Study of the Perturbation Series for the Ground State of a Many-Fermion System. III*

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We derive, through sixth order in the Fermi momentum, the many-fermion energy expansion. This expansion is compared with actual energies previously computed for the square-well potential. We introduce an approximation scheme, valid over a significant range of conditions, which includes, we think, infinite nuclear matter. We show that saturation is almost certainly a singular point of the many-fermion energy curve.

1. INTRODUCTION

IN this paper we attempt to synthesize, in relatively easily calculable form, the results of our two previous papers.[1,2] We found in those studies that the Brueckner approximation, even though it sums more terms than the ladder approximation, is not a more accurate approximation than the ladder approximation. In fact, in the low-to-moderate density range, the sum of the complete potential perturbation theory may even lie on the opposite side of the ladder approximation from the Brueckner approximation. The reason lies primarily in the neglect of the ring diagrams.

Since nuclear density is only a small fraction of closest packing density for the currently accepted diameter of the nuclear hard core, we have extended the power series in the Fermi momentum found by Huang and Yang[3] to include the sixth order in k_F. A discussion of this expansion is given in the second section. We find that in addition to ordinary powers of k_F, terms in $k_F \ln k_F$ also appear. In contrast to the terms found by Huang and Yang, the additional terms are not completely determined by the two-body scattering parameters but also depend on many-body aspects (off-diagonal terms) of the scattering matrix.

In the third section we evaluate the additional terms for the soft, repulsive, square-well potential and compare the results obtained with those obtained by other methods previously. Reasonable accuracy is obtained for the potentials investigated to a density of nearly $k_F = 1.0$ for strong interactions and more for weak ones.

In the final section we discuss the calculation of the energy in the infinite-nuclear-matter problem and outline a calculational method, which combines aspects of the Brueckner K-matrix approach and the Monte Carlo procedures used in I and II, that should accurately reproduce the many-fermion energy over a significant range of conditions. We discuss the analytic structure of the many-body energy and the inclusion of spin-dependent forces and isotopic spin.

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[1] G. A. Baker, Jr., J. L. Gammel, and B. J. Hill, Phys. Rev. 132, 1373 (1963); hereinafter referred to as I.

[2] G. A. Baker, Jr., B. J. Hill, and R. J. McKee, Jr., Phys. Rev. 135, A922 (1964); hereinafter referred to as II.

[3] K. Huang and C. N. Yang, Phys. Rev. 105, 767 (1957).

2. THE FERMI-MOMENTUM EXPANSION

Our procedure for obtaining the expansion of the ground-state energy of a many-fermion system in terms of the Fermi momentum is to select from the perturbation expansion in the potential all those terms which can contribute through the desired order in k_F and sum them to all orders in the potential strength. The basis of our selection procedure is the observation that the creation of an independent hole momentum at a potential vertex implies that the contribution of the diagram must vanish at least as one higher power of k_F in the limit as k_F goes to zero. The reason for this result is that the volume in momentum space is proportional to k_F^3; however, there is one more denominator which may reduce the volume factor by k_F^2 leaving a net result of k_F^1. The only vertex which neither creates nor destroys a hole is the particle-particle scattering vertex (vertex A in the notation of I and II) illustrated in Fig. 1.

Since the leading terms in the energy are of the order k_F^3, it follows that if we agree to iterate[4] every relevant vertex by applying an indefinite number of vertex A's, we may obtain $(2+n)$th order in k_F by considering the nth in potential strength to find the basic graphs. It will be convenient, following Brueckner, to denote the iterated vertexes with a K and the uniterated ones with a V.

By examination of the catalog of graphs given in I we may classify the basic graphs for the first six orders in k_F as follows: Those proportional to k_F^3 are elaborations of B1. Those proportional to k_F^5 are elaborations of R3 and H3. Those proportional to k_F^6 are elaborations of F3, I.6, IA.1, IA.2, IA.3, II.3=II.4, II.5, II.7 =II.12, II.8=II.11, II.9, II.10, IIA.1, IIA.2=IIA.4, IIA.3, IIA.5, IIA.6, III.2, III.9+10. The other dia-

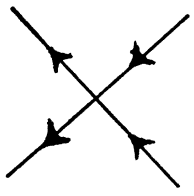

FIG. 1. Vertex which does not lead to an increase in the power of the Fermi momentum multiplying a diagram.

[4] K. A. Brueckner, *The Many-Body Problem*, edited by C. de Witt (John Wiley & Sons, New York, 1959), pp. 65 *et seq.*

grams in that catalog are either included as part of the elaboration of the above, or (IV.4–7) are of higher order in k_F. These sequences must be expanded in k_F to give the Fermi-momentum expansion. For instance, the ladder diagrams, B1, B2, B3, I.1, $\cdots$ all vanish as k_F^3 goes to zero; however, they also contribute terms like k_F^4, k_F^5, k_F^6, etc. Starting with the elaborations of F3 terms proportional to $k_F^6 \ln k_F$ are obtained.

Our procedure for obtaining the higher order contributions from the ladder series is to consider a typical term, to rewrite it as a sum of terms whose behavior as a function of k_F is simple, and then to identify and sum all the infinite sequences of them which occur. We will consider for simplicity an ordinary, central force, although, as we will point out later these results may be easily generalized. Let us now consider, for example, the third-order (in the potential) ladder term, B3. Its contribution to the energy is

$$\mathrm{B3}=3M^3/[2^{11}\pi^{10}k_F^3\hbar^6]\int_{\substack{|\frac{1}{2}\mathbf{p}+\mathbf{k}|\le k_F,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_0|>k_F,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_1|>k_F,\\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|\le k_F,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_0|>k_F,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_1|>k_F}} d\mathbf{p}d\mathbf{k}d\mathbf{k}_0d\mathbf{k}_1 \times\frac{v(\mathbf{k}_0-\mathbf{k})v(\mathbf{k}_0-\mathbf{k}_1)[v(\mathbf{k}_1-\mathbf{k})-\frac{1}{2}v(\mathbf{k}_1+\mathbf{k})]}{(k_0^2-k^2)(k_1^2-k^2)}; \quad (2.1)$$

$$\mathrm{B3}=3M^3/[2^{11}\pi^{10}\hbar^6]\Big\{k_F^3(\tfrac{4}{3}\pi)^2\int d\mathbf{k}_0d\mathbf{k}_1\frac{v(\mathbf{k}_0)v(\mathbf{k}_0-\mathbf{k}_1)\frac{1}{2}v(k_1)}{k_0^2k_1^2}$$

$$+k_F^4\int d\mathbf{k}_0\frac{v(k_0)\frac{1}{2}v(k_0)}{k_0^2}\Big[\int_{\substack{|\frac{1}{2}\mathbf{p}+\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_1|>1,\\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_1|>1}} d\mathbf{p}d\mathbf{k}d\mathbf{k}_1v(0)[(k_1^2-k^2)^{-1}-(k_1^{-2})]$$

$$-\int d\mathbf{p}d\mathbf{k}d\mathbf{k}_1[1-F(k_F\mathbf{p},k_F\mathbf{k}_1)]v(0)/k_1^2\Big]$$

$$+k_F^5\int_{\substack{|\frac{1}{2}\mathbf{p}+\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_0|>1,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_1|>1,\\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_0|>1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_1|>1}} d\mathbf{p}d\mathbf{k}d\mathbf{k}_0d\mathbf{k}_1\tfrac{1}{2}v(0)^3[(k_0^2-k^2)^{-1}-(k_0^{-2})][(k_1^2-k^2)^{-1}-(k_1^{-2})]$$

$$+k_F^5\int d\mathbf{p}d\mathbf{k}d\mathbf{k}_0d\mathbf{k}_1[1-F(k_F\mathbf{p},k_F\mathbf{k}_0)][1-F(k_F\mathbf{p},k_F\mathbf{k}_1)]\frac{\frac{1}{2}v(0)^3}{k_0^2k_1^2}$$

