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Infrared Photons and Gravitons*

STEVEN WEINBERG†
Department of Physics, University of California, Berkeley, California
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It is shown that the infrared divergences arising in the quantum theory of gravitation can be removed by the familiar methods used in quantum electrodynamics. An additional divergence appears when infrared photons or gravitons are emitted from noninfrared external lines of zero mass, but it is proved that for infrared gravitons this divergence cancels in the sum of all such diagrams. (The cancellation does not occur in massless electrodynamics.) The formula derived for graviton bremsstrahlung is then used to estimate the gravitational radiation emitted during thermal collisions in the sun, and we find this to be a stronger source of gravitational radiation (though still very weak) than classical sources such as planetary motion. We also verify the conjecture of Dalitz that divergences in the Coulomb-scattering Born series may be summed to an innocuous phase factor, and we show how this result may be extended to processes involving arbitrary numbers of relativistic or nonrelativistic particles with arbitrary spin.

I. INTRODUCTION

THE chief purpose of this article is to show that the infrared divergences in the quantum theory of gravitation can be treated in the same manner as in quantum electrodynamics. However, this treatment apparently does not work in other non-Abelian gauge theories, like that of Yang and Mills. The divergent phases encountered in Coulomb scattering will incidentally be explained and generalized.

It would be difficult to pretend that the gravitational infrared divergence problem is very urgent. My reasons for now attacking this question are:

(1) Because I can. There still does not exist any satisfactory quantum theory of gravitation, and in lieu of such a theory it would seem well to gain what experience we can by solving any problems that can be solved with the limited formal apparatus already at our disposal. The infrared divergences are an ideal case of this sort, because we already know all about the coupling of a very soft graviton to any other particle,[1] and about the external graviton line wave functions[1] and internal graviton line propagators.[2]

(2) Because something might go wrong, and that would be interesting. Unfortunately, nothing does go wrong. In Sec. II we see that the dependence on the infrared cutoffs of real and virtual gravitons cancels just as in electrodynamics.

However, there is a more subtle difficulty that might have been expected. Ordinary quantum electrodynamics would contain unremovable logarithmic divergences if the electron mass were zero, due to diagrams in which a soft photon is emitted from an external electron line with momentum parallel to the electron's.[3] There are no charged massless particles in the real world, but hard neutrinos, photons, and gravitons do carry a gravitational "charge," in that they can emit soft gravitons. In Sec. III we show that diagrams in which a soft graviton is emitted from some other hard massless particle line do contain divergences like the $\ln m_e$ terms in massless electrodynamics, but that these divergences cancel when we sum all such diagrams.[4] However, this cancellation is definitely due to the details of gravitational coupling, and does not save theories (like Yang and Mills's) in which massless particles can emit soft massless particles of spin one.

(3) Because in solving the infrared divergence problem we obtain a formula for the emission rate and spectrum of soft gravitons in arbitrary collision processes, which may (if our experience in electrodynamics is a guide) be numerically the most important gravitational radiative correction. In Sec. IV this formula is used to calculate the soft gravitational inner bremsstrahlung in an arbitrary nonrelativistic collision, and the result is then used to estimate the thermal gravitational radiation from the sun. The answer is several

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[1] S. Weinberg, Phys. Rev. **135**, B1049 (1965).

[2] See, e.g., S. Weinberg, Phys. Rev. **138**, B988 (1965). The graviton propagator given in Eq. (2.20) of the present article is not just the vacuum expectation value of a time-ordered product, but includes the effects of instantaneous "Newton" interactions that must be added to the interaction to maintain Lorentz invariance, and further, it does *not* include certain non-Lorentz-invariant gradient terms which disappear because the gravitational field is coupled to a conserved source. This disappearance has so far only been proved for graviton lines linking particles on their mass shells, and in fact this is the one impediment which keeps us from claiming that we possess a completely satisfactory quantum theory of gravitation. In using (2.20) we are to some extent relying on an act of faith, but this faith seems particularly well-founded in our present context because we use (2.20) here to link particle lines with momenta only infinitesimally far from their mass shells. See also S. Weinberg, in *Brandeis 1964 Summer Lectures on Theoretical Physics* (Prentice-Hall, Inc., New York, 1965).

[3] The extra divergences in massless quantum electrodynamics have long been known to many theorists. Recently, it has been noted by T. D. Lee and M. Nauenberg, Phys. Rev. **133**, B1549 (1964), that these divergences cancel if transition rates are computed only between suitable ensembles of final *and initial* states. [See also T. Kinoshita, J. Math. Phys. **3**, 650 (1962)]. However, these ensembles include not only indefinite numbers of very soft quanta but also hard massless particles with indefinite energies, and I remain unconvinced that transition rates between such ensembles are the only ones that can be measured and need be finite.

[4] I understand that this cancellation has also been found by R. P. Feynman.

orders of magnitude greater than from more usually considered sources, like planetary motion.

The formalism derived in Sec. II is used in Sec. V to calculate the divergent final- and initial-state interaction phases in arbitrary scattering processes.

II. REAL AND VIRTUAL INFRARED DIVERGENCES

This section shows how to treat the infrared divergences arising from very soft real and virtual gravitons. In order to keep the discussion as perspicuous as possible, I repeat the conventional treatment[5,6] of infrared divergences in electrodynamics (correcting a mistake in Ref. 5), with explanations at each step of how the same arguments apply to gravitation. It is not really quite correct to treat gravitation and electromagnetism as mutually exclusive phenomena, but it will be made obvious that in a combined theory the infrared photons and gravitons simply supply independent correction factors to transition rates.

1. One Soft Photon or Graviton

If we attach a soft-photon line with momentum q to an outgoing charged-particle line in a Feynman diagram, we must supply one extra charged-particle propagator with momentum $p+q$ and one extra vertex for the transition $p+q\rightarrow p$. If the soft-photon line is attached to an incoming charged-particle line, the extra propagator is for momentum $p-q$ and the transition is $p\rightarrow p-q$. For instance, if the charged particle has zero spin, these factors are[7]

$$i(2\pi)^4 e(2p^\mu+\eta q^\mu)[-i(2\pi)^{-4}] \times[(p+\eta q)^2+m^2-i\epsilon]^{-1}, \quad (2.1)$$

where $\eta=+1$ or -1 for an outgoing or incoming charged particle. In the limit $q\rightarrow 0$ Eq. (2.1) becomes (because $p^2+m^2=0$):

$$e\eta p^\mu/[p\cdot q-i\eta\epsilon]. \quad (2.2)$$

Although (2.1) applies only for zero spin, the limiting form (2.2) is well known[8] to hold for any spin.

Diagrams with the soft-photon line attached to an internal charged-particle line lack the denominator $p\cdot q$, and therefore are negligible for $q\rightarrow 0$. Hence the effect of attaching one soft-photon line to an arbitrary diagram is simply to supply an extra factor,

$$\sum_n e_n\eta_n p_n{}^\mu/[p_n\cdot q-i\eta_n\epsilon], \quad (2.3)$$

the sum running over all external lines in the original diagram.

