Noncovariance of the Dirac Monopole

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Starting from the equations which express the divergence of the Maxwell field tensor and its dual in terms of the electric and magnetic current densities, a field theory of the Dirac magnetic monopole is constructed. It is shown that such a theory is incompatible with the requirement of Lorentz invariance if the usual number of degrees of freedom of the electromagnetic field is to be preserved. It is further demonstrated by an explicit construction of the generators of spatial rotations that, independent of the question of Lorentz invariance, the usual argument for the quantization of magnetic charge is not consistent with rotational invariance. A soluble field-theoretic model is given which clearly displays the difficulties of Lorentz invariance inherent in any theory of the Dirac monopole. The mass spectrum of the Maxwell field in this model is shown by direct calculation to be explicitly noncovariant if and only if both the electric and magnetic couplings are nonvanishing.

I. INTRODUCTION

THE great proliferation of ever larger symmetry groups witnessed in the past several years has now made fairly commonplace the view that one of the major problems of high-energy physics is the discovery of a group large enough (and flexible enough) to accommodate the known particles and resonances. Thus the simplicity which the physicist has come to expect (and even demand) of nature has most recently been sought almost exclusively in terms of symmetries rather than detailed dynamics.

One outgrowth of such a philosophy has been a revival[1] of the old symmetry argument for the existence of Dirac's magnetic monopole.[2] According to this view the usual form of Maxwell's equations for the field tensor $F^{\mu\nu}$ and its dual $\bar{F}^{\mu\nu}\equiv\frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}$,

$$\partial_\nu F^{\mu\nu}=e_0 j^\mu\,, \tag{1a}$$

$$\partial_\nu \bar{F}^{\mu\nu}=0\,, \tag{1b}$$

display a lack of symmetry which must be remedied through the replacement of (1b) by

$$\partial_\nu \bar{F}^{\mu\nu}=g_0 j^{\mu\prime}\,, \tag{2}$$

where $j^{\mu\prime}(x)$ is a conserved "magnetic" current which now provides a "monopole" source for the magnetic field.

It is usually stated that a consistent quantization of such a monopole theory is possible only if a very definite relation exists between the electric and magnetic coupling constants. Since, however, there has thus far been no complete field-theoretical formulation of Dirac's monopole, previous derivations of this relation have been semiclassical arguments which are consequently not in agreement with respect to the question of renormalization. Thus, it is not at all clear at present whether the usual constraint

$$g^2/4\pi=\tfrac{1}{4}n^2(e^2/4\pi)^{-1}\,,\quad n=1,2,\cdots \tag{3}$$

is to be required of the renormalized or bare coupling constants. Unless one is willing to take the questionable step of requiring that the appropriate renormalization constants be rational, it is clear that this ambiguity already suggests a possible complication in the formulation of a theory of the monopole and must cast considerable doubt on the utility of (3) in the analysis of experimental results.[3]

In view of these rather uncertain foundations it is somewhat surprising that the monopole has long enjoyed an immunity to attack by theorists. This situation has recently been remedied by Zwanziger[4] who showed that the monopole requires the existence of singularities which are not usually admissible in an S-matrix theory. Unfortunately, however, this argument fails in the important case in which the theory is assumed invariant under the parity operation. Furthermore, Weinberg[5] has subsequently suggested from arguments based on perturbation theory that the monopole may well face even greater difficulties associated with the more fundamental test of Lorentz invariance. It is the object of this paper to further examine this question and to show that is is indeed impossible to formulate a Lorentz-invariant field theory of the monopole.

In the following section we carry out a radiation gauge decomposition of Maxwell's equations in the presence of both electric and magnetic coupling. While it is convenient for this analysis to introduce a set of potentials into the theory, it is to be emphasized that no loss of generality ensues from this device as the entire procedure could alternatively be carried out by using explicitly nonlocal functions of the electric- and magnetic-field strengths. Section III presents a construction of an energy momentum tensor which generates the

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[1] K. W. Ford, Sci. Am. **209**, No. 6, 122 (1963).

[2] P. A. M. Dirac, Proc. Roy. Soc. (London) **A133**, 60 (1931); Phys. Rev. **74**, 817 (1948).

[3] W. V. R. Malkus, Phys. Rev. **83**, 899 (1951); M. Fidecaro, G. Finocchiaro, and G. Giacomelli, Nuovo Cimento **22**, 657 (1961); E. Amaldi, G. Baroni, H. Bradner, L. Hoffman, A. Manfredini, G. Vanderhaege, and H. G. de Carvalho, Notas Fis., Centro Brasil. Pesquisas Fis. **8**, No. 15 (1961); E. M. Purcell, G. B. Collins, T. Fujii, J. Hornbostel, and F. Turkot, Phys. Rev. **129**, 2326 (1963); E. Goto, H. H. Kolm, and K. W. Ford, *ibid.* **132**, 387 (1963).

[4] D. Zwanziger, Phys. Rev. **137**, B647 (1965).