$$+k_F^5\int_{|\frac{1}{2}\mathbf{p}+\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|\le1} d\mathbf{p}d\mathbf{k}d\mathbf{k}_0d\mathbf{k}_1\Big\{\frac{\frac{1}{2}\bar{T}''(k=0)k^2}{k_0^2k_1^2}-\frac{\frac{1}{2}k^2[v(0)v(k_1)v(k_1)-v(k_0)v(\mathbf{k}_0-\mathbf{k}_1)v(k_1)]}{k_0^4k_1^2}$$

$$-\frac{\frac{1}{2}k^2[v(k_0)v(k_0)v(0)-v(k_0)v(\mathbf{k}_0-\mathbf{k}_1)v(k_1)]}{k_0^2k_1^4}\Big\}-\frac{1}{k_F^3}\int d\mathbf{p}d\mathbf{k}d\mathbf{k}_0d\mathbf{k}_1\Big\{[1-F(\mathbf{p},\mathbf{k}_0)F(\mathbf{p},\mathbf{k}_1)]$$

$$\times\Big[\frac{v(\mathbf{k}_0)v(\mathbf{k}_0-\mathbf{k}_1)\frac{1}{2}v(\mathbf{k}_1)-\frac{1}{2}v(0)^3}{k_0^2k_1^2}\Big]-[1-F(\mathbf{p},\mathbf{k}_0)]\frac{v(0)v(k_1)\frac{1}{2}v(k_1)}{k_0^2k_1^2}-[1-F(\mathbf{p},\mathbf{k}_1)]\frac{v(0)v(k_0)\frac{1}{2}v(k_0)}{k_0^2k_1^2}$$

$$+[1-F(\mathbf{p},\mathbf{k}_0)][1-F(\mathbf{p},\mathbf{k}_1)]\frac{\frac{1}{2}v(0)^3}{k_0^2k_1^2}\Big\}+\frac{1}{k_F^3}\int_{\substack{|\frac{1}{2}\mathbf{p}+\mathbf{k}|\le k_F,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_0|>k_F,\\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|\le k_F,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_0|>k_F,\\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_1|>k_F,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_1|>k_F}} d\mathbf{p}d\mathbf{k}d\mathbf{k}_0d\mathbf{k}_1$$

$$\times\Big\{\frac{v(\mathbf{k}_0-\mathbf{k})v(\mathbf{k}_1-\mathbf{k}_0)[v(\mathbf{k}_1-\mathbf{k})-\frac{1}{2}v(\mathbf{k}_1+\mathbf{k})]-\frac{1}{2}v(0)^3}{(k_0^2-k^2)(k_1^2-k^2)}+[v(0)v(k_1)\tfrac{1}{2}v(k_1)+v(k_0)\tfrac{1}{2}v(k_0)v(0)$$

$$-v(k_0)v(\mathbf{k}_0-\mathbf{k}_1)\tfrac{1}{2}v(k_1)-\tfrac{1}{2}v(0)^3-\tfrac{1}{2}\bar{T}''(k=0)k^2]/(k_0^2k_1^2)+\frac{k^2}{2k_0^4k_1^2}[v(0)v(k_1)v(k_1)-v(k_0)v(\mathbf{k}_0-\mathbf{k}_1)v(k_1)]$$

$$+\frac{k^2}{2k_0^2k_1^4}[v(k_0)v(k_0)v(0)-v(k_0)v(\mathbf{k}_0-\mathbf{k}_1)v(k_1)]\Big\}, \quad (2.2)$$

where $F(\mathbf{p},\mathbf{k})$ is 1 if $|\frac{1}{2}\mathbf{p}+\mathbf{k}|>k_F$ and $|\frac{1}{2}\mathbf{p}-\mathbf{k}|>k_F$ and zero otherwise, and $\bar{T}''(k=0)$ is the second derivative with respect to the magnitude of $\mathbf{k}$, evaluated for $|\mathbf{k}|=0$, of $T=v(\mathbf{k}_0-\mathbf{k})v(\mathbf{k}_1-\mathbf{k}_0)[v(\mathbf{k}_1-\mathbf{k})-\frac{1}{2}v(\mathbf{k}_1+\mathbf{k})]$, averaged over the angles of $\mathbf{k}_0$ and $\mathbf{k}_1$. The terms corresponding to $\bar{T}'(k=0)$ vanish by symmetry.

The coefficients of k_F^3, k_F^4, and k_F^5 are potential-dependent constants. Following Yang and Huang[3] we will now observe that these constants are closely related to those which appear in the expansion of the zero-energy scattering length and several other potential-dependent parameters. For this purpose we note[5] the solution of the integral equation

$$\Psi_p(\mathbf{r})=e^{i\mathbf{p}\cdot\mathbf{r}}-\frac{\lambda}{4\pi}\int\frac{e^{-q|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}U(r')\Psi_p(\mathbf{r}')d\mathbf{r}' \tag{2.3}$$

can be expanded (in momentum representation) in powers of λ as

$$\int e^{-i\mathbf{p}_1\cdot\mathbf{r}}\Psi_p(\mathbf{r})d\mathbf{r}=(2\pi)^3\Big[\delta(\mathbf{p}_1-\mathbf{p})-\lambda(p_1^2+q^2)^{-1}u(\mathbf{p}-\mathbf{p}_1)+\lambda^2(p_1^2+q^2)^{-1}\int u(\mathbf{p}_1-\mathbf{p}_2)(p_2^2+q^2)^{-1}u(\mathbf{p}_2-\mathbf{p})d\mathbf{p}_2$$

$$-\lambda^3(p_1^2+q^2)^{-1}\int u(\mathbf{p}_1-\mathbf{p}_2)(p_2^2+q^2)^{-1}u(\mathbf{p}_2-\mathbf{p}_3)(p_3^2+q^2)^{-1}u(\mathbf{p}_3-\mathbf{p})d\mathbf{p}_2d\mathbf{p}_3+\cdots, \tag{2.4}$$

with the notation

$$u(\mathbf{q}_1-\mathbf{q}_2)=(2\pi)^{-3}\int e^{-i\mathbf{q}_1\cdot\mathbf{r}}U(r)e^{i\mathbf{q}_2\cdot\mathbf{r}}d\mathbf{r}). \tag{2.5}$$

From the well-known integral expression for the scattering length ($p=q=0$)

$$a=(1/4\pi)\int U(r)\Psi_0(\mathbf{r})d\mathbf{r} \tag{2.6}$$

and expansion (2.4), it follows, as Yang and Huang[3] have shown, that the coefficient of k_F^3 in (2.2) is the third-order term (in potential strength) in the expansion of a constant times the scattering length. The remainder of the scattering-length series similarly appears in higher order. The coefficients of k_F^4 contains the two terms which result from the expansion of a constant times the square of the scattering length, as found previously by Yang and Huang.[3] The rest of the terms in this expansion appear in the higher order terms. The first two integral coefficients of k_F^5 are a constant factor times the first (third-order) term in the expansion of a constant times the scattering length cubed. The last integral coefficient of k_F^5 contains two types of terms. In order to sum them to all orders let us consider the auxiliary quantity,

$$\Theta_\pm(k_F)=[M/\hbar^2][3^2/(2^5\pi^3k_F^6)]\int_{|\frac{1}{2}\mathbf{p}+\mathbf{k}|\le k_F,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|\le k_F}e^{\mp i\mathbf{k}\cdot\mathbf{r}}U(r)\Psi_k(\mathbf{r})d\mathbf{r}d\mathbf{k}d\mathbf{p}, \tag{2.7}$$

where q is taken as zero here. We note that

$$a=\tfrac{1}{2}\Theta_\pm(0)=\Theta_+(0)-\tfrac{1}{2}\Theta_-(0), \tag{2.8}$$

and define

$$b_1=\tfrac{1}{2}[\Theta_+''(0)-\tfrac{1}{2}\Theta_-''(0)], \tag{2.9}$$

where Θ_+ comes from the direct, and Θ_- from the exchange, contribution. The quantity b_1 is seen, by means of expansion (2.4) to be exactly a constant times the sum of the $\bar{T}''(k=0)$ terms which appear in every order of the perturbation expansion.