If we attach a soft-graviton line to an external spin-zero line, the extra factors are[2]

$$\tfrac{1}{2}i(2\pi)^4(8\pi G)^{1/2}(2p^\mu+\eta q^\mu)(2p^\nu+\eta q^\nu) \times[-i(2\pi)^{-4}][(p+\eta q)^2+m^2-i\epsilon]^{-1}, \quad (2.4)$$

where μ, ν are the graviton polarization indices. For $q\rightarrow 0$ this gives

$$(8\pi G)^{1/2}\eta p^\mu p^\nu/[p\cdot q-i\eta\epsilon]. \quad (2.5)$$

The limiting form (2.5) is actually valid whatever the spin of the external line to which we attach the graviton.[1] For example, if this line is outgoing and has spin $\frac{1}{2}$, then we have instead of Eq. (2 4) the factor

$$-\tfrac{1}{4}(2\pi)^4(8\pi G)^{1/2}\{(2p^\mu+q^\mu)\gamma^\nu+(2p^\nu+q^\nu)\gamma^\mu\} \times[-i(2\pi)^{-4}]\left[\frac{-i(p^\lambda+q^\lambda)\gamma_\lambda+m}{(p+q)^2+m^2-i\epsilon}\right]. \quad (2.6)$$

But (2.6) appears multiplied on the left with a Dirac spinor $\bar{u}$ such that

$$\bar{u}[ip^\lambda\gamma_\lambda+m]=0.$$

Thus, moving the propagator numerator to the left of the vertex function, we are left with an anticommutator equal to (2.5) in the limit $q\rightarrow 0$. For general spin the same conclusion can be reached on grounds of Lorentz invariance,[1] without embroiling oneself in higher spin formalisms. The normalization factor $(8\pi G)^{1/2}$ is chosen so that an arbitrary nonrelativistic two-particle scattering amplitude will have a one-graviton-exchange pole with the correct residue to correspond to a potential Gm_1m_2/r.

The dominance of the $1/(p\cdot q)$ pole in (2.5) implies that the effect of attaching one soft-graviton line to an arbitrary diagram is to supply a factor equal to the sum of (2.5) over all external lines in the diagram

$$(8\pi G)^{1/2}\sum_n \eta_n p_n{}^\mu p_n{}^\nu/[p_n\cdot q-i\eta_n\epsilon]. \quad (2.7)$$

2. Many Soft Photons or Gravitons

It is well known that the effect of attaching several soft-photon lines to an arbitrary diagram is to supply a product of factors of the form (2.3), one for each soft photon. For we note that if N soft photons are emitted from an outgoing (incoming) charged-particle line with photon r last (first), photon s next-to-last (second), etc., then the charged particle propagators will contribute a multiple pole factor

$$[p\cdot q_r-i\eta\epsilon]^{-1}[p\cdot(q_r+q_s)-i\eta\epsilon]^{-1}\cdots,$$

but this must be summed over the $N!$ permutations $12\cdots N\rightarrow rs\cdots$, and the sum is just

$$[p\cdot q_1-i\eta\epsilon]^{-1}[p\cdot q_2-i\eta\epsilon]^{-1}\cdots.$$

[5] J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), Chap. 16.

[6] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961).

[7] The notation used is that of Ref. 5. In particular, $\hbar=c=1$, and $p\cdot q\equiv\mathbf{p}\cdot\mathbf{q}-p^0q^0$.

[8] For $j=\frac{1}{2}$ see Ref. 5 or 6. For general spin see Ref. 1.

For example, for $N=2$

$$[p\cdot q_1-i\eta\epsilon]^{-1}[p\cdot(q_1+q_2)-i\eta\epsilon]^{-1}+[p\cdot q_2-i\eta\epsilon]^{-1}[p\cdot(q_2+q_1)-i\eta\epsilon]^{-1}=[p\cdot q_1-i\eta\epsilon]^{-1}[p\cdot q_2-i\eta\epsilon]^{-1}.$$

The result may be proved for general N by an easy mathematical induction. The same factorization occurs trivially when soft photons are emitted from different legs.

The pole structure created by inserting the soft-graviton factors (2.7) is the same as for the soft-photon factors (2.3), so by precisely the same reasoning the effect of attaching N soft-graviton lines to an arbitrary Feynman diagram is just to multiply the matrix element by N factors (2.7). It is this factorizability that will allow us to obtain the sum of an unlimited number of very complicated Feynman diagrams.

3. Virtual Infrared Divergences

We will define an infrared virtual photon or graviton as one which connects two external lines and carries energy less than Λ, where Λ is some convenient dividing point chosen low enough to justify the approximations made above in subsections (1) and (2). By "connecting external lines" we mean that the infrared line may join onto a line that has already emitted soft real quanta or virtual infrared quanta, but not onto one which, by real or virtual emission, has acquired a momentum far off its mass shell. In addition to the cutoff $|\mathbf{q}|\leqq\Lambda$ which just defines the infrared lines, we will also impose a cutoff $|\mathbf{q}|\geqq\lambda$ in order to display the logarithmic divergences as powers of $\ln\lambda$. We take λ very small (in particular, $\lambda\ll\Lambda$) so this cutoff only affects the infrared lines because it is only these that give infrared divergences for $\lambda=0$.

The effect of adding N virtual infrared-photon lines to a diagram that does not already involve any infrared lines is to multiply the matrix element by N pairs of the factors (2.3), each pair connected by a photon propagator

$$\frac{-i}{(2\pi)^4}\frac{g_{\mu\nu}}{q^2-i\epsilon}, \qquad (2.8)$$

and then sum over polarization indices and integrate over q's. In addition we must divide by $2^N N!$, because we saw in Subsec. 2 that the external-line poles factor only if we sum over all places to which we may attach the two ends of each infrared virtual-photon line, and this includes spurious sums over the $N!$ permutations of the lines and over the two directions that each line might be thought to flow. The result is then

$$\frac{1}{N!}\left[\frac{1}{2}\int_\lambda^\Lambda d^4q\,A(q)\right]^N, \qquad (2.9)$$

where

$$A(q)=\frac{-i}{(2\pi)^4[q^2-i\epsilon]}\times\sum_{nm}\frac{e_ne_m\eta_n\eta_m(p_n\cdot p_m)}{[p_n\cdot q-i\eta_n\epsilon][-p_m\cdot q-i\eta_m\epsilon]}. \qquad (2.10)$$

The limits on the integral in (2.9) refer to $|\mathbf{q}|$. Note that we have changed the sign of $p_m\cdot q$ in the second denominator in (2.10), because if we define q as the momentum emitted by line n then q must be absorbed by line m. Summing over N, we conclude that the S matrix for an arbitrary process may be expressed as

$$S_{\beta\alpha}=S_{\beta\alpha}{}^0\exp\left(\frac{1}{2}\int_\lambda^\Lambda d^4q\,A(q)\right), \qquad (2.11)$$

where $S_{\beta\alpha}{}^0$ is the S matrix without infrared virtual photons.