[5] S. Weinberg, Phys. Rev. **138**, B988 (1965).

group of translations and spatial rotations on the fundamental field variables but fails to yield the generators of pure Lorentz transformations. It is subsequently shown that it is not possible to modify this energy momentum tensor so as to admit the construction of a consistent set of generators of the Lorentz group. Finally, in Sec. IV we illustrate these general results by presenting a soluble field-theoretical model which has both electric and magnetic couplings. Although the model is fully consistent for vanishing e_0 or g_0, in the case $e_0g_0\neq 0$ the excitation spectrum of the Maxwell field is shown by direct calculation to be explicitly noncovariant.

II. FORMULATION OF THE THEORY

The basic equations (1a) and (2) which define the fundamental ingredients of a monopole theory may be split quite naturally into two distinct sets of equations. Of these the first,

$$\partial_k F^{0k}=e_0 j^0\,, \tag{4a}$$

$$\partial_k \bar{F}^{0k}=g_0 j_5{}^0\,, \tag{4b}$$

serves to express the longitudinal parts of the electric and magnetic fields in terms of the appropriate charge density while the remaining set,

$$\partial_0 F^{0k}=\partial_l F^{kl}-e_0 j^k\,, \tag{5a}$$

$$\partial_0 \bar{F}^{0k}=\partial_l \bar{F}^{kl}-g_0 j_5{}^k\,, \tag{5b}$$

consists of the equations of motion for the true degrees of freedom of the Maxwell field. In writing (4b) and (5b) we have introduced the convenient notation $j_5{}^\mu(x)$ for the magnetic current in accord with our intention to construct a parity-conserving theory describing the coupling of a pseudovector magnetic current density. While the final results do not depend upon the assumption of parity conservation, this seems to be the more interesting case and serves to illustrate all the essential points.

We shall construct the current $j^\mu(x)$ for a spin one-half field in the usual way:

$$j^\mu(x)=\tfrac{1}{2}\psi\beta\gamma^\mu q\psi\,,$$

where q is the imaginary antisymmetrical charge matrix which acts in the two-dimensional internal space of the Hermitian field $\psi(x)$.[6] With regard to the construction of $j_5{}^\mu(x)$ it is crucial to note that the assumption that $j_5{}^\mu(x)$ be formed without the use of derivatives is an essential restriction without which the usual Pauli moment coupling term could be used to provide a perfectly consistent realization of (1a) and (2). While it is well to note explicitly this additional assumption, the basic philosophy of the monopole must in any event require us to retain as much as possible the formal similarity in the construction of $j^\mu(x)$ and $j_5{}^\mu(x)$.

[6] We use a Majorana representation of the Dirac algebra and the metric $(1,1,1,-1)$.

In order to preserve the invariance of the theory with respect to both parity and charge conjugation,[7] it is necessary to introduce a second Fermi field $\psi'(x)$ and define the current by

$$j_5{}^\mu(x)=\tfrac{1}{2}\psi'\beta\gamma_5\gamma^\mu q'\psi'\,,$$

where q' is the symmetrical matrix

$$q'=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

The operation of electric charge conjugation may now be introduced by requiring

$$E\psi(x)E^{-1}=q'\psi(x)\,,$$
$$E\psi'(x)E^{-1}=\psi'(x)\,,$$

so that

$$Ej^\mu(x)E^{-1}=-j^\mu(x)\,,$$
$$Ej_5{}^\mu(x)E^{-1}=j_5{}^\mu(x)\,.$$

Similarly, magnetic charge conjugation is defined by

$$M\psi(x)M^{-1}=\psi(x)\,,$$
$$M\psi'(x)M^{-1}=iq\psi'(x)\,,$$

with the consequent result

$$Mj^\mu(x)M^{-1}=j^\mu(x)\,,$$
$$Mj_5{}^\mu(x)M^{-1}=-j_5{}^\mu(x)\,.$$

Thus the transformation induced by the product operator $C=EM$ on both current operators, together with the prescription

$$CF^{\mu\nu}(x)C^{-1}=-F^{\mu\nu}(x)\,,$$

ensures the invariance of (1a) and (2) under the combined operations of electric and magnetic charge conjugation.

The equations (4) together with the decomposition of the electric and magnetic fields into their three-dimensional transverse and longitudinal parts,

$$F^{0k}(x)=F_T{}^{0k}(x)+F_L{}^{0k}(x)=F_T{}^{0k}(x)-\partial_k\Lambda(x),$$
$$\bar{F}^{0k}(x)=\bar{F}_T{}^{0k}(x)+\bar{F}_L{}^{0k}(x)=\bar{F}_T{}^{0k}(x)-\partial_k\Lambda'(x),$$

yield the identification

$$\Lambda(x)=-\nabla^{-2}e_0j^0=\int d^3x'\frac{e_0j^0(x')}{4\pi|\mathbf{x}-\mathbf{x}'|}\,,$$

$$\Lambda'(x)=-\nabla^{-2}g_0j_5{}^0=\int d^3x'\frac{g_0j_5{}^0(x')}{4\pi|\mathbf{x}-\mathbf{x}'|}\,.$$

[7] By charge conjugation we mean the product of electric charge conjugation (E) and magnetic charge conjugation (M). See also N. F. Ramsay, Phys. Rev. **109**, 225 (1958). It should be emphasized that although a theory describing a particle having both electric and magnetic charge can be constructed if the requirement of invariance under charge conjugation is dropped, the basic inconsistency problem remains unaltered.

