The formulas

$$xP_j'=P_{j-1}'+jP_j=P_{j+1}'-(j+1)P_j \tag{5.3}$$

are often useful in such calculations. We get

$$jT^j(\mathbf{p}):T^j(\mathbf{q}\cdots\mathbf{qa})=c_j[(\mathbf{a}\cdot\mathbf{p})P_j'+(\mathbf{q}\cdot\mathbf{a})(jP_j-xP_j')]=c_j[(\mathbf{a}\cdot\mathbf{p})P_j'-(\mathbf{a}\cdot\mathbf{q})P_{j-1}']. \tag{5.4}$$

Other relations follow by repeated application of this differential method. See, for example, Eqs. (3) and (4), of Table I.

Formulas like these, but with $\mathbf{p}=\mathbf{q}$ are worth considering separately. Note that

$$T^j(\mathbf{p}):T^j(\mathbf{p}\cdots\mathbf{pa}_1\mathbf{a}_2\cdots\mathbf{a}_n)=T^n(\mathbf{p}):T^n(\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_n)(c_j/c_n)(p^2)^{j-n}. \tag{5.5}$$

The constant in (5.5) is determined by the condition that the equation be valid for $\mathbf{a}_1=\mathbf{a}_2=\cdots=\mathbf{a}_n=\mathbf{p}$. Setting $\mathbf{p}\to\mathbf{p}+\mathbf{b}$ and proceeding as before, we find

$$jT^j(\mathbf{p}\cdots\mathbf{pb}):T^j(\mathbf{p}\cdots\mathbf{pa}_1\cdots\mathbf{a}_n)+(j-n)T^j(\mathbf{p}):T^j(\mathbf{p}\cdots\mathbf{pba}_1\cdots\mathbf{a}_n)=(c_j/c_n)\{nT^n(\mathbf{p}\cdots\mathbf{b}):T^n(\mathbf{a}_1\cdots\mathbf{a}_n)+2(j-n)\mathbf{p}\cdot\mathbf{b}T^n(\mathbf{p}):T^n(\mathbf{a}_1\cdots\mathbf{a}_n)\}. \tag{5.6}$$

Equations (5) through (8) of Table I are special cases of these formulas.

TABLE II. Reduction of tensor products to Legendre functions and ordinary scalar products for tensors of half-integral spin.

1. $T^j(\mathbf{p}):(\mathbf{n}\cdot\mathbf{S})T^j(\mathbf{q})=c_j(-i\mathbf{n}\cdot\mathbf{p}\times\mathbf{q})P_j'(x)$
2. $T^j(\mathbf{p}):(\mathbf{n}\cdot\mathbf{S})T^j(\mathbf{p})=0$
3. $T^j(\mathbf{p}):(\mathbf{n}\cdot\mathbf{S})T^j(\mathbf{p}\cdots\mathbf{pa})=c_j(\tfrac{1}{2}i)(j+1)\mathbf{n}\cdot\mathbf{a}\times\mathbf{p}$
4. $T^j(\mathbf{p}\cdots\mathbf{pb})(\mathbf{n}\cdot\mathbf{S})T^j(\mathbf{p}\cdots\mathbf{pa})=c_ji(2j)^{-1}(j+1) \times\{(j-1)[(\mathbf{n}\cdot\mathbf{p}\times\mathbf{b})(\mathbf{p}\cdot\mathbf{a})-(\mathbf{p}\cdot\mathbf{b})(\mathbf{n}\cdot\mathbf{p}\times\mathbf{a})]-\mathbf{n}\cdot\mathbf{b}\times\mathbf{a}\}$
5. $T^{j+1/2}(\mathbf{p})^*:T^{j+1/2}(\mathbf{q}) =c_j(2j+1)^{-1}[(j+1)P_j(x)-i(\boldsymbol{\sigma}\cdot\mathbf{p}\times\mathbf{q})P_j'(x)]$
6. $T^{j+1/2}(\mathbf{p})^*:T^{j+1/2}(\mathbf{p})=c_j(2j+1)^{-1}(j+1)$
7. $T^{j+1/2}(\mathbf{p})^*:T^{j+1/2}(\mathbf{p}\cdots\mathbf{pa})=c_j(2j+1)^{-1}(j+1)(\mathbf{p}\cdot\mathbf{a}+\tfrac{1}{2}i\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{p})$

2. Half-Integral Spin. (Table II)

The interpretation of the angular-momentum operator **S** as a generator of rotations means that

$$T^j(\mathbf{p}):\mathbf{S}U^j=-i(\mathbf{p}\times\nabla_p)\{T^j(\mathbf{p}):U^j\} \quad (5.7)$$

for any U^j. In particular

$$T^j(\mathbf{p}):(\mathbf{n}\cdot\mathbf{S})T^j(\mathbf{q})=-i(\mathbf{n}\cdot\mathbf{p}\times\mathbf{q})P_j'(x)c_j. \quad (5.8)$$

Alternatively, it follows from (2.6) that

$$(\mathbf{n}\cdot\mathbf{S})T^j(\mathbf{q})=jT^j(\mathbf{q}\cdots\mathbf{q}\ i\mathbf{n}\times\mathbf{q}). \quad (5.9)$$

This leads again to (5.8) via (5.4).

