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Dashen-Lee Sum Rules for Magnetic Moments*

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This work stresses the importance of the contribution of an isotopic-spin-$\frac{1}{2}$ pion-nucleon intermediate state for the Dashen-Lee sum rules which are obtained from the commutation relations between various combinations of the isoscalar and isovector magnetic-moment operators M^S and M^V. In particular, it is shown that the M^S-M^V sum rule can be brought into agreement with experiment by introducing this contribution. Furthermore, it does not have much influence on the "good" M^V-M^V sum rule $(\mu_p{}^2/2m)^2=\langle r\rangle_p{}^2/6$, where μ_p, m, and $\langle r\rangle_p$ are the total magnetic moment, mass, and charge radius of the proton, respectively. The implications of these results for the commutation relations between space components of current densities are also discussed.

IN the past months a great deal of attention has been devoted to the study of equal-time commutation relations between charges, current densities, and moments of current densities.

By taking matrix elements of the commutator between physical states and inserting a complete set of intermediate states, one obtains various sum rules. In particular, Fubini *et al.*[1] obtained general sum rules from the commutators of charges with charges and current densities, and give elegant dispersion-theoretical interpretations of the resulting sum rules. Adler[2] and Weisberger,[3] by commuting the space integrals of the fourth component of the axial-vector current density, obtain sum rules for the axial-vector coupling constant in β decay which turn out to be in good agreement with the experimental value.

A somewhat different approach has been taken by Dashen and Gell-Mann[4] and Lee,[5] who derive sum rules from the equal-time commutation relations of two magnetic-moment operators by considering as intermediate states only those contained in the **56** representation of $SU(6)$. As we shall see later, while the sum rule obtained by commuting two isovector magnetic-moment operators is in good agreement with experiment, the one obtained from the commutation relation between an isovector and an isoscalar magnetic-moment operator is in strong disagreement with experiment.

We would like to stress the particular importance of sum rules obtained from the Dashen-Lee commutation relations; up to now, these have been the only ones which provide a test for the postulated commutation relations between space components of the current densities. It is critical then to see if the "bad" isoscalar-isovector sum rule can be brought into agreement with experiment by including further intermediate states, and particularly, the isotopic-spin-$\frac{1}{2}$ pion-nucleon state. The result, obtained here by means of an essentially qualitative calculation, is that this state is sufficient to saturate the sum rule.

We also investigate the effect of the π-N intermediate state in the "good" isovector-isovector sum rule. After isolating the contribution of the one-nucleon intermediate state, we find that most of the contribution of the π-N states comes from the **33** resonance; the other contributions, which produced a large effect in the M^S-M^V sum rule, mostly cancel in this case. Hence we obtain a result which essentially is the one obtained by Dashen and Gell-Mann[4] and Lee.[5]

The importance of the continuum intermediate states has also been pointed out by Schnitzer,[6] in connection with similar calculations involving the meson systems.

The Dashen-Lee commutation relations between magnetic-moment operators are given by

$$[M_i{}^\alpha,M_j{}^\beta]=\tfrac{1}{4}if\alpha\beta\gamma\int d^3r\,(r^2\delta_{ij}-r_ir_j)J_0{}^\gamma +\tfrac{1}{4}id\alpha\beta\gamma'\int d^3r\;\epsilon_{ijk}r_k(\mathbf{r}\cdot\mathbf{A}^{\gamma'})\,, \quad (1)$$

where

$$M_i{}^\alpha=\frac{1}{2}\int d^3r\;\epsilon_{ijk}r_jJ_k{}^\alpha. \quad (2)$$

$i, j, k=1, 2, 3$; $\alpha, \beta, \gamma, \gamma'=1\cdots8$ are $SU(3)$ indexes; $J_\mu{}^\alpha$ and $A_\mu{}^\alpha$ ($\mu=1, 2, 3, 4$) are, respectively, the vector and axial-vector current operators. $f_{\alpha\beta\gamma}$ and $d_{\alpha\beta\gamma}$ are the F and D matrix elements given by Gell-Mann.[7] If one now takes $i=j=3$ and

$$M^\alpha=\tfrac{1}{2}\sqrt{2}(M^1+iM^2)=M^+,\quad M^\beta=\tfrac{1}{2}\sqrt{2}(M^1-iM^2)=M^-$$