The last two terms are part of the expansion of

$$\int d\mathbf{r}d\mathbf{r}'d\mathbf{k}\Psi_0(r)U(r)[(1-e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')})/k^4]U(r')\Psi_0(\mathbf{r}'), \tag{2.10}$$

as can be shown by use of the expansion (2.4) and by considering the corresponding terms in higher order. The

[5] See, for example, T. Y. Wu and T. Ohmura, *Quantum Theory of Scattering* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey 1962), Sec. B.

integration over $\mathbf{k}$ can be done and yields $\pi^2|\mathbf{r}-\mathbf{r}'|$, and it is convenient to define the parameter

$$b_2=[M/\hbar^2]^2(3/320\pi^2)\int d\mathbf{r}d\mathbf{r}'\Psi_0(\mathbf{r})U(r)|\mathbf{r}-\mathbf{r}'|U(r')\Psi_0(\mathbf{r}'), \tag{2.11}$$

which differs by a constant factor from the sum to all orders of the relevant terms in the ladder diagrams.

By extending the procedure used to obtain the coefficients of k_F through fifth order, or by taking the limit of the last two integrals in (2.2) as k_F goes to zero, we determine that the coefficient of k_F^6 is comprised of a constant times the scattering length to the fourth power and another constant times $a(b_1-\frac{1}{2}b_2)$. Evaluating these constants we may expand in powers of k_F the energy contribution of ladder diagrams as

$$\Delta E_L/N=(\hbar^2/M)[(1/3\pi)k_F^3a-g_1k_F^4a^2+(1/3\pi)(b_1+b_2)k_F^5+g_2k_F^5a^3+k_F^6h_1a(b_1-\tfrac{1}{2}b_2)-g_3k_F^6a^4+\cdots, \tag{2.12}$$

where

$$g_1=\frac{3}{8\pi}\left\{\int_{\substack{|\frac{1}{2}\mathbf{p}+\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_1|>1,\\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_1|>1}} d\mathbf{p}d\mathbf{k}d\mathbf{k}_1[(k_1^2-k^2)^{-1}-k_1^{-2}]-\int d\mathbf{p}d\mathbf{k}d\mathbf{k}_1[1-F(k_F\mathbf{p},k_F\mathbf{k}_1)]\right\}(2\pi^2)^{-2}, \tag{2.13}$$

$$g_2=\frac{3}{8\pi}\left\{\int_{\substack{|\frac{1}{2}\mathbf{p}+\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_1|>1,\\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_1|>1,\\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_2|>1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_2|>1}} d\mathbf{p}d\mathbf{k}d\mathbf{k}_1d\mathbf{k}_2[(k_1^2-k^2)^{-1}-k_1^{-2}][(k_2^2-k^2)^{-1}-k_2^{-2}]\right.$$
$$\left.+\int d\mathbf{p}d\mathbf{k}d\mathbf{k}_1d\mathbf{k}_2[1-F(k_F\mathbf{p},k_F\mathbf{k}_1)][1-F(k_F\mathbf{p},k_F\mathbf{k}_2)]/(k_1^2k_2^2)\right\}(2\pi^2)^{-3}, \tag{2.14}$$

$$g_3=\frac{3}{8\pi}\left\{\int_{\substack{|\frac{1}{2}\mathbf{p}+\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_1|>1,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_3|>1,\\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|\le1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_1|>1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_3|>1,\\ |\frac{1}{2}\mathbf{p}+\mathbf{k}_2|>1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}_2|>1}} d\mathbf{p}d\mathbf{k}d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3[(k_1^2-k^2)^{-1}-k_1^{-2}][(k_2^2-k^2)^{-1}-k_2^{-2}][(k_3^2-k^2)^{-1}-k_3^{-2}]\right.$$
$$\left.-\int d\mathbf{p}d\mathbf{k}d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3[1-F(k_F\mathbf{p},k_F\mathbf{k}_1)][1-F(k_F\mathbf{p},k_F\mathbf{k}_2)][1-F(k_F\mathbf{p},k_F\mathbf{k}_3)]/(k_1^2k_2^2k_3^2)\right\}(2\pi^2)^{-4}, \tag{2.15}$$

$$h_1=\frac{5}{8\pi}\left\{\int_{\substack{|\frac{1}{2}\mathbf{p}+\mathbf{k}|\le1,\ |\frac{1}{2}p+k_1|>1,\\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|\le1,\ |\frac{1}{2}p-k_1|>1}} d\mathbf{p}d\mathbf{k}d\mathbf{k}_1 2k^2((k_1^2-k^2)^{-1}-k_1^{-2})\right.$$
$$\left.-\int d\mathbf{p}d\mathbf{k}d\mathbf{k}_1[1-F(k_F\mathbf{p},k_F\mathbf{k}_1)](1+k^2/k_1^2)\right\}(2\pi^2)^{-2}. \tag{2.16}$$

We may easily reduce these constants to double integrals. The result is

$$g_j=\frac{3}{\pi^{j+1}}\int_0^2 p\,dp\left[p\int_0^{1-\frac{1}{2}p}k^2dk(\tau_1^j+(-\tau_2)^j)+\int_{1-p/2}^{(1-\frac{1}{4}p^2)^{1/2}}k\,dk(1-\tfrac{1}{4}p^2-k^2)(\tau_1^j+(-\tau_2)^j)\right], \tag{2.17}$$

$$h_1=\frac{5}{2\pi^2}\int_0^2 p\,dp\left[p\int_0^{1-p/2}k^2dk(2k^2\tau_1-\tau_3-k^2\tau_2)+\int_{1-p/2}^{(1-\frac{1}{4}p^2)^{1/2}}k\,dk(1-\tfrac{1}{4}p^2-k^2)(2k^2\tau_1-\tau_3-k^2\tau_2)\right], \tag{2.18}$$

where

$$\tau_1=\frac{1}{p}\left[(k^2+\tfrac{1}{4}p^2-1)\ln\left(\frac{1+p+\frac{1}{4}p^2-k^2}{1-\frac{1}{4}p^2-k^2}\right)+(1-\tfrac{1}{4}p^2)\ln\left(\frac{1+\frac{1}{2}p}{1-\frac{1}{2}p}\right)\right]+k\ln\left(\frac{1+\frac{1}{2}p+k}{1+\frac{1}{2}p-k}\right),$$

$$\tau_2=1+\tfrac{1}{2}p+\frac{(1-\frac{1}{4}p^2)}{p}\ln\left(\frac{1+\frac{1}{2}p}{1-\frac{1}{2}p}\right), \tag{2.19}$$

$$\tau_3=\tfrac{2}{3}+\tfrac{1}{2}p-p^2/24.$$

Although these could be carried further analytically we have preferred to evaluate (2.17) and (2.18) numerically. We obtain

$$g_1=-5.566\times10^{-2},\quad g_2=6.347\times10^{-2},$$
$$g_3=-3.824\times10^{-2},\quad h_1=-2.968\times10^{-2}. \tag{2.20}$$

We remark that all the terms $g_j\,k_F^{j+3}\,a^{j+1}$ occur in the ladder series. Because of the last ln term in τ_1 there appears in the integrand for g_j a term proportional to $(\ln p)^j$ (for p small). This term leads to a contribution of the order of $j!$, and hence there is at least a subsequence of terms in the k_F expansion which is a divergent series. Whether or not this divergence is canceled by other terms, we do not know, but we feel that it is not, except in very special cases.