Formulas (3) and (4) of Table II are obtained by setting

$$(\mathbf{n}\cdot\mathbf{S})T^j(\mathbf{p}\cdots\mathbf{pa}) =(j-1)T^j(\mathbf{p}\cdots i\mathbf{n}\times\mathbf{p}\ \mathbf{a})+T^j(\mathbf{p}\cdots\mathbf{p}\ i\mathbf{n}\times\mathbf{a}) \quad (5.10)$$

and applying Table I.

We have already used the notation

$$T^{j+1/2}(\mathbf{ab}\cdots\mathbf{c};u)=\mathcal{P}_\sigma T^j(\mathbf{ab}\cdots\mathbf{c})|u\rangle \quad (5.11)$$

to indicate the combination of an integral spin j with a spinor of spin $\frac{1}{2}$ to form $j+\frac{1}{2}$. The complex-conjugate relation (with a different spinor) is

$$\langle v|T^j(\mathbf{ab}\cdots\mathbf{c})\mathcal{P}_\sigma=T^{j+1/2}(\mathbf{ab}\cdots\mathbf{c};v)^* \quad (5.12)$$

since P_σ is Hermitian. In what follows, we write $T^{j+1/2}(\mathbf{ab}\cdots\mathbf{c})$, $T^{j+1/2}(\mathbf{p})$, etc., without explicitly mentioning the spinor. Then $T^{j+1/2}(\mathbf{ab}\cdots)^*:T^{j+1/2}(\mathbf{a'b'}\cdots)$ is a 2×2 matrix in spin space. For example,

$$T^{j+1/2}(\mathbf{p})^*:T^{j+1/2}(\mathbf{q})=T^j(\mathbf{p}):\mathcal{P}_\sigma T^j(\mathbf{q}) =\left\{\frac{j+1}{2j+1}P_j(x)-\frac{i\boldsymbol{\sigma}\cdot\mathbf{p}\times\mathbf{q}}{2j+1}P_j'(x)\right\}c_j. \quad (5.13)$$

The remainder of Table II follows directly.

3. Matrix Elements of Tensor Operators. (Table III)

We first obtain an expression for $T^j(\mathbf{p}):T^k(\mathbf{S})T^j(\mathbf{q})$. We begin by considering the quantity

$$Z=T^j(\mathbf{p}):(\mathbf{n}_1\cdot\mathbf{S})(\mathbf{n}_2\cdot\mathbf{S})\cdots(\mathbf{n}_k\cdot\mathbf{S})T^j(\mathbf{q}) \quad (5.14)$$

and then make Z traceless and symmetric with respect to the **n**'s. Note that

$$Z=\{\mathbf{n}_k\cdot(-i\mathbf{p}\times\nabla_p)\mathbf{n}_{k-1}\cdot(-i\mathbf{p}\times\nabla_p)\cdots \times\mathbf{n}_1\cdot(-i\mathbf{p}\times\nabla_p)\}c_jP_j(x). \quad (5.15)$$

Now $\mathbf{n}_1\cdot(-i\mathbf{p}\times\nabla_p)$ applied to any $f(x)$ yields $X_1f'(x)$ where $X_1=-i\mathbf{n}_1\cdot(\mathbf{p}\times\mathbf{q})$. Also, $\mathbf{n}_2\cdot(-i\mathbf{p}\times\nabla_p)$ applied to X_1 causes **p** in X_1 to be replaced by $-i(\mathbf{p}\times\mathbf{n}_2)$ and yields X_{12};

$$X_{12}=(-i)^2(\mathbf{p}\times\mathbf{n}_2)\cdot(\mathbf{n}_1\times\mathbf{q}) =(-i)^2[(\mathbf{p}\cdot\mathbf{n}_1)(\mathbf{q}\cdot\mathbf{n}_2)-(\mathbf{n}_1\cdot\mathbf{n}_2)\mathbf{p}\cdot\mathbf{q}]. \quad (5.16)$$

The $\mathbf{n}_1\cdot\mathbf{n}_2$ part can be dropped because we intend to make the result traceless in the **n**'s. Similarly, the result of $\mathbf{n}_3\cdot(-i\mathbf{p}\times\nabla_p)$ on X_{12} can be dropped entirely. One obtains, in a more or less obvious notation,

$$c_j^{-1}Z\equiv X_1X_2\cdots X_kP_j^{(k)}(x)+\sum X_{12}X_3\cdots X_k \times P_j^{(k-1)}(x)+\sum X_{12}X_{34}X_5X_6\cdots X_kP_j^{(k-2)}+\cdots +\sum X_{12}X_{34}X_{56}\cdots P_j^{\frac{1}{2}k}\quad (\text{or } P_j^{\frac{1}{2}(k+1)}). \quad (5.17)$$