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[1] S. Fubini and G. Furlan, Physics **1**, 229 (1965); S. Fubini, G. Furlan, and C. Rossetti, International Center for Theoretical Physics (Trieste) Report No. IC/65/58 (unpublished) and CERN Report No. 65/998/5 Th. 578 (unpublished).
[2] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965).
[3] W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965).
[4] R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 142 (1965); *ibid.* **17**, 145 (1965).
[5] B. W. Lee, Phys. Rev. Letters **14**, 676 (1965).
[6] H. J. Schnitzer, CERN Report No. 65/1006/5—Th. 582 (unpublished).
[7] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

from Eq. (1), one has

$$[M_3{}^+,M_3{}^-]=\frac{1}{6}\int d^3r\, r^2J_0{}^3-\frac{1}{12}\int d^3r\, Q_{33}J_0{}^3, \quad (3)$$

where $Q_{ij}=3r_{ij}-r^2$.

Now, taking Eq. (3) between proton states, and inserting a complete set of intermediate states, we get the sum rule

$$\sum_{\mathfrak{N}}\langle p|M_3{}^+|\mathfrak{N}\rangle\langle\mathfrak{N}|M_3{}^-|p\rangle -\sum_{\mathfrak{N}'}\langle p|M_3{}^-|\mathfrak{N}'\rangle\langle\mathfrak{N}'|M_3{}^+|p\rangle=\tfrac{1}{12}\langle r\rangle_p{}^2,$$

where $\langle r\rangle_p$ is the charge radius of the proton. Keeping only the contribution due to the N and N^* particles, one obtains

$$(25/18)(\mu_p/2m)^2-C_{VV}(\tfrac{3}{2})=\tfrac{1}{12}\langle r\rangle_p{}^2, \quad (4)$$

where $C_{VV}(\frac{3}{2})=\langle p|M_3{}^3|N_+{}^*\rangle\langle N_+{}^*|M_3{}^3|p\rangle$, m is the nucleon mass, μ_p is the total magnetic moment of the proton, and we have used the $SU(6)$ result $\mu_n=-\frac{2}{3}\mu_p$.

Taking for $C_{VV}(\frac{3}{2})$ the $SU(6)$ predictions,[8] we get[9]

$$(\mu_p/2m)^2=\tfrac{1}{6}\langle r\rangle_p{}^2. \quad (5)$$

Equation (5) is rather well satisfied by experiment. Other results that can be derived from Eq. (3) are $Q_{33}=0$, and the radius of the neutron $\langle r\rangle_n=0$. The interest of those results has been particularly stressed by Dashen and Gell-Mann.[4]

The last term on the right-hand side of Eq. (1) gives additional sum rules. For example, taking

$$M_i{}^\alpha=\tfrac{1}{2}\sqrt{2}(M_1{}^3+iM_2{}^3) \quad\text{and}\quad M_j{}^\beta=\tfrac{1}{2}\sqrt{2}(M_1{}^8-iM_2{}^8),$$

we obtain

$$[M_+{}^3,M_-{}^8]=\tfrac{1}{12}\sqrt{3}\int d^3r\, r_3(\mathbf{r}\cdot\mathbf{A}^3). \quad (6)$$

Putting

$$\mathbf{A}=iF_A(k^2)\bar{\psi}\boldsymbol{\gamma}\gamma_5\psi-F_p(k^2)\bar{\psi}\mathbf{k}\gamma_5\psi,$$

where k_μ is the four-momentum transfer, taking Eq. (6) again between proton states, and noticing that the N^* intermediate state does not appear in this case, we get

$$(5/18)\sqrt{3}(\mu_p/2m)^2=-\tfrac{1}{12}\sqrt{3}[\tfrac{1}{6}\langle r\rangle_A{}^2F_A(0)+B_A(0)], \quad (7)$$

where

$$B_A(0)=(1/m)[(1/4m)F_A(0)+F_p(0)],$$

and

$$\tfrac{1}{6}\langle r\rangle_A{}^2=-(1/F_A(0))(dF_A(k^2)/dk^2)_{k=0}.$$

To obtain this formula, one uses

$$\langle\mathfrak{N}|M|\mathfrak{N}'\rangle=u_{\mathfrak{N}}[i\boldsymbol{\nabla}_k\times\mathbf{j}(k)]_{k=0}u_{\mathfrak{N}'}, \quad (8)$$

where $u_{\mathfrak{N}}$ and $u_{\mathfrak{N}'}$ are two-component Pauli spinors, and $\mathbf{j}(k)$ is the Fourier transform of the current in Eq. (2).