The rate for $\alpha\to\beta$ is given by the absolute square of (2.11),

$$\Gamma_{\beta\alpha}=\Gamma_{\beta\alpha}{}^0\exp\left\{\operatorname{Re}\int_\lambda^\Lambda d^4q\,A(q)\right\}. \qquad (2.12)$$

The real part of the integral arises wholly from the $i\pi\delta(q^2)$ term in the photon propagator [for details, see Sec. V], so

$$\operatorname{Re}\int_\lambda^\Lambda d^4q\,A(q)=-\frac{1}{2(2\pi)^3}\int_\lambda^\Lambda d^4q\,\delta(q^2)\times\sum_{nm}\frac{e_ne_m\eta_n\eta_m(p_n\cdot p_m)}{(p_n\cdot q)(p_m\cdot q)}=-A\ln(\Lambda/\lambda), \qquad (2.13)$$

where A is the positive dimensionless constant

$$A\equiv\int d^2\Omega\,A(\hat{q}), \qquad (2.14)$$

$$A(\hat{q})\equiv\frac{1}{2(2\pi)^3}\sum_{nm}\frac{e_ne_m\eta_n\eta_m(p_n\cdot p_m)}{[E_n-\mathbf{p}_n\cdot\hat{q}][E_m-\mathbf{p}_m\cdot\hat{q}]}. \qquad (2.15)$$

The integral in (2.14) is elementary, and yields

$$A=-\frac{1}{8\pi^2}\sum_{mn}\eta_n\eta_me_ne_m\beta_{nm}{}^{-1}\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right), \qquad (2.16)$$

where β_{nm} is the relative velocity of particles n and m in the rest frame of either:

$$\beta_{nm}\equiv\left[1-\frac{m_n{}^2m_m{}^2}{(p_n\cdot p_m)^2}\right]^{1/2}. \qquad (2.17)$$

Using (2.13) in (2.12), we find

$$\Gamma_{\beta\alpha}=\Gamma_{\beta\alpha}{}^0(\lambda/\Lambda)^A. \qquad (2.18)$$

The same combinatorics apply to gravitons, and yield an expression for the infrared virtual-graviton corrections to any process $\alpha\to\beta$

$$S_{\beta\alpha}=S_{\beta\alpha}{}^{0}\exp\left\{\frac{1}{2}\int_{\lambda}^{\Lambda}d^4q\,B(q)\right\}, \tag{2.19}$$

where S^0 is the S matrix without virtual infrared gravitons, and $B(q)$ is the result of joining a pair of factors (2.7) with a graviton propagator. The effective graviton propagator joining a $(\mu\nu)$ vertex with a $(\rho\sigma)$ vertex is known[2] to be

$$\frac{-i}{2(2\pi)^4}\frac{\{g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho}-g_{\mu\nu}g_{\rho\sigma}\}}{q^2-i\epsilon}. \tag{2.20}$$

Therefore we find

$$B(q)=\frac{-8\pi Gi}{(2\pi)^4[q^2-i\epsilon]}\times\sum_{nm}\frac{\eta_n\eta_m\{(p_n\cdot p_m)^2-\frac{1}{2}m_n{}^2m_m{}^2\}}{[p_n\cdot q-i\eta_n\epsilon][-p_m\cdot q-i\eta_m\epsilon]}. \tag{2.21}$$

The rate for $\alpha\to\beta$ is the absolute square of (2.19)

$$\Gamma_{\beta\alpha}=\Gamma_{\beta\alpha}{}^{0}\exp\left\{\mathrm{Re}\int_{\lambda}^{\Lambda}d^4q\,B(q)\right\}. \tag{2.22}$$

The real part of the integral comes only from the $+i\pi\delta(q^2)$ term in the graviton propagator, so

$$\mathrm{Re}\int_{\lambda}^{\Lambda}d^4q\,B(q)=\frac{-8\pi G}{2(2\pi)^3}\int_{\lambda}^{\Lambda}d^4q\,\delta(q^2)\times\sum_{nm}\frac{\eta_n\eta_m\{(p_n\cdot p_m)^2-\frac{1}{2}m_n{}^2m_m{}^2\}}{[p_n\cdot q][p_m\cdot q]}=-B\ln(\Lambda/\lambda), \tag{2.23}$$

where B is the positive dimensionless constant

$$B\equiv\int d^2\Omega\,B(\hat{q}) \tag{2.24}$$

$$B(\hat{q})\equiv\frac{8\pi G}{2(2\pi)^3}\sum_{nm}\frac{\eta_n\eta_m\{(p_n\cdot p_m)^2-\frac{1}{2}m_n{}^2m_m{}^2\}}{[E_n-\mathbf{p}_n\cdot\hat{q}][E_m-\mathbf{p}_m\cdot\hat{q}]}. \tag{2.25}$$

The solid-angle integration (2.24) yields

$$B=\frac{G}{2\pi}\sum_{nm}\eta_n\eta_m m_n m_m\frac{1+\beta_{nm}{}^2}{\beta_{nm}(1-\beta_{nm}{}^2)^{1/2}}\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right) \tag{2.26}$$

with β_{nm} the relative velocity (2.17). Using (2.23) in (2.22), we find the cutoff dependence of the rate is

$$\Gamma_{\beta\alpha}=\Gamma_{\beta\alpha}{}^{0}(\lambda/\Lambda)^B. \tag{2.27}$$

Since $B>0$ this shows that *all* processes have zero rate in the limit $\lambda\to 0$, just as all charged-particle processes have zero rate for $\lambda\to 0$ in electrodynamics. The paradox is resolved in both cases by taking into account the infrared divergences attributable to emission of real soft photons and gravitons.

4. Real Infrared Divergences

The S-matrix element for emitting N real soft photons in a process $\alpha\to\beta$ is given by multiplying the nonradiative S matrix by N factors of form (2.3), and then contracting each of these factors with the appropriate "wave function"

$$(2\pi)^{-3/2}(2|\mathbf{q}|)^{-1/2}\epsilon_\mu{}^*(\mathbf{q},h), \tag{2.28}$$

where $\mathbf{q}$ is the photon momentum, $h=\pm1$ is its helicity, and ϵ_μ is the corresponding polarization vector.[1] We therefore find for the radiative transition amplitude

$$S_{\beta\alpha}{}^{ph}(12\cdots N)=S_{\beta\alpha}\prod_{r=1}^{N}(2\pi)^{-3/2}(2|\mathbf{q}_r|)^{-1/2}\times\sum_n\frac{\eta_n e_n[p_n\cdot\epsilon^*(\mathbf{q}_r,h_r)]}{[p_n\cdot q_r]}. \tag{2.29}$$