Here, $P_j^{(n)}$ means the nth derivative of P_j. Counting carefully, we find that (we put $\mathbf{p}\times\mathbf{q}=\mathbf{r}$)

$$T^j(\mathbf{p}):T^k(\mathbf{S})T^j(\mathbf{q})=(-i)^kc_j \times\{T^k(\mathbf{r})P_j^{(k)}(x)+T^k(\mathbf{rr}\cdots\mathbf{rpq})(\tfrac{1}{2}k(k-1))P_j^{(k-1)}(x) +T^k(\mathbf{r}\cdots\mathbf{rpqpq})(\tfrac{1}{2}k(k-1))(\tfrac{1}{2}(k-2)(k-3)) \times(2!)^{-1}P_j^{(k-2)}(x)+\cdots\} \quad (5.18)$$

with $\frac{1}{2}k$ terms in the sum if k is even and $\frac{1}{2}(k+1)$ terms if k is odd. We can also write the series starting from the other end;

$$T^j(\mathbf{p}):T^{2n}(\mathbf{S})T^j(\mathbf{q}) =(-1)^nc_j(2n-1)!![T^{2n}(\mathbf{pqpq}\cdots\mathbf{pq})P_j^{(n)}(x) +nT^{2n}(\mathbf{pqpq}\cdots\mathbf{pq}\ \mathbf{p}\times\mathbf{q}\ \mathbf{p}\times\mathbf{q})P_j^{(n+1)}(x)+\cdots] \quad (5.19)$$

and

$$T^j(\mathbf{p}):T^{2n-1}(\mathbf{S})T^j(\mathbf{q}) =i(-1)^nc_j(2n-1)!![T^{2n-1}(\mathbf{pqpq}\cdots\mathbf{pq}\ \mathbf{p}\times\mathbf{q})P_j^{(n)}(x) +\tfrac{1}{3}nT^{2n-1}(\mathbf{pq}\cdots\mathbf{pq}\ \mathbf{p}\times\mathbf{q}\ \mathbf{p}\times\mathbf{q}\ \mathbf{p}\times\mathbf{q})P_j^{(n+1)}(x)+\cdots]. \quad (5.20)$$

The value of the nth derivative of the Legendre function at $x=1$ is

$$P_j^{(n)}(1)=\frac{(j+n)!}{(j-n)!}\frac{1}{2^nn!}. \quad (5.21)$$

Thus, putting $\mathbf{p}=\mathbf{q}$ in (5.19) yields

$$T^j(\mathbf{p}):T^{2n}(\mathbf{S})T^j(\mathbf{p})=a_{jn}T^{2n}(\mathbf{p}), \quad (5.22)$$

where

$$a_{jn}=(-1)^nc_jP_j^{(n)}(1)(2n-1)!!. \quad (5.23)$$

By applying the differential technique to (5.22) or, if necessary, to (5.19), (5.20), one obtains formulas (2) through (6) of Table III.

So far in this section, we have written **S** for the spin operator on systems of spin j; that is $\mathbf{S}\equiv\mathbf{S}^j$. Noting again that $\mathbf{S}^{j+1/2}=\mathbf{S}^j+\frac{1}{2}\boldsymbol{\sigma}$, we have

$$T^k(\mathbf{S}^{j+1/2})=T^k(\mathbf{S}^j+\tfrac{1}{2}\boldsymbol{\sigma})=T^k(\mathbf{S}^j)+\frac{k}{2}T^k(\mathbf{S}^j\cdots\mathbf{S}^j\boldsymbol{\sigma}) \quad (5.24)$$

since any tensors with two or more $\boldsymbol{\sigma}$'s must vanish. Moreover,

$$T^{j+1/2}(\mathbf{p})\equiv\mathcal{P}_\sigma T^j(\mathbf{p})\equiv T^j(\mathbf{p}\cdots\mathbf{pp}'), \quad (5.25)$$

where

$$\mathbf{p}'=[(j+1)\mathbf{p}+ji\boldsymbol{\sigma}\times\mathbf{p}](2j+1)^{-1}. \quad (5.26)$$

With these representations and (2), (5) of Table III, one can derive the remaining formulas of this table.

The results in Table III will be used in the moment analyses of reactions in which a resonance decays into

TABLE III. Matrix elements of tensor operators.