If we now take $F_A(0)=-1.2$, assume that $\langle r\rangle_A$ is equal to[10] $\langle r\rangle_p$, and use the Goldberger-Treiman relation[11] for F_p, we have a difference of a factor 20 between the right-hand side and the left-hand side of Eq. (7); more precisely, we obtain

$$(\mu_p/2m)^2=0.85/m_\pi{}^2. \quad (9)$$

We see immediately that if we could take the pion mass of the order of magnitude of the ρ-meson mass, that is, if the $SU(6)$ symmetry were not broken, Eq. (7) would reduce essentially to Eq. (5). In fact, however, the symmetry is broken, and we think this has the consequence that we can no longer confine the intermediate states to those contained in the **56** representation of $SU(6)$. In the sum rule Eq. (7), we must now include "leakage" into other isospin-$\frac{1}{2}$ intermediate states.

Because of the smallness of the mass of the pion, the most natural state to consider is a pion-nucleon continuum state of isotopic spin $\frac{1}{2}$, whose contribution to Eq. (7) we call $C_{VS}(\frac{1}{2})$.

Because of the fact that we have to deal with a virtual photon (time-like with $\mathbf{k}=0$) in the amplitudes Eq. (8), we will use the pion electroproduction amplitudes, extrapolated to the region of negative photon four-momentum squared $k^2=-\lambda^2$. For an order-of-magnitude estimate of this contribution, we use the static-limit Born amplitudes given by Fubini, Nambu, and Wataghin.[12] Then

$$\langle\mathfrak{N}|M^3|\mathfrak{N}'\rangle=u_{\mathfrak{N}}\tfrac{1}{3}\sqrt{3}[\boldsymbol{\nabla}_k\times(\mathbf{H}^++2\mathbf{H}^-)]_{k=0}u_{\mathfrak{N}'},$$
$$\langle\mathfrak{N}|M^8|\mathfrak{N}'\rangle=u_{\mathfrak{N}}3[\boldsymbol{\nabla}_k\times\mathbf{H}^0]_{k=0}u_{\mathfrak{N}'},$$

where $\mathbf{H}^+$, $\mathbf{H}^-$, $\mathbf{H}^0$ are the usual isovector and isoscalar amplitudes, defined in Ref. 8.[13] We assume that the electromagnetic form factor of the pion, as well as the isovector electromagnetic form factor of the nucleon are dominated by the ρ-meson pole, and the isoscalar electromagnetic form factor of the nucleon is dominated by the ω-meson and ϕ-meson poles.[14] We take

$$F_\pi(-\lambda^2)\simeq F_N{}^V(-\lambda^2)\simeq m_\rho{}^2/(m_\rho{}^2-\lambda^2-i\Gamma_\rho m_\rho),$$
$$F_N{}^S(-\lambda^2)\simeq m_\omega{}^2m_\phi{}^2/(m_\phi{}^2-\lambda^2-i\Gamma_\phi m_\phi)(m_\omega{}^2-\lambda^2-i\Gamma_\omega m_\omega),$$

where Γ_ω, Γ_ρ, Γ_ϕ are the full widths of the ω, ρ, and ϕ mesons. Our estimate for $C_{VS}(\frac{1}{2})$ obtained in this way

[8] M. A. B. Bég, B. W. Lee, and A. Pais, Phys. Rev. Letters **13**, 514 (1964).

[9] This result has also been obtained, in the framework of the $M(12)$ symmetry, by P. G. O. Freund and R. Oehme, Phys. Rev. Letters **14**, 1085 (1965).

[10] There is some experimental evidence for this, from the CERN neutrino experiments. See M. M. Block *et al.*, Phys. Letters **12**, 281 (1964), and J. K. Bienlein *et al.*, Phys. Letters **13**, 80 (1964).

[11] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958). The value given for $F_p(k^2)$ by these authors is essentially confirmed for $k^2=m_\mu{}^2$ by the experiments on muon capture.

[12] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1958).

[13] More precisely, Eqs. (14)–(14)′ of Ref. 8.

[14] The ϕ-meson pole contribution might account for the "soft core" term in the isoscalar electromagnetic form factor given by L. N. Hand, D. G. Miller, and R. Wilson, Rev. Mod. Phys. **35**, 353 (1963).

is approximately

$$C_{VS}(\tfrac{1}{2})\simeq\frac{\mu_s}{\pi m_\pi^2}\left(\frac{f^2}{4\pi}\right)\int\frac{d\lambda^2}{\lambda(m+\lambda)}F_\pi^*(-\lambda^2)F_N{}^S(-\lambda^2)\,,$$

with

$$\mu_s=\tfrac{1}{2}(\mu_p+\mu_n)\,.$$

Assuming now that the main contribution to the integral comes from the peaks in the form factors at $\lambda=m_\rho$, m_ω, m_ϕ, we have

$$C_{VS}(\tfrac{1}{2})\simeq0.38/m_\pi^2$$

which, when combined with the nucleon intermediate state, gives

$$(\mu_p/2m)^2\simeq0.05/m_\pi^2\,.$$

Of course, this surprisingly good result should not be taken too seriously (because of the qualitative nature of the calculation) but the order of magnitude of $C_{VS}(\frac{1}{2})$ is most encouraging.