The S-matrix element for emitting N real soft gravitons in a process $\alpha\to\beta$ is similarly obtained by multiplying the S matrix for $\alpha\to\beta$ by N factors of form (2.7) and then contracting each of these factors with the appropriate graviton "wave function"[1]

$$(2\pi)^{-3/2}(2|\mathbf{q}|)^{-1/2}\epsilon_\mu{}^*(\mathbf{q},\pm1)\epsilon_\nu{}^*(\mathbf{q},\pm1), \tag{2.30}$$

where $\mathbf{q}$ is the graviton momentum, $h=\pm2$ is its helicity, and ϵ_μ is the same as in (2.28). We therefore find the graviton emission-matrix element

$$S_{\beta\alpha}{}^{gr}(12\cdots N)=S_{\beta\alpha}\prod_{r=1}^{N}(2\pi)^{-3/2}(2|\mathbf{q}_r|)^{-1/2}(8\pi G)^{1/2}\times\sum_n\frac{\eta_n[p_n\cdot\epsilon^*(\mathbf{q}_r,\frac{1}{2}h_r)]^2}{[p_n\cdot q_r]}. \tag{2.31}$$

The rate for emitting N soft photons or gravitons with momenta near $\mathbf{q}_1\cdots\mathbf{q}_N$ is given by squaring (2.29) or (2.31), summing over helicities, and dividing by $N!$ because photons and gravitons are bosons. This gives

$$\Gamma_{\beta\alpha}{}^{ph}(\mathbf{q}_1\cdots\mathbf{q}_N)d^3q_1\cdots d^3q_N=(1/N!)\Gamma_{\beta\alpha}\prod_{r=1}^{N}\mathcal{A}(\mathbf{q}_r)d^3q_r, \tag{2.32}$$

$$\Gamma_{\beta\alpha}{}^{gr}(\mathbf{q}_1\cdots\mathbf{q}_N)d^3q_1\cdots d^3q_N=(1/N!)\Gamma_{\beta\alpha}\prod_{r=1}^{N}\mathcal{B}(\mathbf{q}_r)d^3q_r, \tag{2.33}$$

where $\Gamma_{\beta\alpha}=|S_{\beta\alpha}|^2$, and

$$\mathcal{A}(\mathbf{q})=(2\pi)^{-3}(2|\mathbf{q}|)^{-1}\sum_{nm}\frac{\eta_n\eta_m e_n e_m p_n{}^\mu p_m{}^\nu\Pi_{\mu\nu}(\mathbf{q})}{(p_n\cdot q)(p_m\cdot q)}, \tag{2.34}$$

$$\mathcal{B}(\mathbf{q})=(2\pi)^{-3}(2|\mathbf{q}|)^{-1}(8\pi G)\times\sum_{nm}\frac{\eta_n\eta_m p_n{}^\mu p_n{}^\nu p_m{}^\rho p_m{}^\sigma \Pi_{\mu\nu\rho\sigma}(\mathbf{q})}{(p_n\cdot q)(p_m\cdot q)}. \quad (2.35)$$

Here $\Pi_{\mu\nu}$ and $\Pi_{\mu\nu\rho\sigma}$ are the polarization sums

$$\Pi_{\mu\nu}(\mathbf{q})=\sum_\pm \epsilon_\mu(\mathbf{q},\pm)\epsilon_\nu{}^*(\mathbf{q},\pm), \quad (2.36)$$

$$\Pi_{\mu\nu\rho\sigma}(\mathbf{q})=\sum_\pm \epsilon_\mu(\mathbf{q},\pm)\epsilon_\nu(\mathbf{q},\pm)\epsilon_\rho{}^*(\mathbf{q},\pm)\times\epsilon_\sigma{}^*(\mathbf{q},\pm). \quad (2.37)$$

We recall that[2]

$$\Pi_{\mu\nu}(\mathbf{q})=g_{\mu\nu}+q_\mu\lambda_\nu+q_\nu\lambda_\mu, \quad \lambda^\mu\equiv\{-\mathbf{q},|\mathbf{q}|\}/2|\mathbf{q}|^2. \quad (2.38)$$

The $q\lambda$ terms do not contribute to $\mathcal{A}(\mathbf{q})$ because charge is conserved:

$$q_\mu\sum_n \eta_n e_n p_n{}^\mu/(p_n\cdot q)=\sum_n \eta_n e_n=0,$$

so (2.34) becomes

$$\mathcal{A}(\mathbf{q})=(2\pi)^{-3}(2|\mathbf{q}|)^{-1}\sum_{nm}\frac{\eta_n\eta_m e_n e_m(p_n\cdot p_m)}{(p_n\cdot q)(p_m\cdot q)}$$

or

$$\mathcal{A}(\mathbf{q})=A(\hat{q})/|\mathbf{q}|^3, \quad (2.39)$$

where $A(\hat{q})$ is given by (2.15). We also recall that[2]

$$\Pi_{\mu\nu\rho\sigma}(\mathbf{q})=\tfrac{1}{2}\{\Pi_{\mu\rho}(\mathbf{q})\Pi_{\nu\sigma}(\mathbf{q})+\Pi_{\mu\sigma}(\mathbf{q})\Pi_{\nu\rho}(\mathbf{q})-\Pi_{\mu\nu}(\mathbf{q})\Pi_{\rho\sigma}(\mathbf{q})\}. \quad (2.40)$$

But again the λq terms in Π do not contribute, this time because energy and momentum are conserved:

$$q_\mu\sum_n \eta_n p_n{}^\mu p_n{}^\nu/(p_n\cdot q)=\sum_n \eta_n p_n{}^\nu=0,$$

so $\Pi_{\mu\nu}$ in (2.40) is effectively just $g_{\mu\nu}$, and (2.35) becomes

$$\mathcal{B}(\mathbf{q})=(2\pi)^{-3}(2|\mathbf{q}|)^{-1}(8\pi G)\times\sum_{nm}\frac{\eta_n\eta_m\{(p_n\cdot p_m)^2-\frac{1}{2}m_n{}^2m_m{}^2\}}{(p_n\cdot q)(p_m\cdot q)}$$

or

$$\mathcal{B}(\mathbf{q})=B(\hat{q})/|\mathbf{q}|^3, \quad (2.41)$$

where $B(\hat{q})$ is given by (2.25).