The consideration of the equations of motion (5) is somewhat more complex and is most conveniently performed by the introduction of a vector potential. In the absence of the magnetic current $j_5^\mu(x)$ this is usually done by using the divergenceless character of the magnetic field to write $\bar{F}^{0k}(x)$ as the curl of a transverse vector,

$$\bar{F}^{0k}=\epsilon^{klm}\partial_l A_m,$$

while for $e_0=0$ the equation

$$\partial_k F^{0k}=0$$

can similarly be used to introduce the pseudovector potential $B_k(x)$,

$$F^{0k}=\epsilon^{klm}\partial_l B_m. \tag{6}$$

Thus, one can readily verify that for $e_0=0$ a consistent monopole theory can be described by the Lagrangian

$$\mathcal{L}_M=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\tfrac{1}{4}F_{\alpha\beta}(\partial_\mu B_\nu-\partial_\nu B_\mu)\epsilon^{\mu\nu\alpha\beta} -g_0 j_5^\mu B_\mu+\mathcal{L}'(\psi'), \tag{7}$$

in direct analogy to the $g_0=0$ case for which one has the more familiar Lagrangian[8]

$$\mathcal{L}_E=\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+e_0 j^\mu A_\mu+\mathcal{L}'(\psi). \tag{8}$$

This leads in a natural way to the question of the number of degrees of freedom to be allowed the electromagnetic field if one is to describe a nontrivial monopole theory. In particular we note that the equations of motion implied by Eq. (7),

$$F_M^{\mu\nu}=\epsilon^{\mu\nu\alpha\beta}\partial_\alpha B_\beta,$$
$$\partial_\nu \bar{F}_M^{\mu\nu}=g_0 j_5^\mu,$$

together with the equations resulting from the Lagrangian (8),

$$F_E^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu,$$
$$\partial_\nu F_E^{\mu\nu}=e_0 j^\mu,$$

allow the construction of a tensor $F^{\mu\nu}\equiv F_E^{\mu\nu}+F_M^{\mu\nu}$ which satisfies the equations (1a) and (2). Since each of these two theories is (kinematically) completely consistent, an essentially trivial monopole theory is realizable. However, this not unknown result has been accomplished only at the expense of a doubling of the usual number of degrees of freedom of the electromagnetic field, thus implying the existence of distinct "electric" and "magnetic" photons. Since these photons furthermore do not interact with each other (thereby negating any argument for the quantization of charge), we shall immediately reject this formulation in favor of an approach not requiring a photon doublet.

To this end we introduce the transverse potential $A_k(x)$ by writing in analogy to ordinary electrodynamics

$$\bar{F}_T^{0k}=\epsilon^{klm}\partial_l A_m.$$

Use of Eq. (5b) enables one to write

$$F_T^{0k}=-\partial_0 A_k+g_0\epsilon^{klm}\partial_l\nabla^{-2}j_5^m$$

which, together with (5a), yields

$$-\partial^2 A_k=e_0 j_T^k+g_0\epsilon^{klm}\partial_0\partial_l\nabla^{-2}j_5^m.$$

One can now construct in a straightforward way the Lagrangian appropriate to this system

$$\begin{aligned}\mathcal{L}=\tfrac{1}{2}i\psi\beta\gamma^\mu\partial_\mu\psi-\tfrac{1}{2}m\psi\beta\psi+\tfrac{1}{2}i\psi'\beta\gamma^\mu\partial_\mu\psi' \\ +\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu) \\ +e_0 j^\mu A_\mu+g_0F^{0k}\epsilon^{klm}\partial_l\nabla^{-2}j_5^m \\ -\tfrac{1}{2}g_0F^{lm}\epsilon^{klm}\partial_k\nabla^{-2}j_5^0,\end{aligned} \tag{9}$$

where we have taken the bare mass of the ψ' field zero in order to have $\partial_\mu j_5^\mu=0$. Since the magnetic current interaction appears to have been slighted by our choice of the potential $A_k(x)$ rather than the $B_k(x)$ of Eq. (6), it is well to emphasize here that one could equally well use this latter potential to obtain our result. The essential point to note is that the theory described by the Lagrangian (9) is inconsistent only in the case $e_0g_0\neq 0$.