It is both instructive and useful to reduce the formulas for b_1 and b_2 to simpler form. It turns out that b_1 can be directly related to the experimental two-body phase shifts, as of course can the scattering length a. First, we can rewrite (2.3) as a differential equation:

$$(-\nabla^2+q^2)\Psi_p(\mathbf{r})+U(r)\Psi_p(\mathbf{r})=(p^2+q^2)e^{i\mathbf{p}\cdot\mathbf{r}}. \tag{2.21}$$

For $q=0$, the boundary conditions are that $\Psi_p(\mathbf{r})$ be finite at $\mathbf{r}=0$ and that the expression $|\Psi_p(\mathbf{r})-e^{i\mathbf{p}\cdot\mathbf{r}}|$ tend to zero like r^{-1} as r tends to infinity.[5] To obtain b_1 it is convenient to make a partial-wave decomposition of $\Psi_p(\mathbf{r})$. Thus if

$$\Psi_p(\mathbf{r})=\sum_{l=0}^{\infty}(2l+1)i^l\psi_l(r)P_l(\cos\theta), \tag{2.22}$$

where P_l are the Legendre polynomials and θ the angle between $\mathbf{p}$ and $\mathbf{r}$; it follows that

$$\int d\mathbf{r}e^{\mp i\mathbf{k}\cdot\mathbf{r}}U(r)\Psi_k(\mathbf{r})$$
$$=4\pi\sum_{l=0}^{\infty}(2l+1)(\pm1)^l\int_0^{\infty}j_l(kr)U(r)\psi_l(r)r^2dr. \tag{2.23}$$

The terms proportional to k^2 now come from the $l=0$ and $l=1$ terms alone. If we use the asymptotic form

$$\psi_l(r)=j_l(kr)-A_l(k)k^l/r^{l+1} \tag{2.24}$$

and the Hermiticity of the $l=0$ part of the scattering matrix for the ladder series, we can easily show that

$$\tfrac{1}{2}\Theta_{\pm}''(0)=\tfrac{3}{5}[A_0''(0)\pm3A_1(0)], \tag{2.25}$$

where, if in two-body scattering for the $l=0$ and 1 phase shifts

$$k\cot\delta_0=-1/a+\tfrac{1}{2}r_0k^2-\cdots,$$
$$k^3\cot\delta_1=-1/A_1(0)+\cdots, \tag{2.26}$$

then

$$A_0''(0)=-\tfrac{1}{2}a^2r_0=-\frac{1}{3}\int_0^{\infty}r^4U(r)\psi_0(r)dr, \tag{2.27}$$

$$b_1=-(3/20)a^2r_0+(27/10)A_1(0). \tag{2.28}$$

We have not been able to express b_2 in terms of phase shifts alone, and believe that it is not so expressible, as we found in I and II that different potentials with identical two-body phase shifts yielded different many-body energies in ladder approximation. For spherically symmetrical potentials we can reduce b_2 to

$$b_2=\frac{1}{20}\int r^2dr(r')^2dr'\psi_0(r)U(r)(3r_>+r_<^2/r_>)$$
$$\times U(r')\psi_0(r'), \tag{2.29}$$

where $r_>$ and $r_<$ are the greater and lesser, respectively, of r and r'.

We wish to point out that all these results also hold for the velocity-dependent forces considered in II because the potential factors in the interior (i.e., not the first or last) of a ladder term do not depend on the hole-state momenta but only on $\mathbf{k}_i-\mathbf{k}_{i+1}$ and $\mathbf{k}_i+\mathbf{k}_{i+1}$ or $\mathbf{q}_i-\mathbf{q}_{i+1}$ and $\mathbf{m}-\mathbf{n}+\mathbf{q}_i+\mathbf{q}_{i+1}$ in the notation of II.

We now turn to those terms whose contributions start in fifth order in k_F. The sum of those diagrams which are elaborations of H3 contributes a term proportional to $k_F^5a^3\hbar^2/M$ in leading order; we may calculate the coefficient by evaluating Eq. (2.7) of I for the arguments of the potential set to zero. The coefficient (obtained by Monte Carlo integration) is $5.74\times10^{-3}\pm1.0\times10^{-5}$. By proceeding as we did in (2.2) we may show that there are two terms contributed by the H3 sequence in sixth order in k_F. One term comes from differentiating the contributions of the K matrices which now stand at each vertex of H3. This term contributes $3(5.74\times10^{-3})(-3\pi g_1)a^4k_F^6\hbar^2/M=9.03\times10^{-3}a^4\times k_F^6\hbar^2/M$. The other contribution is of the form ab_2. It is

$$-[20/(27\pi^2)]k_F^6ab_2\hbar^2/M. \tag{2.30}$$

The other terms which start in fifth order are elaborations of R3. We obtain, proceeding as above, contributions of

$$(-2.863\times10^{-2}\pm1.7\times10^{-5})k_F^5a^3\hbar^2/M$$
$$+3(-2.863\times10^{-2})(-3\pi g_1)k_F^6a^4\hbar^2/M$$
$$+k_F^6c_1\hbar^2/M. \tag{2.31}$$

The term c_1 arises in a similar manner to the ab_2 term in H3; however, in the calculation of c_1 we must consider the effects of a hole-filled-state interaction in the presence of an excited Fermi sea. In this situation, solutions of (2.3) or (2.21) occur with q not equal to zero. Let us define

$$K(k)=\int_0^{\infty}j_0(kr)U(r)\psi_0(r)r^2dr, \tag{2.32}$$

where $\psi_0(r)$ is that of (2.21)–(2.22) for $l=0$, $p=0$, $q=0$, and

$$\hat{K}_{\pm}(k)=\frac{1}{4\pi}\int e^{\mp\frac{1}{2}i\mathbf{k}\cdot\mathbf{r}}U(r)\hat{\psi}_{\frac{1}{2}\mathbf{k}}(\mathbf{r})d\mathbf{r}, \tag{2.33}$$

where $\hat{\psi}_{\frac{1}{2}\mathbf{k}}$ is the solution of (2.21) with $\mathbf{p}=\frac{1}{2}\mathbf{k}$, and $q^2=\frac{3}{4}k^2$. If we expand

$$\hat{\psi}_{\frac{1}{2}\mathbf{k}}(\mathbf{r})=\sum_{l=0}^{\infty}(2l+1)i^l\hat{\psi}_l(r)P_l(\cos\theta)\,, \tag{2.34}$$

then

$$\hat{K}_{\pm}(k)=\sum_{l=0}^{\infty}(2l+1)(\pm 1)^l\int_0^{\infty}j_l(\tfrac{1}{2}kr)U(r)\hat{\psi}_l(r)r^2dr\,. \tag{2.35}$$

We may now write down c_1. It is

$$c_1=-\frac{2}{9\pi^4}\int d\mathbf{k}\{K^2(k)[\hat{K}_-(k)-2\hat{K}_+(k)] +K^2(0)K(k)\}/k^4\,. \tag{2.36}$$

We shall now turn to those sequences of terms whose contributions begin in sixth order in k_F. The self-energy corrections which are elaborations of the final third-order graph F3 are of this type. If we write this contribution as $k_F{}^6c_2$, then by application of the techniques used above we can show that

$$c_2=\frac{2}{9\pi^4}\int d\mathbf{k}K^2(k)[2\hat{K}_+(k)-\hat{K}_-(k)-K(0)]/k^4\,. \tag{2.37}$$

As $\hat{K}_+'(0)=\hat{K}_-'(0)\neq 0$, the integrals (2.36)–(2.37) for c_1 and c_2 diverge logarithmically as k goes to zero. This behavior is symptomatic of the appearance of a $k_F{}^6\ln k_F$ term in the expansion. We can isolate this term by replacing the k_F dependence in the lower limit of the integral and calculating the various contributions which arise in c_1+c_2. They are

$$c_a=\frac{4}{\pi^4k_F{}^6}\int_{\substack{|\frac{1}{2}\mathbf{p}+\mathbf{k}'|\leq k_F,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}|>k_F,\\ |\frac{1}{2}\mathbf{p}-\mathbf{k}'|\leq k_F,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|>k_F}} d\mathbf{p}d\mathbf{k}d\mathbf{k}'\frac{[K(0)]^2\hat{K}_+'(0)}{k^3(1+\beta k)}\,, \tag{2.38}$$

$$b_3=\frac{2}{9\pi^4}\int d\mathbf{k}\left\{K^2(k)[4\hat{K}_+(k)-2\hat{K}_-(k)-K(0)]-K^2(0)K(k)-\frac{2K^2(0)\hat{K}_+'(0)k}{1+\beta k}\right\}k^{-4}\,, \tag{2.39}$$