1. $T^j(\mathbf{p}):T^{2n}(\mathrm{S})T^j(\mathbf{p})=a_{jn}T^{2n}(\mathbf{p})$; [$a_{jn}$ given in Eq. (5.23)]
2. $T^j(\mathbf{p}):T^{2n}(\mathrm{S})T^j(\mathbf{p}\cdots\mathbf{pa})=a_{jn}\{(n/j)T^{2n}(\mathbf{p}\cdots\mathbf{pa})$ $+((j-n)/j)(\mathbf{p}\cdot\mathbf{a})T^{2n}(\mathbf{p})\}$
3. $T^j(\mathbf{p}\cdots\mathbf{pb}):T^{2n}(\mathrm{S})T^j(\mathbf{p}\cdots\mathbf{pa})=a_{jn}\{(n/j(n+1))$ $\times[(j+1)T^{2n}(\mathbf{p}\cdots\mathbf{pab})-(j-n)(\mathbf{p}\cdot\mathbf{b})T^{2n}(\mathbf{p}\cdots\mathbf{pa})$ $-(j-n)(\mathbf{p}\cdot\mathbf{a})T^{2n}(\mathbf{p}\cdots\mathbf{pb})]+((j-n)(2n+1)/2j^2(n+1))$ $\times T^{2n}(\mathbf{p})[(j+n+1)\mathbf{a}\cdot\mathbf{b}+(j-n-1)(\mathbf{p}\cdot\mathbf{a})(\mathbf{p}\cdot\mathbf{b})]\}$
4. $T^j(\mathbf{p}):T^{2n-1}(\mathrm{S})T^j(\mathbf{p})=0$
5. $T^j(\mathbf{p}):T^{2n-1}(\mathrm{S})T^j(\mathbf{p}\cdots\mathbf{pa})=ia_{jn}T^{2n-1}(\mathbf{p}\cdots\mathbf{p}\,\mathbf{p}\times\mathbf{a})$
6. $T^j(\mathbf{p}\cdots\mathbf{pb}):T^{2n-1}(\mathrm{S})T^j(\mathbf{p}\cdots\mathbf{pa})=ia_{jn}\{((n-1)/j)$ $\times[T^{2n-1}(\mathbf{p}\cdots\mathbf{pb}\,\mathbf{p}\times\mathbf{a})-T^{2n-1}(\mathbf{p}\cdots\mathbf{p}\,\mathbf{p}\times\mathbf{b}\,\mathbf{a})]$ $+((j-n)/j)[(\mathbf{p}\cdot\mathbf{b})T^{2n-1}(\mathbf{p}\cdots\mathbf{p}\,\mathbf{p}\times\mathbf{a})$ $-T^{2n-1}(\mathbf{p}\cdots\mathbf{p}\,\mathbf{p}\times\mathbf{b})(\mathbf{p}\cdot\mathbf{a})]+j^{-1}T^{2n-1}(\mathbf{p}\cdots\mathbf{p}\,\mathbf{b}\times\mathbf{a})\}$
7. $T^{j+1/2}(\mathbf{p})^*:T^{2n}(\mathrm{S})T^{j+1/2}(\mathbf{p})=a_{jn}((j+n+1)/(2j+1))T^{2n}(\mathbf{p})$
8. $T^{j+1/2}(\mathbf{p})^*:T^{2n-1}(\mathrm{S})T^{j+1/2}(\mathbf{p})=a_{jn}((2n-1)(j+n)/2n(2j+1))$ $\times[(j+1)T^{2n-1}(\mathbf{p}\cdots\mathbf{p}\boldsymbol{\sigma})+(n-j-1)(\mathbf{p}\cdot\boldsymbol{\sigma})T^{2n-1}(\mathbf{p})]$

two particles with spins of zero and one-half or spins of zero and one. The derivations given are of a less elementary nature than those of the preceding tables, but they express the data of the moment analysis in almost final form. Simpler derivations, which do not give the over-all normalizations, may suggest themselves to the reader. Formulas (5) and (7) are obvious by inspection, for example, apart from normalization. In fact, we shall not make use of the normalizations, but merely wished to show how they might be obtained.

VI. CONNECTION BETWEEN TENSORS AND THE FUNCTIONS OF THE ROTATION GROUP

1. Spherical Coordinates

Let $\mathbf{n}^1$, $\mathbf{n}^2$, and $\mathbf{n}^3$ be mutually orthonormal basis vectors. The Cartesian components of a unit vector $\mathbf{p}$ in this basis are just $p_m=\mathbf{n}^m\cdot\mathbf{p}$ with $p_1=\sin\theta\cos\varphi$, etc.

The following set of basis vectors is often called "spherical:"

$$\mathbf{n}^+=-(\sqrt{2})^{-1}(\mathbf{n}^1+i\mathbf{n}^2) \qquad (6.1\text{a})$$

$$\mathbf{n}^0=\mathbf{n}^3 \qquad (6.1\text{b})$$

$$\mathbf{n}^-=(\sqrt{2})^{-1}(\mathbf{n}^1-i\mathbf{n}^2). \qquad (6.1\text{c})$$

The spherical components of $\mathbf{p}$ are $p_M=(\mathbf{n}^M)^*\cdot\mathbf{p}$, $M=+,0,-$:

$$p_+=(\mathbf{n}^+)^*\cdot\mathbf{p}=-(\sqrt{2})^{-1}(p_1-ip_2)=-(\sqrt{2})^{-1}\sin\theta\, e^{-i\varphi} \qquad (6.2\text{a})$$

$$p_0=(\mathbf{n}^0)^*\cdot\mathbf{p}=p_3=\cos\theta \qquad (6.2\text{b})$$

$$p_-=(\mathbf{n}^-)^*\cdot\mathbf{p}=(\sqrt{2})^{-1}(p_1+ip_2)=(\sqrt{2})^{-1}\sin\theta\, e^{+i\varphi}. \qquad (6.2\text{c})$$