It can now be readily demonstrated that the equations implied by (9),

$$F^{0k}=-\partial_0 A_k-\partial_k A^0+g_0\epsilon^{klm}\partial_l\nabla^{-2}j_5^m,$$
$$F^{lm}=\partial_l A_m-\partial_m A_l+g_0\epsilon^{klm}\partial_k\nabla^{-2}j_5^0,$$
$$\left[\gamma^\mu\left(\frac{1}{i}\partial_\mu-e_0qA_\mu\right)+m\right]\psi=0,$$
$$\left[\gamma^\mu\frac{1}{i}\partial_\mu+\gamma_5\gamma_k q'\epsilon^{klm}\partial_l\nabla^{-2}F^{0m}+\gamma^0 g_0\gamma_5 q'\nabla^{-2}j_5^0\right]\psi'=0,$$

by using the potential $B_k(x)\equiv-\epsilon^{klm}\partial_l\nabla^{-2}F^{0m}$ and $B^0(x)\equiv g_0\nabla^{-2}j_5^0$ can be given the more symmetrical form[9]

$$\partial_\nu F^{\mu\nu}=e_0 j^\mu, \tag{10a}$$

$$\partial_\nu \bar{F}^{\mu\nu}=g_0 j_5^\mu, \tag{10b}$$

$$\left[\gamma^\mu\left(\frac{1}{i}\partial_\mu-e_0qA_\mu\right)+m\right]\psi=0, \tag{10c}$$

$$\gamma^\mu\left(\frac{1}{i}\partial_\mu-g_0q'\gamma_5B_\mu\right)\psi'=0. \tag{10d}$$

Finally, we obtain from (9) the only nonvanishing equal-time commutation relations among the canonical

[8] It is well to note here that a straightforward application of the action principle [J. Schwinger, Phys. Rev. 91, 713 (1953)] yields the commutation relation

$$\delta(x^0-x^{0\prime})[E_k(x),H_l(x')]=-i\epsilon^{klm}\partial_m\delta(x-x')$$

for each of the two theories described by Eqs. (7) and (8).

[9] These equations are essentially identical to those obtained by Cabibbo and Ferrari, Nuovo Cimento 23, 1147 (1962).

variables $\psi(x)$, $\psi'(x)$, $F_T{}^{0k}(x)$, and $A_k(x)$:

$$\{\psi(x),\psi(x')\}\delta(x^0-x^{0\prime})=\delta(x-x'),$$
$$\{\psi'(x),\psi'(x')\}\delta(x^0-x^{0\prime})=\delta(x-x'),$$
$$[F_T{}^{0k}(x),A^l(x')]\delta(x^0-x^{0\prime})=i\delta_{kl}{}^T(x-x'),$$

a list which may be supplemented by the derived relations

$$[\bar{F}_T{}^{0\kappa}(x),B^l(x')]\delta(x^0-x^{0\prime})=-i\delta_{kl}{}^T(x-x'),$$
$$[F_T{}^{0k}(x),\bar{F}_T{}^{0l}(x')]\delta(x^0-x^{0\prime})=-i\epsilon_{klm}\partial_m\delta(x-x'),$$
$$[A^k(x),B^l(x')]\delta(x^0-x^{0\prime})=-i\epsilon_{klm}\partial_m\nabla^{-2}\delta(x-x'),$$

to give a more symmetrical result. It is well to note that our reluctance to increase the number of degrees of freedom of the Maxwell field has made the potentials $A_k(x)$ and $B_k(x)$ canonical variables, a point which is intimately related to the basic inconsistency of the theory.

Although we have now succeeded in writing the equations of motion of the monopole in the seemingly covariant form (10), it will nonetheless be shown that this theory fails to yield a consistent set of generators of the Lorentz group. We shall now focus attention on the proof of this important result.

III. PROOF OF NONCOVARIANCE

We shall approach the question of the covariance of the theory constructed in the previous section by seeking an explicit operator realization of the energy-momentum tensor $T^{\mu\nu}$ such that the generators

$$P^\mu=\int d^3x\, T^{0\mu}(x), \tag{11a}$$

$$J^{\mu\nu}=\int d^3x[x^\mu T^{0\nu}(x)-x^\nu T^{0\mu}(x)], \tag{11b}$$

satisfy the structure relations

$$[P^\mu,P^\nu]=0,$$
$$-i[P_\lambda,J_{\mu\nu}]=g_{\lambda\nu}P_\mu-g_{\lambda\mu}P_\nu,$$
$$-i[J_{\kappa\lambda},J_{\mu\nu}]=g_{\mu\lambda}J_{\nu\kappa}-g_{\nu\lambda}J_{\mu\kappa}-g_{\mu\kappa}J_{\nu\lambda}+g_{\nu\kappa}J_{\mu\lambda}$$

of the inhomogeneous Lorentz group. It is clear from (11) that, for the verification of these commutation relations, it is sufficient to carry out the construction of $T^{0\mu}(x)$.