$$c_b=\frac{4}{\pi^4}\int_{\substack{|\frac{1}{2}\mathbf{p}+\mathbf{k}'|\leq 1,\ |\frac{1}{2}\mathbf{p}+\mathbf{k}|>1,\\ |\frac{1}{2}\mathbf{p}-\mathbf{k}'|\leq 1,\ |\frac{1}{2}\mathbf{p}-\mathbf{k}|>1}} d\mathbf{p}d\mathbf{k}d\mathbf{k}'K^2(0)\hat{K}_+'(0)k[(k^2-k'^2)^{-2}-k^{-4}]\,, \tag{2.40}$$

$$=(2.00\times 10^{-2}\pm 1.0\times 10^{-4})K^2(0)\hat{K}_+'(0)\,.$$

The quantities b_3 and c_b are now convergent and the $\ln k_F$ part comes from c_a alone. Evaluating c_a analytically we get

$$c_a=\frac{16}{9\pi^3}K^2(0)\hat{K}_+'(0)\left\{\ln(k_F\beta) +3\int_0^2 p(1-\tfrac{1}{2}p)^2(1+\tfrac{1}{4}p)[1+\tfrac{1}{2}p-(1-\tfrac{1}{4}p^2)^{1/2}] \times\left[1-\left(\frac{1-\frac{1}{2}p}{1+\frac{1}{2}p}\right)^{1/2}\right]dp\right\} \tag{2.41}$$

plus terms which vanish as k_F goes to (2.41) zero. We have evaluated the integrals in (2.41) by Simpson's rule. It should be pointed out that the division between the $k_F{}^6\ln k_F\beta$ term and the $k_F{}^6$ term (c_a+b_3) is somewhat arbitrary because of the scale factor β.

One may easily show (since $\hat{K}_+'(0)$ comes from the $l=0$ terms alone) by considering the wave function in the region beyond the range of the potential as a linear combination of a solution (even in k) of Eq. (2.21) and a solution of the homogeneous part of Eq. (2.21) (even in k), that

$$\hat{K}_+'(0)=\tfrac{1}{2}\sqrt{3}a^2\,. \tag{2.42}$$

Thus we get

$$c_a=[8\sqrt{3}/(9\pi^3)]\ln(k_F\beta)+1.443\times 10^{-2}\,. \tag{2.43}$$

The contribution of the remainder of the sequences is of the form $k_F{}^6a^4$. We have calculated by Monte Carlo the values of the coefficients from the formulas of I and listed them in Table I. We remark that IIA.1 and II.5 separately are not of this form, but their sum is.

TABLE I. Values of the coefficients.

Diagram	Coefficient	Standard deviation
I.6	-3.39×10^{-4}	4.2×10^{-6}
IA.1	-1.81×10^{-3}	1.4×10^{-5}
IA.2	-8.41×10^{-4}	3.6×10^{-6}
IA.3	-1.67×10^{-3}	9.0×10^{-6}
II.3	8.38×10^{-4}	6.4×10^{-6}
II.4[a]	8.38×10^{-4}	6.4×10^{-6}
II.5+IIA.1	2.9×10^{-4}	2.0×10^{-5}
II.7	-9.73×10^{-4}	4.6×10^{-6}
II.8	-1.900×10^{-3}	9.0×10^{-6}
II.9	4.96×10^{-4}	4.8×10^{-6}
II.10	1.74×10^{-4}	2.8×10^{-6}
II.11[a]	-1.900×10^{-3}	9.0×10^{-6}
II.12[a]	-9.73×10^{-4}	4.6×10^{-6}
IIA.2	1.68×10^{-3}	4.7×10^{-5}
IIA.3	-2.29×10^{-4}	2.3×10^{-6}
IIA.4[a]	1.68×10^{-3}	4.7×10^{-5}
IIA.5	1.756×10^{-3}	6.2×10^{-6}
IIA.6	1.674×10^{-3}	8.7×10^{-6}
III.2	-1.589×10^{-3}	1.1×10^{-5}
III.9+10	2.404×10^{-3}	1.3×10^{-5}
Total	-3.94×10^{-4}	7.0×10^{-5}

[a] Identical with a previous diagram but must be included in the total.

We point out that the insertion of all A-type vertices in IIA.1 and II.5 leads to full three-body scattering. However, as IIA.1 and II.5, as well as the A-type vertex insertions in them, differ only in the order in which holes are filled (which permutation changes the overall sign of the contribution found by setting the hole momenta equal to zero), this three-body effect cancels exactly to leading order (k_F^6). There remains, however, a residual $k_F^6 a^4$ contribution. It might be thought that one should also permit the $\mathbf{n}-\mathbf{q}$ line in II.6 or III.1 to interact via an A vertex with the pair of filled state lines connecting the second and third vertices, and hence include full three-body terms in the elaboration of R3 or F3; however, these terms in fact are the same terms that we are counting as elaborations of IIA.1 and II.5, and hence we would be including them an extra time.

Combining our previous results we obtain for the Fermi-momentum expansion

$$\begin{aligned}\Delta E/N = (\hbar^2/M)[&(3\pi)^{-1}k_F^3 a + 5.566\times10^{-2}k_F^4 a^2 \\ &+4.058\times10^{-2}k_F^5 a^3 + (3\pi)^{-1}(b_1+b_2)k_F^5 \\ &+3.624\times10^{-2}k_F^6 a^4 - 2.968\times10^{-2}k_F^6 a b_1 \\ &+6.021\times10^{-2}k_F^6 a b_2 + k_F^6 b_3 \\ &+[8\sqrt{3}/(9\pi^3)]a^4 k_F^6 \ln(k_F\beta)+\cdots. \end{aligned} \quad (2.44)$$

Through sixth order in the Fermi momentum, the many-body energy series depends on four potential-dependent parameters, a, b_1, b_2, and b_3. Of these, a and b_1 are directly measurable from two-body scattering and b_2 and b_3 relate to other aspects which do not follow directly from the scattering phase shifts.

3. THE SOFT, REPULSIVE SQUARE-WELL POTENTIAL

As an illustration of the calculation of the Fermi-momentum expansion, we choose a soft, repulsive square-well potential because most of the calculation can be done analytically, and because there are data available (paper I) with which we can compare our present results.

The first step is to solve for the $l=0$ wave function from (2.21), $p=q=0$. It is, for this case,

$$\begin{aligned}\psi_0(r) &= \frac{\sinh[(MVr^2/\hbar^2)^{1/2}]}{(MVr^2/\hbar^2)^{1/2}\cosh[(MVc^2/\hbar^2)^{1/2}]}, && r\le c, \\ &= (1-a/r), && r>c, \end{aligned} \quad (3.1)$$

where M is the mass, V the potential strength, c the width of the potential, and $\hbar$ Planck's constant. The scattering length is, by (2.6), [or (3.1)],

$$a = c(1-\tanh\rho/\rho), \quad (3.2)$$

where

$$\rho = (MVc^2/\hbar^2)^{1/2}. \quad (3.3)$$

Next, using asymptotic form (2.24) to compute $A_1(0)$ and (2.27) to compute $A_0''(0)$ we compute from (2.28) for b_1,

$$\begin{aligned} b_1 = (9/10)c^3[&1+3(1-\rho\coth\rho)/\rho^2] \\ &-1/10c^3\{1+(6/\rho^2)-[(3/\rho)+(6/\rho^3)]\tanh\rho\}. \end{aligned} \quad (3.4)$$

The computation of $A_1(0)$ requires the solution of the $l=1$ Eq. (2.21) $p=q=0$. We may compute b_2 by substitution of (3.1) into (2.29). The result is

$$\begin{aligned} b_2 = (1/20)c^3[&1-(\tanh\rho)/\rho](3+6\rho^{-2}-6\tanh\rho/\rho) \\ &+[1+6\rho^{-2}-(3+6\rho^{-2})(\tanh\rho)/\rho] \\ &+3\{[1-(\tanh\rho)/\rho]/\rho^2+\tanh^2\rho/\rho^2\}. \end{aligned} \quad (3.5)$$