We can immediately understand the relation of the result to the hypothetical limit of exact symmetry: If the pion had the mass of the ρ meson, the form-factor poles would lie at threshold in the πN channel, and since the π is produced in a p wave, $C_{VS}(\frac{1}{2})$ would vanish. The same argument holds for many higher intermediate states, like nucleon plus ρ meson.

We can go back now to the "good" sum rule Eq. (4), and see if an isospin-$\frac{1}{2}$ pion-nucleon intermediate state gives a big contribution. With the same type of calculation we did for $C_{VS}(\frac{1}{2})$, we obtain

$$C_{VV}(\tfrac{1}{2})\simeq0.1/m_\pi^2\,.$$

The fact that $C_{VV}(\frac{1}{2})$ is smaller than $C_{VS}(\frac{1}{2})$ has a simple explanation: the small width of the ω makes the form-factor peak in the integral for $C_{VS}(\frac{1}{2})$ larger than that in $C_{VV}(\frac{1}{2})$. Furthermore, the presence of the ϕ-meson pole enhances $C_{VS}(\frac{1}{2})$ by a factor $m_\phi^2/(m_\phi^2-m_\rho^2)$ compared with $C_{VV}(\frac{1}{2})$. We can go further now, and try to evaluate also the $C_{VV}(\frac{3}{2})$ contribution to the sum rule Eq. (4), always using the Fubini-Nambu-Wataghin[12] amplitudes. Since the dominant contribution to $C_{VV}(\frac{1}{2})$ is proportional to[15]

$$\int\frac{d\lambda}{\lambda}\frac{m_\rho{}^4}{(m_\rho{}^2-\lambda^2)^2+m_\rho{}^2\Gamma_\rho{}^2}\sin^2\delta_{33}\,,$$

we can put

$$C_{VV}(\tfrac{3}{2})=C_{VV}(33)+C_{VV}(m\rho)\,,$$

where $C_{VV}(33)$ is the contribution of the **33** resonance, and $C_{VV}(m\rho)$ comes from the resonance in the form factor. In this way, we obtain

$$C_{VV}(33)\simeq0.02/m_\pi^2\,,$$

which is of the order of magnitude of the value of $C_{VV}(\frac{3}{2})$ given by the $SU(6)$ predictions, and

$$C_{VV}(m\rho)=0.11/m_\pi^2$$

which is just of the right size to cancel the $C_{VV}(\frac{1}{2})$ contribution.

Therefore, we have obtained again the result Eq. (5), and in some sense we have an explanation of the fact that it is a good approximation to take only N and N^* intermediate states in the commutation relation Eq. (3).

It can also be seen from these calculations that the relation Eq. (1) is probably right, in the sense that terms involving gradients of delta functions[16] do not contribute to the commutator. If one wants more quantitative results, one has to overcome the two main sources of uncertainty of this calculation. First of all, the static model is not a very good approximation when $\lambda\geq500$ MeV. This difficulty can in principle be overcome using recent calculations[17] or experimental data on pion electroproduction. The more serious difficulty comes from the fact that we are using electromagnetic form factors in the time-like region of the photon, about which nothing is known experimentally. Of course, the best source of information will be the electron-positron colliding-beam experiments that will, we hope, be available in the near future.

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[15] We take $\sin^2\delta_{33}=\frac{1}{4}\Gamma_{33}{}^2/[(\lambda-m_{33})^2+\frac{1}{4}\Gamma_{33}{}^2]$ at the **33** resonance and at $\lambda\simeq m_\rho$ we estimate $\sin^2\delta_{33}$ from the phase-shift analysis of pion-nucleon scattering.

[16] J. Schwinger, Phys. Rev. Letters **3**, 296 (1959). See also Ref. 4.

[17] See, for instance, I. M. Barbour, Nuovo Cimento **27**, 1382 (1963); P. Dennery, Phys. Rev. **124**, 2000 (1961). Unfortunately, these authors are mainly interested in the $J=\frac{3}{2}$ angular-momentum state of the pion-nucleon system.