For $T^{0k}(x)$ we shall show that the appropriate operator is

$$T^{0k}=F^{0l}F^{kl}+\tfrac{1}{2}\psi\left(\frac{1}{i}\partial_k-e_0qA_k\right)\psi+\tfrac{1}{2}\psi'\left(\frac{1}{i}\partial_k-g_0\gamma_5q'B_k\right)\psi'$$
$$+\tfrac{1}{2}\partial_l(\psi\tfrac{1}{2}\sigma_{kl}\psi)+\tfrac{1}{2}\partial_l(\psi'\tfrac{1}{2}\sigma_{kl}\psi')-F_L{}^{0l}\epsilon^{klm}\bar{F}_L{}^{0m}, \tag{12}$$

where

$$\sigma_{kl}=\tfrac{1}{4}i[\gamma^k,\gamma^l].$$

It is convenient in working with P_k and J_{kl} to use in place of (12) the form

$$F_T{}^{0l}(\partial_kA_l-\partial_lA_k)+\tfrac{1}{2}\psi\frac{1}{i}\partial_k\psi+\tfrac{1}{2}\psi'\frac{1}{i}\partial_k\psi'$$
$$+\tfrac{1}{2}\partial_l(\psi\tfrac{1}{2}\sigma_{kl}\psi)+\tfrac{1}{2}\partial_l(\psi'\tfrac{1}{2}\sigma_{kl}\psi'), \tag{13}$$

which differs from it by an inconsequential divergence term. Since (13) is identical to the momentum density operator in the absence of coupling, it is clear that (12) leads immediately to the structure relations appropriate to the three-dimensional inhomogeneous rotation group. More generally, one can show by straightforward calculation that P_k and J_{kl} generate the group of spatial translations and rotations upon all the basic field operators of the theory.

Before turning to the question of pure Lorentz transformations we shall briefly remark upon the last term of Eq. (12). It is clear that this expression exactly cancels any contribution to the momentum density arising from purely static electric and magnetic fields. Thus it is significant that there can be no intrinsic angular momentum associated with a stationary electron-monopole pair, in direct contradiction of one of the more elegant formulations[10] of the argument for the quantization of charge. According to this latter view the classical angular-momentum density $\mathbf{r}\times(\mathbf{E}\times\mathbf{H})$ of the electromagnetic field should give rise to a net angular momentum for an electron-monopole pair with respect to the relative direction they define in space. Thus in the classical limit

$$e_0j^0(x)=e\delta(x)\delta(y)\delta(z-a),$$
$$g_0j_5{}^0(x)=g\delta(x)\delta(y)\delta(z),$$

one might expect an angular momentum about the z axis.

$$J_3=\int[\mathbf{r}\times(\mathbf{E}\times\mathbf{H})]_3d^3x$$
$$=-2eg\int\frac{d^3x}{(4\pi)^2}\frac{1}{[x^2+y^2+(z-a)^2]^{1/2}}\frac{\partial}{\partial z}\frac{1}{r}$$
$$=-\frac{eg}{4\pi}\int_{-1}^{1}x\,dx\int_0^\infty[r^2+a^2-2rax]^{-1/2}dr$$
$$=-\frac{eg}{4\pi}\left[\int_0^a\frac{dr}{a}\int_{-1}^{1}x\,dx\sum_0^\infty\left(\frac{r}{a}\right)^lP_l(x)\right.$$
$$\left.+\int_a^\infty\frac{dr}{r}\int_{-1}^{1}x\,dx\sum_0^\infty\left(\frac{a}{r}\right)^lP_l(x)\right]$$
$$=-\frac{eg}{4\pi},$$

[10] This formulation is due to M. Fierz, Helv. Phys. Acta **17**, 27 (1944).

the quantization of which leads immediately to the well-known result

$$eg=2n\pi.$$

Although one could, in principle, hope to retain this argument by seeking an alternative form for P_k and J_{kl}, it is an immediate consequence of Schur's lemma that these operators are unique and that the last term of (12) is essential to the preservation of rotational invariance.[11] We have thus uncovered serious objections to the quantization of charge independent of the more complex question of Lorentz invariance.

Having now established the rotational invariance of the theory (at the expense of giving up magnetic charge quantization) we must now go on to consider the pure Lorentz transformations to display the fundamental inconsistency of the theory. To this end we propose for the operator $T^{00}(x)$ the form

$$T^{00}(x)=\tfrac{1}{2}\psi\beta\gamma^k\left(\frac{1}{i}\partial_k-e_0qA_k\right)\psi+\tfrac{1}{2}m\psi\beta\psi$$
$$+\tfrac{1}{2}\psi'\beta\gamma^k\left(\frac{1}{i}\partial_k-g_0q'\gamma_5B_k\right)\psi'+\tfrac{1}{2}[(F^{0k})^2+(\bar{F}^{0k})^2]. \quad (14)$$

As the first and most obvious property of $T^{00}(x)$ we must require that P^0 generate the development in time of all operators in the theory. Thus

$$\int d^3x'[T^{00}(x'),\chi(x)]=\frac{1}{i}\partial_0\chi(x)$$

for any operator $\chi(x)$, a result which is readily established by straightforward calculation. Another important condition is the requirement that J^{0k} as defined by (11b) transform $F^{\mu\nu}$ as a second-rank tensor, i.e.,

$$-i[J^{0k},F^{\mu\nu}]=(x^0\partial_k-x^k\partial^0)F^{\mu\nu}+g^{\mu0}F^{k\nu}$$
$$-g^{k\mu}F^{0\nu}+g^{k\nu}F^{0\mu}-g^{\nu0}F^{k\mu}.$$

Again one can directly verify this commutation relation as well as the corresponding vector transformation properties of both $j^\mu(x)$ and $j_5{}^\mu(x)$ thus establishing the covariance of Eqs. (10a) and (10b) for the $T^{00}(x)$ of Eq. (14).