The calculation of b_3 is more difficult and we have not completed it analytically, but we have reduced it to a numerical integration. To obtain b_3 we need several subsidiary quantities. We need $K(k)$ from (2.32) and (3.1), it is

$$K(k) = c\left[\frac{[\rho^2\sin(kc)](kc)-\cos(kc)\rho\tanh\rho}{\rho^2+k^2c^2}\right]. \quad (3.6)$$

We need, further, the solutions, for all l, of (2.21) with $p=\frac{1}{2}k$ and $q^2=\frac{3}{4}k^2$. They are

$$\hat{\psi}_l(r) = \frac{k^2c^2}{\rho^2+k^2c^2}j_l(\tfrac{1}{2}kr)+\frac{\rho^2 R_l}{\rho^2+k^2c^2}j_l(i(\rho^2+\tfrac{3}{4}k^2c^2)^{1/2}r/c), \quad (3.7)$$

where

$$R_l = \frac{\tfrac{1}{2}i\sqrt{3}kch_l^{(1)\prime}(\tfrac{1}{2}i\sqrt{3}kc)j_l(\tfrac{1}{2}kc)-\tfrac{1}{2}kch_l^{(1)}(\tfrac{1}{2}i\sqrt{3}kc)j_l'(\tfrac{1}{2}kc)}{\tfrac{1}{2}i\sqrt{3}kch_l^{(1)\prime}(\tfrac{1}{2}i\sqrt{3}kc)j_l(i(\rho^2+\tfrac{3}{4}k^2c^2)^{1/2}r/c)-i(\rho^2+\tfrac{3}{4}k^2c^2)^{1/2}h_l^{(1)}(\tfrac{1}{2}i\sqrt{3}kc)j_l'(i(\rho^2+\tfrac{3}{4}k^2c^2)^{1/2}r/c)}.$$

From the $\hat{\psi}_l(r)$ we may compute the quantities $\hat{K}_l(k)$. They are, by (2.35)

$$\hat{K}_l(k) = \int_0^\infty j_l(\tfrac{1}{2}kr)U(r)\hat{\psi}_l(r)r^2dr. \quad (3.8)$$

They may be used to compute, via

$$\hat{K}_\pm(k) = \sum_{l=0}^{\infty}(2l+1)(\pm1)^l\hat{K}_l(k), \quad (3.9)$$

the $\hat{K}_{\pm}(k)$ as

$$\hat{K}_{\pm}(k)=c\frac{(k^2c^2)\rho^2}{\rho^2+k^2c^2}\left[\tfrac{1}{3}\text{ for }+\text{ and }\frac{j_1(kc)}{kc}\text{ for }-\right]+c\left(\frac{\rho^2}{k^2c^2+\rho^2}\right)^2\sum_{l=0}^{\infty}(2l+1)(\pm 1)^l\left[\frac{H_l j_l(\tfrac{1}{2}kc)-\tfrac{1}{2}kcj_{l-1}(\tfrac{1}{2}kc)}{H_l-J_l}\right]$$
$$\times[-\tfrac{1}{2}kcj_{l-1}(\tfrac{1}{2}kc)+J_l j_l(\tfrac{1}{2}kc)], \quad (3.10)$$

where we define

$$H_l=\tfrac{1}{2}i\sqrt{3}kch_{l-1}^{(1)}(\tfrac{1}{2}i\sqrt{3}kc)/h_l^{(1)}(\tfrac{1}{2}i\sqrt{3}kc),$$
$$J_l=i(\rho^2+\tfrac{3}{4}k^2c^2)^{1/2}j_{l-1}(i(\rho^2+\tfrac{3}{4}k^2c^2)^{1/2})/j_l(i(\rho^2+\tfrac{3}{4}k^2c^2)^{1/2}), \quad (3.11)$$

and j_l and $h_l^{(1)}$ have their standard meaning as spherical Bessel functions. On substituting (3.6) and (3.11) into (2.36) and (2.39) one may obtain b_3 numerically. We have used Simpson's-rule integration to calculate b_3.

In Table II we give a short list of the numerical results for the various parameters.

We can compute the ladder-approximation energy-series coefficients from (2.12), (2.20), and Table II. We give a short table of them in Table III ($\Delta E_L M/\hbar^2 N=A_1k_F^3+A_2k_F^4+\cdots$).

If we compute the ladder energy from a power series of four terms, the results are in excellent agreement with the results reported in I which were calculated by the standard methods, and for weak to medium potentials, by the Padé approximant method. The error increases to about 10% for $k_F=1.0$ with a hard-core potential and decreases rapidly for smaller k_{Fc}. For weaker potentials the range in k_{Fc} increases. For $V=1.0$, the error does not reach 10% until $k_{Fc}=1.5$ and for $V=0.125$ the 10% point is about $k_{Fc}=2.0$.

We can conclude from these results that, in spite of the probable asymptotic nature of the k_F expansion, Eq. (2.12) forms a good summary of the ladder energy for low to moderate densities (such as that of nuclear matter). If the ladder-approximation energy in this density range is desired it can be calculated by (2.12) with a great resulting economy in effort compared to standard methods.

If we now form the difference between the complete energy and the ladder approximation as given by (2.44) and (2.12), we may compare it, for weak to moderate potentials, with the results found in I. When this comparison is made, the same general-error pattern emerges when due regard is taken for the fact that for weak potentials the difference is quite small. For very strong potentials tending towards the hard-core limit we have nothing to compare our results against. However, as long as the corrections calculated to the ladder approximation are small, we expect the results to be accurate.

TABLE II. Square-well parameters.

V	a	b_1	b_2	b_3
0.125	3.968×10^{-2}	5.025×10^{-3}	2.433×10^{-4}	2.64×10^{-7}
0.25	7.577×10^{-2}	1.008×10^{-2}	8.878×10^{-4}	2.08×10^{-6}
1.0	2.384×10^{-1}	4.024×10^{-2}	8.852×10^{-3}	1.24×10^{-4}
2.5	4.189×10^{-1}	9.520×10^{-2}	2.767×10^{-2}	1.37×10^{-3}
5.0	5.629×10^{-1}	1.682×10^{-1}	5.083×10^{-2}	5.54×10^{-3}
10.0	6.849×10^{-1}	2.666×10^{-1}	7.720×10^{-2}	1.48×10^{-2}
100.0	9.000×10^{-1}	5.816×10^{-1}	1.482×10^{-2}	5.83×10^{-2}
10000.0	9.900×10^{-1}	7.762×10^{-1}	1.941×10^{-1}	8.84×10^{-2}
∞	1.0	0.8	0.2	$\cdots$

4. INFINITE NUCLEAR MATTER

There are two further points which need to be discussed before we can use our results for nuclear matter. The first is the inclusion of isotopic spin and of spin-dependent forces and is quite straightforward. The second, which is less straightforward, is the inclusion of attractive forces.

The inclusion of isotopic spin and of spin-dependent forces can be treated by noting that for the ladder terms the exchange is given by

$$\mathrm{Tr}[K(I-P_{ij})], \quad (4.1)$$

where K is a scattering operator, I the identity, and P_{ij} is the permutation operator for the scattered particle. For no isotopic spin, spin-$\frac{1}{2}$ "particles" (such as He^3), the quantity $I-P_{ij}$ is twice the singlet (odd in spin space) projection operator in the limit of zero momentum where $\langle k|K|k\rangle$ and $\langle -k|K|k\rangle$ matrix elements are equal. For the case where there is also isotopic spin, then $I-P_{ij}$ is $6P_S+2P_T$, where P_S and P_T are the singlet and triplet projection operators, respectively. Thus taking into account the three triplet states, we replace, in the k_F^3 term in the energy,

$$a\rightarrow 3(a_S+a_T) \quad (4.2)$$

where a is, in a spin-dependent isotopic-spinless problem, a_S. In like manner, we can treat the k_F^4 term as

$$a^2\rightarrow 3(a_S^2+a_T^2) \quad (4.3)$$

where again, for a spin-dependent, isotopic-spinless problem, we have a_S^2. There are two k_F^5 terms. In the a^3 one the corresponding replacement is made.

TABLE III. Ladder-approximation coefficients.