The rates for emission of N photons or gravitons with energies near $\omega_1\cdots\omega_N$ are given by integrating (2.32) and (2.33) over solid angles. Using (2.39), (2.41), (2.14), and (2.24), we find

$$\Gamma_{\beta\alpha}{}^{ph}(\omega_1\cdots\omega_N)d\omega_1\cdots d\omega_N=\frac{A^N}{N!}\Gamma_{\beta\alpha}\frac{d\omega_1}{\omega_1}\cdots\frac{d\omega_N}{\omega_N}, \quad (2.42)$$

$$\Gamma_{\beta\alpha}{}^{gr}(\omega_1\cdots\omega_N)d\omega_1\cdots d\omega_N=\frac{B^N}{N!}\Gamma_{\beta\alpha}\frac{d\omega_1}{\omega_1}\cdots\frac{d\omega_N}{\omega_N}. \quad (2.43)$$

These formulas show that the integrated photon or graviton emission rate will contain logarithmic infrared divergences. In order to display these divergences quantitatively, it is convenient to calculate the rate for the transition $\alpha\to\beta$ accompanied by any number of soft photons with total energy less than E, and with each individual ω_r greater than the infrared cutoff λ. We use the well-known representation of the step function to write this rate as

$$\Gamma_{\beta\alpha}(\leqq E)=\frac{1}{\pi}\sum_{N=0}^{\infty}\int_\lambda^E d\omega_1\cdots\int_\lambda^E d\omega_N\int_{-\infty}^{\infty}d\sigma\frac{\sin E\sigma}{\sigma}\times\exp\{i\sigma\sum_r\omega_r\}\Gamma_{\beta\alpha}(\omega_1\cdots\omega_N). \quad (2.44)$$

Applying this to (2.42) and (2.43), the photon and graviton emission rates are

$$\Gamma_{\beta\alpha}{}^{ph}(\leqq E)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin E\sigma}{\sigma}\exp\left\{A\int_\lambda^E\frac{d\omega}{\omega}e^{i\omega\sigma}\right\}d\sigma, \quad (2.45)$$

$$\Gamma_{\beta\alpha}{}^{gr}(\leqq E)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin E\sigma}{\sigma}\exp\left\{B\int_\lambda^E\frac{d\omega}{\omega}e^{i\omega\sigma}\right\}d\sigma. \quad (2.46)$$

For $\lambda\to0$ the ω integrals become

$$\int_\lambda^E\frac{d\omega}{\omega}e^{i\omega\sigma}\to\ln(E/\lambda)+\int_0^E\frac{d\omega}{\omega}(e^{i\omega\sigma}-1)+O(\lambda). \quad (2.47)$$

Hence (2.45) and (2.46) give for $\lambda\to0$

$$\Gamma_{\beta\alpha}{}^{ph}(\leqq E)=(E/\lambda)^A b(A)\Gamma_{\beta\alpha}, \quad (2.48)$$

$$\Gamma_{\beta\alpha}{}^{gr}(\leqq E)=(E/\lambda)^B b(B)\Gamma_{\beta\alpha}, \quad (2.49)$$

where $b(x)$ is the real function[5]

$$b(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\sigma\frac{\sin\sigma}{\sigma}\exp\left\{x\int_0^1\frac{d\omega}{\omega}(e^{i\omega\sigma}-1)\right\}$$
$$b(x)\simeq1-\tfrac{1}{12}\pi^2x^2+\cdots. \quad (2.50)$$

Since A and B are positive, the factors $(E/\lambda)^A$ and $(E/\lambda)^B$ become infinite for $\lambda\to0$.

5. Cancellation of Divergences

It now only remains to insert the formulas (2.18) and (2.27) which display the virtual infrared divergences into (2.48) and (2.49). As promised, all dependence on the infrared cutoff λ disappears, leaving us with

$$\Gamma_{\beta\alpha}{}^{ph}(\leqq E)=(E/\Lambda)^A b(A)\Gamma_{\beta\alpha}{}^0, \quad (2.51)$$

$$\Gamma_{\beta\alpha}{}^{gr}(\leqq E)=(E/\Lambda)^B b(B)\Gamma_{\beta\alpha}{}^0. \quad (2.52)$$

We repeat that A is given by (2.16), B by (2.26), $b(x)$ by (2.50), and $\Gamma_{\beta\alpha}{}^0$ is the rate for the process $\alpha\to\beta$ without soft photon and graviton emission and without including virtual infrared photons or gravitons. The quantity Λ is an ultraviolet cutoff that has been used to define what we mean by "infrared," but (2.51) and (2.52) show that a change $\Lambda\to\Lambda'$ just renormalizes

$\Gamma_{\beta\alpha}{}^0$ by a factor $(\Lambda'/\Lambda)^A$ in electrodynamics or $(\Lambda'/\Lambda)^B$ for gravitation theory. Thus it makes no difference how we fix Λ, except that if we wish to estimate $\Gamma_{\beta\alpha}{}^0$ by ignoring *all* radiative corrections it will usually be a good strategy to fix Λ equal to some mass typical of the particles in the reaction $\alpha \to \beta$.

For reasons beyond my ken, Jauch and Rohrlich[5] did not fix Λ, but instead took $\Lambda = E$. They therefore missed the energy-dependent factor $(E/\Lambda)^A$ in (2.51). The factors E^A in (2.51) and E^B in (2.52) correctly represent the shape of the energy spectrum for E ranging from zero (where Γ vanishes) up to some maximum smaller (though not necessarily much smaller) than any energy characterizing the process $\alpha \to \beta$.

6. Remark

It was crucial in the above that the infrared divergences arise only from diagrams in which the soft real or virtual photon or graviton is attached to an external line, with "external line" *not* including the soft real photons or gravitons themselves. In electrodynamics this is true because photons are electrically neutral. In gravitation theory it is justified because the effective coupling constant for emission of a very soft graviton from a graviton (or photon) line with energy E is proportional to E, and the vanishing of this factor prevents simultaneous infrared divergences from a graviton and the line to which it is attached.

But these remarks do not apply to theories involving charged massless particles. In such theories (including the Yang-Mills theory) a soft photon emitted from an external line can itself emit a pair of soft charged massless particles, which themselves emit soft photons, and so on, building up a cascade of soft massless particles each of which contributes an infrared divergence. The elimination of such complicated interlocking infrared divergences would certainly be a Herculean task, and might even not be possible.

We may be thankful that the zero charge of soft photons and the zero gravitational mass of soft gravitons saves the real world from this mess. Perhaps it would not be too much to suggest that it is the infrared divergences that prohibit the existence of Yang-Mills quanta or other charged massless particles. See Sec. III for further remarks in this direction.

7. Another Remark

To lowest order in G, Eq. (2.52) gives the power spectrum of soft gravitons accompanying a reaction $\alpha \to \beta$ as

$$E d\Gamma_{\beta\alpha}(\leqq E) = B\Gamma_{\beta\alpha}{}^0 dE. \qquad (2.53)$$

This formula could also have been derived in classical weak-field gravitational radiation theory. [Note that in cgs units (2.26) will give a dimensionless B only if we divide the right-hand side by $\hbar c$. But $dE = \hbar d\omega$, so $\hbar$ does not appear in (2.53) if written as a formula for power per unit frequency interval.]

There is no infrared divergence in (2.53), but this is because it gives the energy rather than the number of gravitons emitted per second with energy between E and $E+dE$. The power spectrum formula (2.53) is both classical and quantum-mechanical, but the infrared divergences are purely quantum mechanical, because it is only in quantum mechanics that we count individual gravitons as well as their total energy.