It is, however, well known that in the radiation gauge formulation of electrodynamics $A_\mu(x)$ $(\psi(x))$ fails to transform as a vector (spinor) in spite of the covariance of (10c). Thus the question of the behavior of Eqs. (10c) (when $g_0=0$) and (10d) under a Lorentz transformation is considerably more complex, and it is precisely here that a monopole field theory must founder. In the $g_0=0$ limit one has from (14) the result

$$-i[J^{0k},A^l(x)]=(x^0\partial_k-x^k\partial^0)A^l-\delta_{kl}A^0+\partial_l\mathcal{A}_k, \quad (15a)$$
$$-i[J^{0k},\psi(x)]=(x^0\partial_k-x^k\partial^0)\psi-\tfrac{1}{2}\beta\gamma^k\psi$$
$$+\tfrac{1}{2}ie_0q\{\mathcal{A}_k,\psi\}, \quad (15b)$$

where

$$\mathcal{A}_k=\partial_m\nabla^{-2}(x^kF^{0m})-x^k\partial_m\nabla^{-2}F^{0m}$$
$$=[\partial_m\nabla^{-2},x^k]F^{0m}.$$

In spite of the absence of manifest Lorentz invariance in this theory one has the familiar result that the equation of motion

$$\left[\gamma^\mu\left(\frac{1}{i}\partial_\mu-e_0qA_\mu\right)+m\right]\psi=0$$

is covariant with respect to the Lorentz transformation described by (15). A similar situation occurs in the case $e_0=0$. Here the vector potential $B_\mu(x)$ and the spinor $\psi'(x)$ satisfy the commutation relations

$$-i[J^{0k},B^l(x)]=(x^0\partial_k-x^k\partial^0)B_l-\delta_{kl}B^0+\partial_l\mathcal{B}_k, \quad (16a)$$
$$-i[J^{0k},\psi'(x)]=(x^0\partial_k-x^k\partial^0)\psi'$$
$$-\tfrac{1}{2}\beta\gamma^k\psi'+\tfrac{1}{2}ig_0q'\gamma_5\{\mathcal{B}_k,\psi'\}, \quad (16b)$$

where

$$\mathcal{B}_k=-\partial_m\nabla^{-2}(x^k\bar{F}^{0m})+x^k\partial_m\nabla^{-2}\bar{F}^{0m}$$
$$=-[\partial_m\nabla^{-2},x^k]\bar{F}^{0m},$$

a result which establishes the covariance of the equation

$$\gamma^\mu\left(\frac{1}{i}\partial_\mu-g_0q'\gamma_5B_\mu\right)\psi'(x)=0.$$

In the case of nonvanishing electric and magnetic couplings, it is clear that, for the energy density of Eq. (14), the transformation properties of ψ and ψ' are given by (15b) and (16b), respectively. However, because of the presence of the term $g_0j_5{}^kB_k$ in (14) and the noncommutativity of $A_\mu(x)$ with $B_k(x)$, the commutation relation (15a) becomes

$$-i[J^{0k},A_l(x)]=(x^0\partial_k-x^k\partial^0)A^l-\delta_{kl}A^0$$
$$+\partial_l\mathcal{A}_k+g_0\epsilon^{lmn}[\partial_m\nabla^{-2},x^k]j_5{}^n \quad (17)$$

while (16a) now has the form

$$-i[J^{0k},B_l(x)]=(x^0\partial_k-x^k\partial^0)B_l-\delta_{kl}B^0$$
$$+\partial_l\mathcal{B}_k+e_0\epsilon^{lmn}[\partial_m\nabla^{-2},x^k]j^n. \quad (18)$$

The presence of the additional terms in both (17) and (18) has destroyed the covariance of Eqs. (10c) and (10d), as a Lorentz transformation on these equations now has the effect of introducing a direct interaction proportional to e_0g_0 between the fields ψ and ψ'. Thus, provided that the uniqueness of $T^{00}(x)$ can be established, we have the gratifying result that the theory fails the test of Lorentz invariance if and only if $e_0g_0\neq0$.

The question of the uniqueness of $T^{00}(x)$ can be handled by relatively straightforward considerations.

[11] The inclusion of this term reduces P_k and J_{kl} to their free-field forms, thus immediately displaying the rotational invariance of the theory.

First, it is to be noted that, since the equations of motion of the Maxwell field (1a) and (2) are fixed, and the $T^{00}(x)$ of Eq. (14) transforms $F^{\mu\nu}(x)$ as a second-rank tensor, that part of T^{00} which contains terms involving F_T^{0k} and $\bar{F}_T^{0k}$ (as well as A_k and B_k) is unique. That is, any attempt to include additional terms in $T^{00}(x)$ involving the dynamical variables of the Maxwell field must alter either the fundamental equations of motion or the transformation properties of $F^{\mu\nu}(x)$. Thus, we have only to deal with the somewhat more subtle problem of whether covariance can be restored by introducing a direct interaction between the currents j^μ and j_5^μ. It is clear that such a coupling must be proportional to e_0g_0 and involve no dimensional parameters. It is not difficult to verify, however, that there is no rotationally invariant parity-conserving coupling, either local or nonlocal, which can be introduced between these two currents.