Scalar products behave as follows:

$$\begin{aligned}\mathbf{p}\cdot\mathbf{q}&=p_0q_0+(p_+)^*q_++(p_-)^*q_-\\&=p_0q_0-p_-q_+-p_+q_-=g^{MM'}p_Mq_{M'}\end{aligned} \qquad (6.3)$$

with

$$g^{00}=+1\,,\quad g^{-+}=g^{+-}=-1\,,\quad \text{other } g^{MM'}=0. \qquad (6.4)$$

All the vector indices of tensors may be expressed in spherical components. The formulas already derived remain valid if scalar products are taken as described and Kronecker deltas are replaced by $g^{MM'}$. In this scheme, the important thing about a tensor component is the number of $+$, 0, and $-$ subscripts. Let these numbers be M_+, M_0, M_-. The tensor component can be written

$$T_m{}^j\to T^j{}_{M_0M_+M_-}.$$

Such a component has a definite magnetic quantum number M,

$$M=M_+-M_-.$$

For half-integral spins, of course, $M=M_+-M_-\pm\frac{1}{2}$ depending on the spinor index.

Any two tensor's components, like T_{+-} and T_{00}, which have the same M must then be proportional. In fact, by symmetry and tracelessness $(g^{MM'}T_{MM'}\cdots=0)$, we infer

$$T_{+-\cdots}=T_{-+\cdots}=\tfrac{1}{2}T_{00\cdots}. \qquad (6.5)$$

For half-integral spins, $\boldsymbol{\sigma}\cdot T^{j+1/2}=0$ gives the additional proportion

$$T^{j+1/2}{}_{0\cdots,\pm1/2}=\sqrt{2}\,T^{j+1/2}{}_{\pm\cdots,\mp1/2}. \qquad (6.6)$$

2. Spherical Harmonics

Let $\mathbf{p}$ be a unit vector, with components $p_M=(\mathbf{n}^M)^*\cdot\mathbf{p}$ and angular coordinates θ, φ, as above. The spherical harmonic $Y_{jM}(\theta,\varphi)$ is defined as the wave function of state $|jM\rangle$ in the θ,φ representation, i.e.,

$$Y_{jM}(\theta,\varphi)=\langle\theta,\varphi|jM\rangle. \qquad (6.7)$$

Our $T^j(\mathbf{p})$ has been defined the other way around; thus

$$T^j{}_{M_0,M_+,M_-}(\mathbf{p})\sim\langle jM|\theta,\varphi\rangle,\quad M=M_+-M_-. \qquad (6.8)$$

One last elaboration of the tensor notation is going to be useful here: Let $T^{j,M}(\mathbf{n})$ denote $T^j(\mathbf{n}^0\cdots\mathbf{n}^0\mathbf{n}^+\cdots\mathbf{n}^+)$ if $M>0$ and $T^j(\mathbf{n}^0\cdots\mathbf{n}^0\mathbf{n}^-\cdots\mathbf{n}^-)$ if $M<0$ where the number of n^0's is $J-|M|$ and the number of $\mathbf{n}^+$'s (or $\mathbf{n}^-$'s) is $|M|$. It is understood that in this context "$\mathbf{n}$" represents a basis $\mathbf{n}^0$, $\mathbf{n}^+$, $\mathbf{n}^-$ rather than a single vector.

We define a function $\bar{Y}_{jM}(\theta,\varphi)$ for $M\geqq0$

$$\bar{Y}_{jM}(\theta,\varphi)=[T^j{}_{00\cdots0++\cdots+}(\mathbf{p})]^*\equiv T^j(\mathbf{p}):T^{j,M}(\mathbf{n})\,, \qquad (6.9)$$

where there are M_0 zeros and M plus signs in the subscript labeling the element of T^j. Since it is always easy to go from the $M\geqq0$ case to $M\leqq0$, we shall consider only the first case explicitly. For $M=0$, (6.9) gives

$$\bar{Y}_{j0}(\theta,\varphi)=T^j(\mathbf{p}):T^j(\mathbf{n}^0)=c_jP_j(x)\,,\quad x=\cos\theta. \qquad (6.10)$$

This $\bar{Y}_{jM}$ is the required spherical harmonic, yet unnormalized.