III. PHOTON AND GRAVITON EMISSION FROM MASSLESS-PARTICLE LINES

Equations (2.16) and (2.26) seem to indicate that the soft photon and graviton emission rates become logarithmically divergent when the mass of one of the *noninfrared* particles in the reaction is allowed to vanish. This divergence occurs because the denominator factors $(p\cdot q)$ in (2.15) and (2.25) will vanish for $\mathbf{q}$ parallel to $\mathbf{p}$ if p^2 is zero. However, we will show that there is a cancellation of these divergences for gravitons, though *not* for photons.[3]

Suppose we let the mass m_1 of particle one vanish, but hold its momentum $\mathbf{p}_1$ fixed. Then Eq. (2.17) becomes

$$\beta_{1n} = 1 - m_1^2 m_n^2/2(p_1\cdot p_n)^2 + \mathcal{O}(m_1^4), \quad (n\neq 1) \qquad (3.1)$$

where $p_1 = \{\mathbf{p}_1, |\mathbf{p}_1|\}$. For $m_1 \to 0$ Eq. (2.16) gives the infrared-photon spectral index as

$$A = -\frac{e_1^2}{4\pi^2} - \frac{\eta_1 e_1}{4\pi^2}\sum_{n\neq 1} \eta_n e_n \ln\left(\frac{4(p_1\cdot p_n)^2}{m_1^2 m_n^2}\right) - \frac{1}{8\pi^2}\sum_{n,m\neq 1} \eta_n\eta_m e_n e_m \beta_{nm}{}^{-1} \ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right). \qquad (3.2)$$

Using charge conservation, we may write the divergent term as

$$+\frac{\eta_1 e_1}{2\pi^2}\ln m_1 \sum_{n\neq 1} \eta_n e_n = -\frac{e_1^2}{2\pi^2}\ln m_1. \qquad (3.3)$$

Hence quantum electrodynamics would be in serious trouble if any charged particle had zero mass.[3] The Yang-Mills theory is for these purposes just the electrodynamics of charged massless vector mesons, so it also shares this trouble. (We have already remarked that a complete treatment of infrared divergences in massless electrodynamics would be enormously more difficult than for ordinary electrodynamics or gravitation, but we are now only considering processes with infrared photons and no infrared charged particles, and for these our present formalism is adequate.)

There are no charged massless particles, so this divergence in A is of only academic interest. But any massless particles can contribute to the infrared-graviton spectral index B. When $m_1 \to 0$ Eq. (2.26) gives

$$B=-\frac{\eta_1 G}{\pi}\sum_{n\neq 1}\eta_n(p_n\cdot p_1)\ln\left(\frac{4(p_1\cdot p_n)^2}{m_1{}^2m_n{}^2}\right)$$
$$-\frac{G}{2\pi}\sum_{n,m\neq 1}\eta_n\eta_m(p_n\cdot p_m)(1+\beta_{nm}{}^2)\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right). \quad (3.4)$$

Using energy and momentum conservation, we may write the divergent term as

$$+\frac{2\eta_1 G}{\pi}(\ln m_1)\sum_{n\neq 1}\eta_n(p_n\cdot p_1)=-\frac{2G}{\pi}(\ln m_1)p_1{}^2=0. \quad (3.5)$$

Since this vanishes, m_1 in Eq. (3.4) may be replaced with any convenient mass; for instance the first logarithm in (3.4) could be written

$$\ln[4(p_1\cdot p_n)^2/m_n{}^4].$$

I leave it to the reader to show that the same cancellations occur when several particles have vanishing mass.

IV. GRAVITATIONAL RADIATION IN NON-RELATIVISTIC COLLISIONS

The rate of emission of energy in soft gravitational radiation during collisions is

$$P(\leqq\Lambda)=\int_0^\Lambda E\,d\Gamma(\leqq E). \quad (4.1)$$

Here "soft" means that the emitted energy E is less than some cutoff Λ chosen smaller than the energies characteristic of the collision process. The rate $\Gamma(\leqq E)$ for a collision with radiated energy $\leqq E$ was calculated in Sec. II as

$$\Gamma(\leqq E)=(E/\Lambda)^B b(B)\Gamma_0, \quad (4.2)$$

with B given by (2.26), $b(B)$ by (2.50), and Γ_0 equal to the collision rate without real or virtual infrared gravitons. Hence the power (4.1) is

$$P(\leqq\Lambda)=(B/(1+B))b(B)\Lambda\Gamma_0. \quad (4.3)$$

Since B is always very tiny ($\leqq 10^{-38}$) both $1+B$ and $b(B)$ are extremely close to one, and we may write

$$P(\leqq\Lambda)=B\Lambda\Gamma_0. \quad (4.4)$$

Also, to a very good approximation Γ_0 may be calculated as the collision rate ignoring gravitation altogether.

For our present purposes it will be convenient to write Eq. (2.26) for B as

$$B=(G/\pi)\sum_{nm}\eta_n m_n\eta_m m_m f(\beta_{nm}), \quad (4.5)$$

where

$$f(\beta)\equiv\frac{1+\beta^2}{2\beta(1-\beta^2)^{1/2}}\ln\left[\frac{1+\beta}{1-\beta}\right]^{1/2} \quad (4.6)$$

and $\eta_n=+1$ or -1 if n is a final or initial particle; m_n is the mass of particle n, and β_{nm} is the relative velocity of particles n and m:

$$\beta_{nm}\equiv[1-m_n{}^2m_m{}^2/(p_n\cdot p_m)^2]^{1/2}. \quad (4.7)$$

If all particles involved in the collision are nonrelativistic then β_{nm} may be expanded in powers of $\mathbf{v}_n$ and $\mathbf{v}_m$, with $\mathbf{v}\equiv\mathbf{p}/E$. We find

$$\beta_{nm}{}^2=\mathbf{v}_n{}^2+\mathbf{v}_m{}^2-2\mathbf{v}_n\cdot\mathbf{v}_m-\mathbf{v}_n{}^2\mathbf{v}_m{}^2-3(\mathbf{v}_n\cdot\mathbf{v}_m)^2$$
$$+2(\mathbf{v}_n{}^2+\mathbf{v}_m{}^2)(\mathbf{v}_n\cdot\mathbf{v}_m)+\cdots. \quad (4.8)$$

Also, $f(\beta)$ may be expanded in powers of β^2:

$$f(\beta)=1+(11/6)\beta^2+(63/40)\beta^4+\cdots. \quad (4.9)$$

Using (4.8) in (4.9) gives

$$f(\beta_{nm})=1+(11/6)(\mathbf{v}_n{}^2+\mathbf{v}_m{}^2-2\mathbf{v}_n\cdot\mathbf{v}_m)$$
$$+(63/40)(\mathbf{v}_n{}^2+\mathbf{v}_m{}^2)^2-(79/30)(\mathbf{v}_n{}^2+\mathbf{v}_m{}^2)$$
$$\times(\mathbf{v}_n\cdot\mathbf{v}_m)+(4/5)(\mathbf{v}_n\cdot\mathbf{v}_m)^2+\cdots. \quad (4.10)$$