It is perhaps instructive to briefly view here the question of covariance in terms of the commutation relations of $T^{\mu\nu}(x)$ with itself as discussed by Schwinger.[12] In particular he has shown that a sufficient condition for Lorentz invariance is

$$-i[T^{00}(x),T^{00}(x')] = -(T^{0k}(x)+T^{0k}(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}'). \quad (19)$$

In the present case this rather complex condition may be simplified considerably by noting that it is certainly valid for e_0 or g_0 equal to zero. It is an immediate consequence of this observation that (19) is not satisfied for the $T^{0k}(x)$ and $T^{00}(x)$ of Eqs. (12) and (14) in the two important respects:

(i) The commutator fails to generate the last term of (12).
(ii) The commutator of j^kA_k with $j_5{}^kB_k$ is not accommodated by the structure of (19).

It requires only a slight extension of this result to again establish the uniqueness of $T^{00}(x)$ as well as its necessary incompatibility with the structure relations of the Lorentz group, thus further verifying the basic inconsistency inherent to a monopole theory.

IV. A SOLUBLE MODEL

As a simple realization of a theory which illustrates all the essential features discussed in the preceding sections, we shall construct here a soluble model describing the coupling of conserved electric and magnetic currents to the electromagnetic field. To this end it is convenient to introduce the scalar and pseudoscalar fields $\phi(x)$ and $\varphi(x)$, respectively, which, in the absence of coupling, are described by the Lagrangian

$$\mathcal{L}=\phi^\mu\partial_\mu\phi+\tfrac{1}{2}\phi^\mu\phi_\mu+\varphi^\mu\partial_\mu\varphi+\tfrac{1}{2}\varphi^\mu\varphi_\mu. \quad (20)$$

[12] J. Schwinger, Phys. Rev. **127**, 324 (1962).

As a result of the equations of motion,

$$\partial_\mu\phi^\mu=0,\qquad \partial_\mu\varphi^\mu=0, \quad (21)$$

implied by (20), the fields $\phi^\mu(x)$ and $\varphi^\mu(x)$ describe conserved vector and pseudovector currents, a result which is unaltered by the inclusion of any coupling which does not explicitly involve the fields $\phi(x)$ and $\varphi(x)$.[13] Thus we are led to propose the model described by the equations

$$\partial_\nu F^{\mu\nu}=e_0\phi^\mu, \quad (22a)$$

$$\partial_\nu \bar{F}^{\mu\nu}=g_0\varphi^\mu. \quad (22b)$$

In the special case $g_0=0$ such a theory is described by the complete Lagrangian

$$\mathcal{L}=\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+\phi^\mu\partial_\mu\phi+\tfrac{1}{2}\phi^\mu\phi_\mu+e_0A^\mu\phi_\mu$$

which is identical to a model previously considered in the discussion of the connection between gauge invariance and mass.[14] The symmetry between electric and magnetic coupling is displayed by the observation that for $e_0=0$ one has as the appropriate Lagrangian

$$\mathcal{L}=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\tfrac{1}{2}F_{\alpha\beta}(\partial_\mu B_\nu-\partial_\nu B_\mu)\epsilon^{\alpha\beta\mu\nu}+\varphi^\mu\partial_\mu\varphi+\tfrac{1}{2}\varphi^\mu\varphi_\mu-g_0B^\mu\varphi_\mu.$$

In the former case one finds that the two transverse modes of the Maxwell field corresponding to a given momentum combine with the scalar mode of the field $\phi(x)$ to describe the three degrees of freedom of a vector meson with mass e_0^2. It is therefore of no surprise that in the case of purely magnetic coupling the transverse electromagnetic modes together with the field $\varphi(x)$ describe a pseudovector field of mass g_0^2. This, however, suggests that in the case of nonvanishing electric and magnetic coupling there might well result serious complications as $\phi(x)$ and $\varphi(x)$ compete in their efforts to determine the character of the Maxwell field.

Proceeding in direct analogy to the approach developed in Sec. II we can now write the Lagrangian

$$\mathcal{L}=\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)+\phi^\mu\partial_\mu\phi+\tfrac{1}{2}\phi^\mu\phi_\mu+\varphi^\mu\partial_\mu\varphi+\tfrac{1}{2}\varphi^\mu\varphi_\mu+e_0\phi^\mu A_\mu+g_0F^{0k}\epsilon^{klm}\partial_l\nabla^{-2}\varphi^m-\tfrac{1}{2}g_0F^{lm}\epsilon^{klm}\partial_k\nabla^{-2}\varphi^0 \quad (23)$$

appropriate to Eqs. (21) and (22). In addition, one finds from (23) the equations

$$\phi^\mu=-\partial^\mu\phi-e_0A^\mu,$$
$$\varphi^k=-\partial^k\varphi-g_0\epsilon^{klm}\partial_l\nabla^{-2}F^{0m},$$
$$\varphi^0=-\partial^0\varphi+\tfrac{1}{2}g_0\epsilon^{klm}\partial_k\nabla^{-2}F^{lm},$$