In (6.10) we set

$$\begin{aligned}\mathbf{n}^0\to\mathbf{n}^0+\epsilon\mathbf{n}^+,\quad x\to x&+\epsilon\mathbf{n}^+\cdot\mathbf{p}\\&=x-(1/\sqrt{2})\sin\theta e^{i\varphi}\epsilon,\end{aligned} \qquad (6.11)$$

where ϵ is an arbitrary number. This is permissible since

$$(\mathbf{n}^0+\epsilon\mathbf{n}^+)\cdot(\mathbf{n}^0+\epsilon\mathbf{n}^+)=\mathbf{n}^0\cdot\mathbf{n}^0=1\,. \tag{6.12}$$

After the substitution $\mathbf{n}^0\rightarrow\mathbf{n}^0+\epsilon\mathbf{n}^+$ in $T^j(\mathbf{n}^0)$, the coefficient of ϵ^M is seen to be

$$\frac{j!}{(j-M)!M!}T^{j,M}(\mathbf{n})\,.$$

Hence

$$\begin{aligned}\bar{Y}_{jM}&=c_j\frac{(j-M)!}{j!}\left(\frac{-\sin\theta e^{i\varphi}}{\sqrt{2}}\right)^M\left(\frac{d}{dx}\right)^M P_j(x)\\&=c_j\frac{(j-M)!}{j!}\left(-\frac{1}{\sqrt{2}}\right)^M P_j{}^M(x)e^{iM\varphi}\end{aligned} \tag{6.13}$$

which connects up the spherical harmonic with the associated Legendre function.

3. Rotation Functions

Let R be a rotation which carries the basis $\mathbf{n}^1$, $\mathbf{n}^2$, $\mathbf{n}^3$ into the basis $\mathbf{N}^1$, $\mathbf{N}^2$, $\mathbf{N}^3$. This defines a rotation matrix

$$R_{mm'}=\mathbf{n}^m\cdot(R\mathbf{n}^{m'})=\mathbf{n}^m\cdot\mathbf{N}^{m'}\,. \tag{6.14}$$

The rotation can be accomplished by a sequence of three (positive) rotations: by α about the three-axis, by β about the (new) two-axis, by γ about the (new) three-axis. This specifies the Euler angles α, β, and γ in agreement with Wigner, Rose, and Jacob and Wick, but not with all other authors. The R matrix is

$$\begin{bmatrix}-\sin\alpha\sin\gamma+\cos\alpha\cos\gamma\cos\beta & -\sin\alpha\cos\gamma-\cos\alpha\sin\gamma\cos\beta & \cos\alpha\sin\beta\\ \cos\alpha\sin\gamma+\sin\alpha\cos\gamma\cos\beta & -\cos\alpha\cos\gamma-\sin\alpha\sin\gamma\cos\beta & \sin\alpha\sin\beta\\ -\cos\gamma\sin\beta & \sin\gamma\sin\beta & \cos\beta\end{bmatrix}. \tag{6.15}$$

Thus, in the $\mathbf{n}^1$, $\mathbf{n}^2$, $\mathbf{n}^3$ frame, $\mathbf{N}^3$ has polar and azimuthal angles β,α whereas in the $\mathbf{N}^1$, $\mathbf{N}^2$, $\mathbf{N}^3$ frame, $\mathbf{n}^3$ has angles $\beta,\pi-\gamma$. In the spherical coordinate notation, the matrix elements of R are

$$\begin{aligned}R_{MM'}&=(\mathbf{n}^M)^*\cdot R(\mathbf{n}^{M'})=(\mathbf{n}^M)^*\cdot\mathbf{N}^{M'}\\&=MM' \text{ matrix element of}\begin{bmatrix}\frac{1}{2}(1+\cos\beta)e^{i(-\alpha-\gamma)} & -\frac{1}{2}\sqrt{2}\sin\beta e^{-i\alpha} & \frac{1}{2}(1-\cos\beta)e^{i(-\alpha+\gamma)}\\ \frac{1}{2}\sqrt{2}\sin\beta e^{-i\gamma} & \cos\beta & -\frac{1}{2}\sqrt{2}\sin\beta e^{i\gamma}\\ \frac{1}{2}(1-\cos\beta)e^{i(\alpha-\gamma)} & \frac{1}{2}\sqrt{2}\sin\beta e^{i\alpha} & \frac{1}{2}(1+\cos\beta)e^{i(\alpha+\gamma)}\end{bmatrix}\end{aligned} \tag{6.16}$$

with the rows and columns listed in the order +, 0, −.

We now define $r^j{}_{MM'}\equiv r^j{}_{MM'}(\alpha,\beta,\gamma)$ as the matrix element of the rotation operator between states of spin j:

$$\begin{aligned}r^j{}_{MM'}&=T^{j,M}(\mathbf{n})^*:RT^{j,M'}(\mathbf{n})\\&=(-1)^M T^{j,-M}(\mathbf{n}):T^{j,M'}(\mathbf{N})\,.\end{aligned} \tag{6.17}$$

As special cases, we have

$$\begin{aligned}r^j{}_{0M'}&=T^j(\mathbf{n}^0):T^{j,M'}(\mathbf{N})\\&=\bar{Y}_{jM'}(\beta,\pi-\gamma)=(-1)^{M'}\bar{Y}_{JM'}(\beta,\gamma)^*;\end{aligned} \tag{6.18a}$$

$$r^j{}_{M0}=T^{j,M}(\mathbf{n})^*:T^j(\mathbf{N}^0)=\bar{Y}_{jM}(\beta,\alpha)^*\,. \tag{6.18b}$$

These functions are proportional to the $D^j{}_{MM'}(\alpha,\beta,\gamma)$ of Wigner, but not yet normalized. They are complex conjugates of the wave functions of a spherical top.