V	A_1	A_2	A_3	A_4
0.125	4.211×10^{-3}	8.765×10^{-5}	5.630×10^{-4}	-5.681×10^{-6}
1.0	2.530×10^{-2}	3.164×10^{-3}	6.069×10^{-3}	-1.299×10^{-4}
5.0	5.972×10^{-2}	1.764×10^{-2}	3.456×10^{-2}	1.453×10^{-3}
∞	1.061×10^{-1}	5.566×10^{-2}	1.696×10^{-1}	1.746×10^{-2}

The definition of b_1 is modified for spin-dependent forces (no isospin) to be

$$b_1=\tfrac{1}{4}[\tfrac{1}{2}(\Theta_{S+}''(0)+\Theta_{S-}''(0))+\tfrac{3}{2}(\Theta_{T+}''(0)-\Theta_{T-}''(0))], \quad (4.4)$$

which becomes, by (2.25),

$$b_1=-(3/20)a_S^2r_{S0}+(27/10)A_{T1}(0) \quad (4.5)$$

where the subscripts S and T indicate the analogous singlet and triplet quantities. When isotopic spin is included as well, we obtain

$$b_1=\tfrac{1}{4}[\tfrac{1}{2}(4\Theta_{S+}''(0)+2\Theta_{S-}''(0))+\tfrac{3}{2}(4\Theta_{T+}''(0)-2\Theta_{T-}''(0))] \quad (4.6)$$

which becomes, by (2.25),

$$b_1=-(9/20)a_S^2r_{S0}+(9/10)A_{S1}(0)-(9/20)a_T^2r_{T0}+(81/10)A_{T1}(0). \quad (4.7)$$

For the quantity b_2, we must calculate directly from (2.11) and for no isotopic spin we use just the corresponding singlet result. When isotopic spin is also present, we get three times the singlet contribution plus the triplet contribution. In the calculation of the triplet contribution the potential is interpreted as a 3×3 matrix and the wave functions as 3 component row and column vectors, so as to yield a scalar result.

In order k_F^6, we simply form the analogous terms by first multiplying the singlet and triplet parts separately and adding together. So we get, for the isotopic spin case

$$a^4\rightarrow 3(a_S^4+a_T^4), \quad ab_1\rightarrow a_Sb_{S1}+a_Tb_{T1}, \quad ab_2\rightarrow a_Sb_{S2}+a_Tb_{T2}. \quad (4.8)$$

The replacement rules for the nonladder terms are slightly more involved. They can be obtained easily (though the calculation is somewhat tedious) by noting that replacement for a single vertex scattering operator at zero energy is

$$\tfrac{1}{2}(a_T+a_S)I+\tfrac{1}{2}(a_T-a_S)P_{ij}(\sigma), \quad (4.9)$$

where $P_{ij}(\sigma)$ permutes the spins only, and that every time an exchange should occur [see Eqs. (2.6)–(2.16) from paper I for this information], a factor of $(I-P_{ij})$ must be inserted. For the case where there is no isotopic spin a is always replaced by a_S. For the H3 contribution a^3 and a^4 and ab_2 are replaced just as they were for ladder terms.

We calculate that the contributions for R3 [Eq. (2.31)] are replaced by

$$(-2.863\times10^{-2})k_F^5(-\tfrac{3}{4}(a_S+a_T)(14a_Sa_T-5a_T^2-5a_S^2))\times\hbar^2/M+3(-2.863\times10^{-2})(-3\pi g_1)k_F^6\times[-\tfrac{3}{4}(-5a_T^4+6a_T^3a_S+6a_T^2a_S^2+6a_Ta_S^3-5a_S^4)]\times\hbar^2/M+k_F^6c_1\hbar^2/M \quad (4.10)$$

for the isotopic-spin case.

The contributions from the primitive a^4 terms group themselves by classes as assigned in I. The substitutions for the ladder terms hold for class I, i.e.,

$$a^4\rightarrow 3(a_S^4+a_T^4) \quad (4.11)$$

for the isotopic-spin case.

For class IA we get the substitution

$$a^4\rightarrow\tfrac{1}{4}(-6a_T^4+232a_T^3a_S+604a_T^2a_S^2+232a_Ta_S^3-6a_S^4) \quad (4.12)$$

for the isotopic-spin case.

Classes II and IIA go together and we get the substitution

$$a^4\rightarrow-\tfrac{3}{4}(-5a_T^4+6a_T^3a_S+6a_T^2a_S^2+6a_Ta_S^3-5a_S^4) \quad (4.13)$$

for the isotopic-spin case.

Finally, for class III, and in c_1 and c_2 we substitute

$$a^4\rightarrow 3(a_T^2+a_S^2)(2a_T^2+a_S^2) \quad (4.14)$$

for the isotopic-spin case.

To summarize, Eq. (2.38) remains, for the case of spin-dependent forces but without isotopic spin, the same, except that (i) a becomes a_S; (ii) b_1 and b_2 are now computed as described in (4.5) and after (4.7); (iii) the compution of b_3 will be discussed below. For the case of isotopic spin as well Eq. (2.38) becomes

$$\begin{aligned}\Delta E/N=(\hbar^2/M)[&(1/\pi)k_F^3(a_S+a_T)+0.1670k_F^4(a_S^2+a_T^2)+0.2076k_F^5(a_S^3+a_T^3)+(1/3\pi)k_F^5(b_1+b_2)\\&+2.147\times10^{-2}k_F^5(a_S+a_T)(14a_Sa_T-5a_T^2-5a_S^2)+4.693\times10^{-2}k_F^6(a_S^4+a_T^4)\\&+3.121\times10^{-2}k_F^6(-5a_T^4+6a_T^3a_S+6a_T^2a_S^2+6a_Ta_S^3-5a_S^4)-1.08\times10^{-3}k_F^6(-6a_T^4+232a_T^3a_S+604a_Ta_S^3\\&+232a_Ta_S^3-6a_S^4)+5.743\times10^{-2}k_F^6(a_T^2+a_S^2)(2a_T^2+a_S^2)-2.968\times10^{-2}k_F^6(a_Sb_{1S}+a_Tb_{1T})\\&+6.021\times10^{-2}k_F^6(a_Sb_{2S}+a_Tb_{2T})+k_F^6b_3+(4\sqrt{3}/3\pi^3)(a_T^2+a_S^2)(2a_T^2+a_S^2)k_F^6\ln(k_F\beta)+\cdots].\end{aligned} \quad (4.15)$$

The quantity b_3 is to be replaced by

$$\begin{aligned}b_3=\frac{1}{9\pi^4}\int d\mathbf{k}\ \mathrm{Tr}\{&K_{12}(k)(I-P_{12})(\hat{K}_{+,23}(k)P_{23}-\hat{K}_{-,23}(k)I)K_{13}(k)(I-P_{13})+K_{12}(0)(I-P_{12})\\&\times K_{23}(k)(I-P_{23})K_{13}(0)(I-P_{13})+K_{12}(k)[(\hat{K}_{+,23}(k)-K_{23}(0))I-(\hat{K}_{-,23}(k)-K_{23}(0))P_{23}]K_{13}(k)(I-P_{13})\\&-2K_{12}(0)\hat{K}_{+,23}'(0)[I-P_{23}]K_{13}(0)[I-P_{13}][1+\beta k]^{-1}\}k^{-4},\end{aligned} \quad (4.16)$$

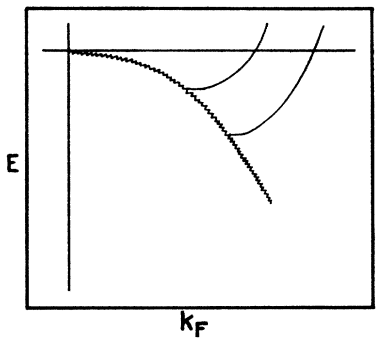

FIG. 2. E versus k_F. The jagged line is the locus of branch points and the other lines are lines of constant V which come in to the locus of branch points with a horizontal slope.

where the K_{ij} are now 64×64 matrices acting in spin and isotopic-spin space between the pair (i,j). In the no-isotopic-spin case, they are 8×8. As the interaction and hence K are usually isotopic-spin independent, one can easily sum out the isotopic spin and reduce the matrices to 8×8 in both cases. The permutation operators used in (4.16) act only in the spin and isotopic-spin space; we have already taken account of the interchange on the momentum in the definition of the K's. This discussion completes the terms in the Fermi-momentum expansion when we are given a spin-dependent potential.