[13] This point has been discussed in some detail by G. S. Guralnik and the author (to be published).
[14] D. G. Boulware and W. Gilbert, Phys. Rev. **126**, 1563 (1962).

which upon introduction of B^μ as defined by

$$F_T^{0k}=-\epsilon^{klm}\partial_l B_m\,,$$
$$\bar{F}_L^{0k}=-\partial_k B^0\,,$$

assume the form

$$\phi^\mu=-\partial^\mu\phi-e_0A^\mu\,,$$
$$\varphi^\mu=-\partial^\mu\varphi-g_0B^\mu. \tag{24}$$

It now requires only simple manipulations on Eqs. (21), (22), and (24) to derive the reduced field equations

$$(-\partial^2+e_0^2)\phi(x)=0\,,$$
$$(-\partial^2+g_0^2)\varphi(x)=0\,,$$
$$(-\partial^2+e_0^2+g_0^2)F_T^{0k}(x)=e_0^2g_0^2\nabla^{-2}F_T^{0k}.$$

While the fields $\phi(x)$ and $\varphi(x)$ each appear to be unaware of the presence of the other, the Maxwell field, unable to choose whether to be a vector or axial-vector meson, experiences a fatally noncovariant distortion of its excitation spectrum if and only if $e_0g_0\neq0$. Thus the breakdown of Lorentz invariance which revealed itself in the general monopole theory through the noncovariance of the equations of motion has been given in this soluble model an explicit and dramatic realization.

In order to illustrate more clearly the correspondence with the general monopole theory, we shall briefly discuss the covariance problem from the viewpoint of the preceding section. In this case one has

$$T^{0k}(x)=\phi^0(-\partial_k\phi-e_0A_k)+\varphi^0(-\partial_k\varphi-g_0B_k)+F^{0l}F^{kl} -e_0g_0(\partial_l\nabla^{-2}\phi^0)\epsilon^{klm}(\partial_m\nabla^{-2}\varphi^0)$$

and (25)

$$T^{00}(x)=\tfrac{1}{2}(\phi^0)^2+\tfrac{1}{2}(\varphi^0)^2+\tfrac{1}{2}(\partial_k\phi+e_0A_k)^2 +\tfrac{1}{2}(\partial_k\varphi+g_0B_k)^2+\tfrac{1}{2}[(F^{0k})^2+(\bar{F}^{0k})^2].$$

In analogy to our previous result the last term of (25) is necessary to preserve rotational invariance while at the same time it eliminates the necessity for the quantization of charge. One further finds that the Hamiltonian P^0 generates the time development of all operators in the theory and that J^{0k} correctly transforms $F^{\mu\nu}(x)$ as a second-rank tensor.

There is however, one noteworthy distinction between this model and the more general theory which arises from the fact that the currents $\phi^\mu(x)$ and $\varphi^\mu(x)$ are linear rather than bilinear in the operators of the charge fields. Thus, in view of the result of Eqs. (17) and (18) together with

$$-i[J^{0k},\phi(x)]=(x^0\partial_k-x^k\partial^0)\phi-e_0\mathcal{A}_k\,,$$
$$-i[J^{0k},\varphi(x)]=(x^0\partial_k-x^k\partial_0)\varphi-g_0\mathcal{B}_k\,,$$

it follows from (24) that $\phi^\mu(x)$ and $\varphi^\mu(x)$ cannot transform as vectors. This has the immediate consequence that none of the equations of this theory are covariant with respect to the Lorentz group if $e_0g_0\neq0$.

With this minor difference all our previous results carry over to this model. The general proofs of the inconsistency directly apply and at the same time we have the further advantage here of being able to explicitly calculate the noncovariant mass spectrum of the Maxwell field.

V. CONCLUSION

Although the pursuit of the magnetic monopole has never attained the distinction of being considered one of the most pressing problems in particle physics, the demonstration of its incompatibility with the axioms of quantum field theory does dispose of what has been at least a mildly annoying problem of electrodynamics. While it goes without saying that the final verdict on this question must be given by the experimentalist, the discovery of such a particle (without a "second photon") would require a drastic reformulation of some of the most fundamental aspects of quantum field theory.

Despite the fact that invariance under parity and charge conjugation has been incorporated into the proof given here, it is to be emphasized that this point is quite inessential, the conclusion resting solely on the assumption of the existence of a Lagrangian and a well-defined set of generators of Lorentz transformations. It is well to note further that this paper has emphasized throughout the symmetry between electric and magnetic couplings, and the basic consistency problem only arises in fitting both into the same theory. While there is thus no fundamental basis for choosing between the two types of coupling (as there is no reason why matter should be locally favored over antimatter), nature, having once decided on a vector coupling, is no longer free to admit a pseudovector interaction.

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