For half-integral spin, we need a spinor basis v^+, v^- and the rotated basis V^+, $V^-=R(v^+,v^-)$. Then $\langle v^M|V^{M'}\rangle$ is the MM'th element of

$$\begin{pmatrix}\cos\frac{1}{2}\beta e^{\frac{1}{2}i(-\alpha-\gamma)} & -\sin\frac{1}{2}\beta e^{\frac{1}{2}i(-\alpha+\gamma)}\\ \sin\frac{1}{2}\beta e^{\frac{1}{2}i(\alpha-\gamma)} & \cos\frac{1}{2}\beta e^{\frac{1}{2}i(\alpha+\gamma)}\end{pmatrix}. \tag{6.19}$$

The rotation function is (case $M\geqq 0$, $M'\geqq 0$)

$$\begin{aligned}r^{j+1/2}{}_{M+1/2,M'+1/2}&=T^{j+1/2,M}(\mathbf{n},v^+)^*:T^{j+1/2,M'}(\mathbf{N},V^+)\\&=\langle v^+|T^{j,M}(\mathbf{n})^*:\left(\frac{j+1}{2j+1}+\frac{\boldsymbol{\sigma}\cdot\mathbf{S}^j}{2j+1}\right)\\&\qquad\times T^{j,M'}(\mathbf{N})|V^+\rangle\,.\end{aligned} \tag{6.20}$$

It is necessary that the spinor basis be aligned with the space basis; that is, $v^\pm$ are eigenstates of $(\mathbf{n}^0\cdot\boldsymbol{\sigma})$ with eigenvalues ±1. Then

$$\boldsymbol{\sigma}\cdot\mathbf{S}^j=\sigma_1(\mathbf{n}^1\cdot\mathbf{S}^j)+\sigma_2(\mathbf{n}^2\cdot\mathbf{S}^j)+\sigma_3(\mathbf{n}^3\cdot\mathbf{S}^j)\,. \tag{6.21}$$

In evaluating the spinor matrix elements of (6.20), it is helpful to note that

$$T^{j,M}(\mathbf{n})^*(\mathbf{n}^0\cdot\mathbf{S})=MT^{j,M}(\mathbf{n})^* \tag{6.22}$$

and

$$\begin{aligned}T^{j,M}(\mathbf{n})^*(\mathbf{n}^-\cdot\mathbf{S})&=-[T^{j,M}(\mathbf{n})(\mathbf{n}^+\cdot\mathbf{S})]^*\\&=-(j-M)T^{j,M+1}(\mathbf{n})^*\end{aligned} \tag{6.23}$$

because $\mathbf{n}^0(\mathbf{n}^+\cdot\mathbf{S})=\mathbf{n}^+$, $\mathbf{n}^+(\mathbf{n}^+\cdot\mathbf{S})=0$. Then

$$\begin{aligned}(2j+1)r^{j+1/2}{}_{M+1/2,M'+1/2}&=(j+M+1)\cos\tfrac{1}{2}\beta r^j{}_{MM'}\\&\quad\times e^{-\frac{1}{2}i(\gamma+\alpha)}+(j-M)\sqrt{2}\sin\tfrac{1}{2}\beta r^j{}_{M+1,M'}e^{-\frac{1}{2}i(\gamma-\alpha)}\,.\end{aligned} \tag{6.24}$$

This result holds regardless of the signs of M, M'. By (6.13), we have

$$T^{j,M}(\mathbf{n})^*:T^j(\mathbf{N}^0)=c_j\frac{(j-M)!}{j!}\frac{e^{-iM\alpha}}{(-\sqrt{2})^M}P_j{}^M(\cos\beta)\,. \tag{6.25}$$

Set $\mathbf{N}^0\rightarrow\mathbf{N}^0+\epsilon\mathbf{N}^+$ in (6.25) and equate coefficients of $\epsilon^{M'}$. Under this substitution, $\alpha\rightarrow\alpha$ and

$$\begin{aligned}\cos\beta=\mathbf{n}^0\cdot\mathbf{N}^0&\rightarrow\mathbf{n}^0\cdot(\mathbf{N}^0+\epsilon\mathbf{N}^+)\\&\rightarrow\cos\beta+\epsilon\sin\beta e^{-i\gamma}/\sqrt{2}\,.\end{aligned} \tag{6.26}$$

Hence, reasoning as in the case of (6.13), we have

$$r^j{}_{MM'}=c_j\frac{(j-M)!(j-M')!}{j!j!}\left(\frac{e^{-i\alpha}}{-\sqrt{2}}\right)^M \times\left(\frac{e^{-i\gamma}\sin\beta}{\sqrt{2}}\right)^{M'}\left(\frac{d}{d\cos\beta}\right)^{M'}P_j{}^M(\cos\beta). \quad (6.27)$$

Particular rotation functions are usually obtained by recursion relations, differential formulas such as (6.27), or by Wigner's explicit series formula. It appears to us that very often the quickest way is by direct calculation of (6.17). For normalization, see below.