Let us now consider the inclusion of an attractive potential outside a central repulsive one. First we will examine the ladder approximation, whose structure is more transparent. In II (Sec. III) it was shown that if the potential is such that

$$[H_r-(1/\Gamma)Pv]\omega\equiv\mathfrak{M}_\Gamma\omega=0\,, \tag{4.17}$$

where $P=0$ for states in the Fermi sea and 1 otherwise, and there is a smallest eigenvalue $\Gamma\geq-\Gamma_0(k_F)$, then the ladder energy expanded in a potential strength parameter λ can be put in the form [Paper II, Eq. (3.13)]

$$\Delta E=c_{-1}\lambda-\lambda\int_{-\Gamma_0(k_F)}\frac{\lambda d\varphi(u)}{1+u\lambda}\,, \tag{4.18}$$

where $\varphi(u)$ is monotonic, nondecreasing. From (4.18) it follows that $\lambda=[\Gamma_0(k_F)]^{-1}$ is a singular point of the energy in ladder approximation. If v is selected of unit strength (just produces a bound state), then

$$\lim_{k_F\to0}\Gamma_0(k_F)=1\,; \tag{4.19}$$

and, as the projection operator in some sense excludes a portion of phase space and therefore raises the energy compared to the corresponding unprojected state, we get $\Gamma_0(k_F)$ to be monotonically decreasing as k_F increases, so the singularity moves to larger λ as k_F increases. This situation is very reasonable physically and we expect that it will persist in the complete theory. We know that for the complete energy there is some form of singularity in the energy curves for constant λ at the saturation density as for higher density the physical system is one phase and for lower densities it is two phase. The energy curve is $E=$constant in the two-phase region and not in the one-phase region. We identify (in the absence of any cooperative phenomena such as superconductivity, superfluidity, etc.) the branch points in the ladder approximation as the ladder approximation to the saturation branch points of the complete energy. Thus we expect a situation as illustrated in Figs. 2 and 3. We remark that there must be a $\lambda>0$ (as shown in Fig. 3) for which condensation does not occur at infinite dilution ($k_F=0$) for a force of finite range as we can always make the attraction small compared to the zero-point energy required for binding by the finite range. Thus we expect to be able to continue to the saturation density by the route shown in Fig. 3 although the portion of the plane above the locus of branch points will be unobtainable. This situation should hold for the Brueckner approximation as well, as the singularities have physical significance. When the ladder or Brueckner approximations are evaluated by the usual numerical techniques, the introduction of a finite mesh for the various integrations involved can be expected to move the branch point to larger values of the strength parameter, hence explaining its non-discovery by Brueckner and Gammel[6] and subsequent workers, all of whom computed only values close to saturation.

We wish to point out that the notion that the boundary between the one- and two-phase regions is the locus of branch points has also been advanced for classical gases with short-range forces.[7]

On the basis of the data accumulated in Papers I and II and herein, and the above discussion, we are now in a position to put forth an approximation for the many-body energy which will at the same time deal with strongly interacting non-dense systems and non-strongly interacting dense systems. Fundamentally, we start with the ladder approximation as ordinarily calculated (see Sec. V, Paper I, for a description). To this approximation we append the three sequences

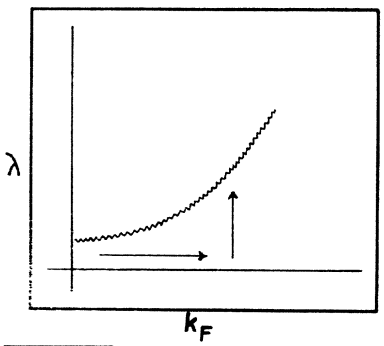

FIG. 3. λ versus k_F. The jagged line is the locus of branch points. The arrows indicate the path of analytic continuation to the point of physical interest.

[6] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

[7] M. E. Fisher, private communication; S. Katsura, Advan. Phys. **12**, 391 (1963) (see p. 416); J. W. Essam and M. E. Fisher, J. Chem. Phys. **38**, 802 (1963); A. F. Andreev, Zh. Eksperim. i Teor. Fiz. **45**, 2064 (1963) [English transl.: Soviet Phys.—JETP **18**, 1415 (1964)].

which start with the third order (in potential strength) and a part of the sum of the fourth-order sequence, which contribute to order k_F^6. We think it should be adequate for the infinite nuclear matter problem.

The program is as follows: First calculate the ladder approximation as ordinarily calculated (Sec. V of Paper I). It will be remembered that off-energy-shell effects do not enter into this calculation. In addition we will need nondiagonal values of K_l which connect relative momenta of less than k_F to relative momenta which may be greater than k_F. These values may be conveniently computed by replacing Paper I, Eq. (5.10) by

$$\langle k'|K_l|k\rangle=\frac{2}{\pi}\int_0^\infty j_l(k'r)V(r)u_{k,l}(r)r^2dr, \quad (4.20)$$

where $|k|\leq k_F$. We will also require the diagonal values of $\hat{K}_l(k)$ under a particular assumption concerning the excitation of the Fermi sea. This assumption can be affected by changing Paper I, Eq. (5.6) to

$$\tfrac{1}{4}\bar{p}^2=k^2+\tfrac{3}{5}k_F^2 \quad (4.21)$$

and Paper I, Eq. (5.8) to

$$\Delta(k)=-3E(k). \quad (4.22)$$

With these quantities computed as a result of an extension of a ladder energy-approximation calculation, we may now evaluate, by Monte Carlo integration, the energy corrections due to the sequences which start with diagrams H3, R3, and F3. These corrections are calculated by placing the appropriate $\langle k'|K|k\rangle$ and $\hat{K}_\pm$ factors in place of the v's which appear in Paper I, Eqs. (2.7)–(2.9), and integrating as described therein. In order to include to some extent the terms starting in fourth order we now define

$$\mathfrak{a}=[3\pi M\Delta E_L/\hbar^2Nk_F^3], \quad (4.23)$$

which goes to the scattering length when k_F goes to zero; then our approximation to the energy is defined by

$$E/N\simeq(\hbar^2/M)\{\mathrm{B1}(K)+\mathrm{H3}(K)+\mathrm{R3}(K)+\mathrm{F3}(K)-3.9\times10^{-4}\mathfrak{a}^4k_F^6\}, \quad (4.24)$$

where by $X(K)$ we mean diagram X with v replaced by K or $\hat{K}_\pm$ as appropriate. The reason for using $\mathfrak{a}$ instead of a in the last term is that, to leading order in k_F, the fourth-order diagrams from which the terms arise contain the product of four K matrices coupled by vanishing momenta. There are, of course, other terms omitted of the same order in k_F as those included by this change; however, in the case of a potential slightly stronger than needed to produce two-body binding, the scattering length a becomes large and repulsive, whereas $\mathfrak{a}$ (at densities somewhat above saturation) remains moderately attractive, which is a physically more reasonable behavior to expect of these terms. This approximation has the properties that the true energy differs from it for small k_F by terms which vanish more rapidly than k_F^6. The first three orders in the potential strength perturbation expansion are treated almost exactly to all orders in k_F and the fourth term is given to within about 5% to $k_F=1.0$ for the square-well potential of I and to better than $k_F=1.5$ for the velocity-dependent potentials of II. The spirit of this approximation is in accord with our previous discussion on the correct path for analytic continuation in the presence of an attractive force. We continue first accurately for weak potentials to the desired density and then continue to stronger potentials. Due to the nature of the ladder approximation, there will be singular points somewhere near the correct location.

The regions in which this approximation is expected to depart from the correct results are the cooperative regions. In addition to the two-phase region discussed previously, one would not expect validity near the hard-core jamming region or where any transition to an ordered state takes place.

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