We are keeping terms up to order v^4 in (4.8)–(4.10), since the lower order terms contribute only to order v^4 in B because of the energy and momentum conservation equations:

$$\sum_n\eta_n m_n(1+\tfrac{1}{2}\mathbf{v}_n{}^2+\tfrac{3}{8}\mathbf{v}_n{}^4+\cdots)=0, \quad (4.11)$$

$$\sum_n\eta_n m_n\mathbf{v}_n(1+\tfrac{1}{2}\mathbf{v}_n{}^2+\cdots)=0. \quad (4.12)$$

Using (4.10), (4.11), and (4.12) in (4.5), we find to lowest nonvanishing order in v

$$B=(G/\pi)[(16/5)Q_{ij}Q_{ij}+(94/15)(Q_{ii})^2], \quad (4.13)$$

where

$$Q_{ij}=\tfrac{1}{2}\sum_n\eta_n m_n v_{ni}v_{nj} \quad (4.14)$$

and repeated Latin indices are summed over 1, 2, 3. Since (4.13) is only correct to order v^4, the velocities in Eq. (4.14) must be subjected to the nonrelativistic conservation laws

$$\sum_n\eta_n m_n(1+\tfrac{1}{2}\mathbf{v}_n{}^2)=0, \quad (4.15)$$

$$\sum_n\eta_n m_n\mathbf{v}_n=0. \quad (4.16)$$

Hence Q_{ii} is just the usual Q value

$$Q_{ii}=-\sum_n\eta_n m_n. \quad (4.17)$$

Also, (4.16) makes Q_{ij} invariant under the Galilean transformation $\mathbf{v}_n\to\mathbf{v}_n+\mathbf{u}$, so B may be computed in any convenient reference frame.

As an example, consider nonrelativistic elastic two-body scattering. We find here

$$Q_{ij}Q_{ij}=\tfrac{1}{2}\mu^2v^4\sin^2\theta_c,$$
$$Q_{ii}=0,$$

where μ is the reduced mass, $v=|\mathbf{v}_1-\mathbf{v}_2|$ is the relative velocity, and θ_c is the scattering angle in the center-of-mass system. Thus Eq. (4.13) gives

$$B=(8G/5\pi)\mu^2v^4\sin^2\theta_c. \quad (4.18)$$

The rate for such collisions per cm^3 per sec is $vn_1n_2(d\sigma/d\Omega)$, where n_1 and n_2 are the number densities of particles 1 and 2. Hence (4.4) and (4.8) give the total power emitted in soft gravitational radiation attributable to 1–2 collisions as

$$P(\leq\Lambda)=\frac{8G}{5\pi}\mu^2v^5n_1n_2V\Lambda\int\left(\frac{d\sigma}{d\Omega}\right)\sin^2\theta_c d\Omega\,, \quad (4.19)$$

with V the volume of the source. For practical purposes we may generally define "soft" radiation by taking the cutoff Λ at half the relative kinetic energy

$$\Lambda\approx\tfrac{1}{4}\mu v^2. \quad (4.20)$$

Everything in the universe is transparent to gravitons, so (4.19) may be used directly to compute the thermal gravitational radiation from any hot body.

We will use these results to estimate the thermal gravitational radiation from the sun. By far the most frequent collisions are the Coulomb collisions between electrons and protons or electrons. In this case we may take

$$\begin{aligned}\mu&=m_e\,, \quad v=(3KT/m_e)^{1/2}\\ n_1&=n_e\,, \quad n_2=n_e+n_p=2n_e.\end{aligned} \quad (4.21)$$

Also, the integral in Eq (4 19) is nothing but the familiar diffusion coefficient, and as is well known[9] can be estimated as

$$\int\left(\frac{d\sigma}{d\Omega}\right)\sin^2\theta_c d\Omega=\frac{8\pi e^2}{(3KT)^2}\ln\Lambda_D\,, \quad (4.22)$$

where e is now unrationalized, and Λ_D is the ratio of the Debye shielding radius (used to cutoff the integral) to the average impact parameter. Putting together (4.19)–(4.22), we find

$$P_\odot=(32/5)G(3KT)^{3/2}m_e^{-1/2}n_e^2V_\odot e^4(\hbar c^5)^{-1}\ln\Lambda_D. \quad (4.23)$$

We did our calculation in natural units with $\hbar=c=1$, but we have supplied a factor $(\hbar c^5)^{-1}$ in Eq. (4.23) to convert it to cgs units. In the sun's core the parameters in (4.23) take the values[10]

$$\begin{aligned}T&\simeq10^7\ {}^\circ\text{K}\,,\\ n_e&\simeq3\times10^{25}\ \text{cm}^{-3}\,,\\ V_\odot&\simeq2\times10^{31}\ \text{cm}^3\,,\\ \ln\Lambda_D&\simeq4.\end{aligned}$$

The solar gravitational radiation power is then

$$P_\odot\simeq6\times10^{14}\ \text{erg/sec}. \quad (4.24)$$

Although this is not much more power than used by the city of Berkeley, it nevertheless compares favorably with the gravitational radiation from previously considered classical sources like planetary motion. A planet of mass m moving in a circular orbit of radius R around a star of mass M emits gravitational radiation with power

$$P=(32/5)Gc^{-5}m^2R^4(GM/R^3)^3. \quad (4.25)$$

For the Jupiter-Sun system this is 7.6×10^{11} erg/sec. Venus and the Earth radiate comparable amounts, and the other planets considerably less, so the thermal gravitational radiation (4.14) from the Sun appears to be the dominant source of gravitational radiation from the solar system. A binary star like Sirius A and B radiates more classically—in this case Eq. (4.25) gives 8×10^{14} erg/sec—but it also radiates more thermally. Thus thermal collisions possibly may provide the most important source of gravitational radiation in the universe. I have no idea what it is good for.

V. PHASE DIVERGENCES

It is well known that the Born series for the Coulomb scattering amplitude gives divergent integrals beyond the first order. Dalitz[11] has studied the scattering by a screened potential

$$V(r)=(e_1e_2/4\pi r)e^{-\lambda r} \quad (5.1)$$

and conjectured that the $\ln\lambda$ term found in second Born approximation might represent the beginning of a series whose sum is a phase factor

$$\exp\left\{\frac{ie_1e_2}{2\pi\beta_{12}}\ln\lambda\right\}\,, \quad (5.2)$$

which would correctly represent the cutoff dependence for $\lambda\to0$, and would not affect the cross section. This is very reasonable, but to my knowledge it has not been proved. I will show here that this conjecture is correct, and has an almost trivial extension to any process involving any number of relativistic or nonrelativistic particles of arbitrary spin.