4. Normalizations

In spherical coordinates, the projection matrix element $\mathcal{P}^j{}_{MM'}$ will vanish unless the magnetic quantum number is "conserved." The simplest case is

$$\mathcal{P}^j{}_{+++\cdots,\,---\cdots}=g^{+-}g^{+-}\cdots=(-1)^j. \quad (6.28)$$

Note too that

$$T^j{}_{+++\cdots+}(\mathbf{p})=(-[\sin\theta e^{-i\varphi}]/\sqrt{2})^j. \quad (6.29)$$

Hence, applying (3.25) with $m=++\cdots$, $m'=--\cdots$, we get

$$c_j=(2j+1)\times\tfrac{1}{2}\int_0^\pi 2^{-j}(\sin\theta)^{2j}d\cos\theta = j!/(2j-1)!! \quad (6.30)$$

in agreement with the calculation of Sec. III.5.

Let us adopt a more explicit notation for the projection matrix elements:

$$\mathcal{P}^j{}_{MM'}\to\mathcal{P}^j(M_+M_-/M_+'M_-'). \quad (6.31)$$

We put

$$\mathcal{P}^j(M0/0M')=\delta_{MM'}(-1)^M c_{jM} \quad (6.32)$$

and seek to determine c_{jM}. The projection operator obeys

$$\sum \mathcal{P}^j(M0/M_+''M_-'')(-1)^M\mathcal{P}^j(M_-''M_+''/0M) = \mathcal{P}^j(M0/0M). \quad (6.33)$$

The sum is over all index sets $\{M_1,M_2,\cdots,M_j\}$ with total magnetic quantum number M. The $(-1)^M$ comes from the way a sum over spherical indices is taken; see (6.3).

Using (6.32), we get

$$c_{jM}{}^2[\text{all terms, properly weighted}](-1)^M = c_{jM}(-1)^M. \quad (6.34)$$

To evaluate the bracket, we first count up all the sequences $\{M_1,M_2,\cdots,M_j\}$ of type $(M,0)$. Then take those of type $(M+1, 1)$ and weight them with a factor of $\frac{1}{4}$ in virtue of (6.5). And so on. The bracket can be expressed as follows:

$$\begin{aligned}[\;]&=\text{coefficient of } X^M \text{ in } 2^{|M|}(1+\tfrac{1}{2}X+\tfrac{1}{2}X^{-1})^j\\ &=\text{coefficient of } X^{M+j} \text{ in } 2^{|M|-j}(1+X)^{2j}\\ &=2^{|M|-j}(2j)!/(j-M)!(j+M)!.\end{aligned} \quad (6.35)$$

Hence

$$c_{jM}=2^{j-|M|}(j-M)!(j+M)!/(2j)!. \quad (6.36)$$

For $M=0$, c_{jM} reduces to c_j.

We can now write, on the basis of rotation invariance,

$$\begin{aligned}\int \bar{Y}_{jM}(\theta,\varphi)\bar{Y}_{jM'}{}^*(\theta,\varphi)d\Omega &=\int(-1)^M T^j{}_{0\cdots++\cdots}(\mathbf{p})T^j{}_{0\cdots--\cdots}(\mathbf{p})d\Omega\\ &=(-1)^M\mathcal{P}^j(M0/0M')d_j\\ &=c_{jM}d_j\delta_{MM'},\end{aligned} \quad (6.37)$$

where d_j is a constant independent of M.

Specializing to $M=M'=0$, we have

$$c_j{}^2\int P_j(\cos\theta)P_j(\cos\theta)d\Omega=c_j{}^2\left(\frac{4\pi}{2j+1}\right)\equiv c_jd_j, \quad (6.38)$$

whence

$$d_j=4\pi c_j/(2j+1). \quad (6.39)$$

The normalized spherical harmonic is

$$Y_{jM}(\theta,\varphi)=((2j+1)/4\pi)^{1/2}(c_jc_{jM})^{-1/2}\bar{Y}_{jM}(\theta,\varphi). \quad (6.40)$$

The same method, applied to the rotation functions, yields

$$\int r^j{}_{MM'}(r^j{}_{MM'})^*d\alpha d\cos\beta d\gamma=\frac{8\pi^2}{2j+1}c_{jM}c_{jM'}. \quad (6.41)$$

Finally, by following this development for

$$r^{j+1/2}{}_{M+1/2,M'+1/2},$$

one finds that (6.41) is valid regardless of whether j, M, and M' are interpreted as integral or half-integral. The connection with the usual rotation function is then

$$D^j{}_{MM'}(\alpha,\beta,\gamma)=c_{jM}{}^{-1/2}c_{jM'}{}^{-1/2}r^j{}_{MM'}(\alpha,\beta,\gamma). \quad (6.42)$$