The divergences in the nonrelativistic Born series for Coulomb scattering are obviously not the same as the ordinary infrared divergences, which depend on retardation effects, i.e., on the $i\pi\delta(q^2)$ term in the photon propagator. However, Eq. (2.11) shows that the full effect of virtual infrared photons is to contribute to the S matrix for any process $\alpha\to\beta$ a factor

$$\frac{S_{\beta\alpha}}{S_{\beta\alpha}{}^0}=\exp\left\{\frac{1}{2}\int_\lambda^\Lambda d^4q\,A(q)\right\}. \quad (5.3)$$

The real part of $A(q)$ gives the familiar infrared-divergence factor $(\lambda/\Lambda)^{A/2}$ which is eventually cancelled by real soft-photon emission processes. But $A(q)$ also has an imaginary part, and we shall find that it is this

[9] See, e.g., L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), Chap. 5.

[10] These parameters apply to the inner $\frac{1}{64}$ of the sun's volume. The first two are from M. Schwarzschild, *Structure and Evolution of the Stars* (Princeton University Press, Princeton, New Jersey, 1958), p. 259. The last is from Table 5.1 of Ref. 9.

[11] R. H. Dalitz, Proc. Roy. Soc. (London) **206**, 509 (1951).

that accounts for the lnλ terms in the nonrelativistic Born series.

Referring back to (2.10), we see that (5.3) may be written as

$$\frac{S_{\beta\alpha}}{S_{\beta\alpha}{}^0}=\exp\left\{\frac{1}{2(2\pi)^3}\sum_{nm} e_n e_m \eta_n \eta_m (p_n\cdot p_m) J_{nm}\right\}, \quad (5.4)$$

$$J_{nm}\equiv i\int^{\Lambda} \frac{d^4q}{[q^2+\lambda^2-i\epsilon][p_n\cdot q-i\eta_n\epsilon][p_m\cdot q+i\eta_m\epsilon]}. \quad (5.5)$$

We are still using an ultraviolet cutoff $|\mathbf{q}|\leqq\Lambda$ to separate the infrared from the noninfrared virtual photons, but in order to facilitate the comparison with (5.2) we are now using a photon mass λ in place of the infrared cutoff $\lambda\leqq|\mathbf{q}|$.

The integrand of (5.5) is analytic in q^0 except at the four poles

$$q^0=\omega-i\epsilon, \qquad q^0=-\omega+i\epsilon,$$
$$q^0=\mathbf{v}_n\cdot\mathbf{q}-i\eta_n\epsilon, \quad q^0=\mathbf{v}_m\cdot\mathbf{q}+i\eta_m\epsilon,$$

where $\omega\equiv(\mathbf{q}^2+\lambda^2)^{1/2}$ and $\mathbf{v}\equiv\mathbf{p}/E$. Also, we may close the q^0 contour with a large semicircle in either the upper or the lower half-planes. If particle n is outgoing and m is incoming then $\eta_n=+1$, $\eta_m=-1$, so by closing the contour in the upper half-plane we avoid the contributions from the poles $q^0=\mathbf{v}_n\cdot\mathbf{q}-i\eta_n\epsilon$ or $q^0=\mathbf{v}_m\cdot\mathbf{q}+i\eta_m\epsilon$. Similarly, if n is incoming and m is outgoing we can avoid these two poles by closing the contour in the lower half-plane. In these two cases it is only the poles at $q^0=\pm(\omega-i\epsilon)$ that contribute, and we find J_{nm} purely real:

$$J_{nm}=-\pi\int^{\Lambda}\frac{d^3q}{\omega(\omega E_n-\mathbf{q}\cdot\mathbf{p}_n)(\omega E_m-\mathbf{q}\cdot\mathbf{p}_m)}$$
$$(\text{for}\quad \eta_n=-\eta_m=\pm1). \quad (5.6)$$

On the other hand, if particles n and m are both outgoing or both incoming, then the poles at $\mathbf{v}_n\cdot\mathbf{q}-i\eta_n\epsilon$ and $\mathbf{v}_m\cdot\mathbf{q}+i\eta_m\epsilon$ lie on opposite sides of the real q^0 axis, and we cannot avoid a contribution from one of them whichever way we close the q^0 contour. We now have, after some elementary integrations,

$$J_{nm}=-\pi\int^{\Lambda}\frac{d^3q}{\omega(\omega E_n-\mathbf{q}\cdot\mathbf{p}_n)(\omega E_m-\mathbf{q}\cdot\mathbf{p}_m)}+\frac{2i\pi^3}{[(p_n\cdot p_m)^2-m_n{}^2m_m{}^2]^{1/2}}\ln\left(\frac{\Lambda^2}{\lambda^2}+1\right)$$
$$(\text{for}\quad \eta_n=\eta_m=\pm1). \quad (5.7)$$

In both (5.6) and (5.7) we find a real term which, if we now reconverted from a photon-mass cutoff to an energy cutoff, would contribute to (5.4) the real factor $(\lambda/A)^{1/2}$ [see (2.14) and (2.15)]. This λ dependence is then cancelled by real-photon emission, and does not interest us in this section. But (5.7) shows that (5.4) will also contain a divergent phase factor

$$\frac{S_{\beta\alpha}}{S_{\beta\alpha}{}^0}=\prod_{nm}{}' \exp\left\{\frac{-i}{16\pi}\frac{e_ne_m}{\beta_{nm}}\ln\left(\frac{\Lambda^2}{\lambda^2}+1\right)\right\}, \quad (5.8)$$

where β_{nm} is the relative velocity

$$\beta_{nm}=[(p_n\cdot p_m)^2-m_n{}^2m_m{}^2]^{1/2}/(-p_n\cdot p_m), \quad (5.9)$$

and the prime reminds us that the product runs over particle pairs with $\eta_n=\eta_m$, i.e., both in the initial or both in the final state. In (5.8) the pairs nm and mn are counted separately, so *each different pair of particles in the initial or final state contributes to the S matrix a phase factor which for* $\lambda\ll\Lambda$ *may be written*

$$\exp\left\{\frac{i}{4\pi}\frac{e_ne_m}{\beta_{nm}}\ln(\lambda/\Lambda)\right\}. \quad (5.10)$$

In two-particle elastic Coulomb scattering this factor occurs in both the initial and final states, and therefore accounts for Dalitz's conjectured phase factor (5.2). *The phase factor (5.10) is correct even if particles n and m are relativistic and/or have spin.*

It hardly needs to be said that a similar result holds for gravitation. Each different particle pair in the initial or final state contributes to the S matrix a divergent phase factor

$$\exp\left\{-i\frac{Gm_nm_m(1+\beta_{nm}{}^2)}{\beta_{nm}[1-\beta_{nm}{}^2]^{1/2}}\ln\frac{\lambda}{\Lambda}\right\}. \quad (5.11)$$

These results might have some practical application to the calculation of scattering by potentials with Coulomb tails. Such potentials may be cut off as in (5.2), and we will then find lnλ terms in each order beyond the first. But however complicated the potential is at small distances, these lnλ terms will always sum to phase factors (5.10), and can therefore be removed in a systematic way.

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