further discussion of the notation]

$$\langle \mathfrak{F}^*(1)\mathfrak{F}(2)\rangle_t = f^*(\hat{y}_1)f(\hat{y}_2)$$

$$\times\int d^3\hat{x}\,d^3r\langle n(\mathbf{x}+\tfrac{1}{2}\mathbf{r},\,t_1)n(\mathbf{x}-\tfrac{1}{2}\mathbf{r},\,t_2)\rangle$$

$$\times\exp\{ip[\tfrac{1}{2}(\hat{y}_1+\hat{y}_2)-\hat{R}]\cdot\mathbf{r}\}\,, \quad (7.13)$$

where $\mathbf{R}$ is a vector from the fixed reference point in the target to a fixed point in the source and $\langle n(\mathbf{x}',t_1)n(\mathbf{x},t_2)\rangle$ is the Van Hove correlation function for the target. The quantity $Q_T(12)$ in Eq. (7.12) is

$$Q_T(12)=\int\frac{d^3Z}{\mathcal{V}_T}\exp[ip(\hat{y}_1-\hat{y}_2)\cdot\mathbf{Z}]\,, \quad (7.14)$$

integrated over the volume of the target.

The quantity (7.9) now becomes

$$\langle\Delta G_{12}\rangle=\langle\Delta G_{12}\rangle_p I_S\,, \quad (7.15)$$

where I_S is defined by Eqs. (2.31), but with $Q(12)$ replaced by $Q_T(12)$ and thus Σ_S replaced by the target area Σ_T in (2.31b), and

$$\langle\Delta G_{12}\rangle_p=\pm\frac{\langle G_1\rangle_0\langle G_2\rangle_0}{\bar{\sigma}_1\bar{\sigma}_2}\int_1\frac{d^3y_1}{\Sigma_1 w_1}\int_2\frac{d^3y_2}{\Sigma_2 w_2}\int d\omega d\omega'$$

$$\times g(\omega)g(\omega')B_1(\omega'-\omega)B_2(\omega-\omega')\exp\{i[(q-q')(y_2-y_1)$$

$$-(\omega-\omega')(T_2-T_1)]\}\,|\langle\mathfrak{F}^*(1)\mathfrak{F}(2)\rangle_t|^2. \quad (7.16)$$

Fluctuations in

$$\langle G_{\rm av}\rangle=T[\langle G_1\rangle\langle G_2\rangle+\langle\Delta G_{12}\rangle] \quad (7.16)$$

may be evaluated from the general analysis of Sec. IV. When the target is large enough that $I_T\ll 1$, we may use Eq. (5.1) to evaluate these. The detector efficiency may be taken into account by including in $\langle G_1\rangle_0$ and $\langle G_2\rangle_0$ an efficiency factor (or factors) η, as in Eq. (2.20b).

PHYSICAL REVIEW VOLUME 140, NUMBER 2B 25 OCTOBER 1965

Position Operators in Relativistic Single-Particle Theories*

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(Received 21 May 1965)

This paper gives formulas for position operators that apply uniformly for all spins of particle. The three-vector $\mathbf{X}$, of the Newton-Wigner and Foldy type, is treated first. Then it is shown that, although $\mathbf{X}$ has complicated Lorentz transformation properties, it is linearly related to a certain four-vector Y_μ which is built up from the Poincaré group generators. The four-vector is the generalization of the classical notion of the component of the position four-vector in the direction perpendicular to the world line of the particle.

I. INTRODUCTION

RECENTLY a formulation of the theory of a free particle with mass and arbitrary spin was given[1] in which there is such a complete parallel with Dirac's theory for an electron-positron that all the known discussions for a spin-$\frac{1}{2}$ particle can be extended to particles with higher spins. The purpose of the present paper is to make this extension for the study of position and to develop formulas for three-vector and four-vector position operators that apply uniformly for all spins.

The special features of the description of free particles developed in Ref. 1 are that there are no auxiliary conditions on the wave function and that the wave-function components are spinors, so that the value of the wave function at a point in space-time in one Lorentz frame determines the value in all Lorentz frames. It is closely related to Weinberg's[2] formulation and Foldy's.[3] In fact there is an operator, which is a generalization of the notion of the rest-to-lab Lorentz transformation, which carries Foldy's wave function into the wave function of Ref. 1. Consequently properties of operators in Foldy's theory can be similarity transformed into the present formulation.

For many of the observable quantities, such as momentum, energy, and angular momentum, the corresponding operators are simply the inhomogeneous Lorentz group generators. The situation is not so straightforward since position and other considerations have to be made. Desirable properties for a position operator $\mathbf{X}$ are that (i) it should be Hermitian with respect to the appropriate Lorentz-invariant inner product for each spin; (ii) it should fulfil the commutation rules

$$[X_i,X_j]=0\,, \quad (1)$$

$$[X_i,p_j]=i\delta_{ij}\,, \quad (2)$$

* This research was done in the Ames Laboratory of the U. S. Atomic Energy Commission.

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[2] S. Weinberg, Phys. Rev. **133**, B1318 (1964).

[3] L. L. Foldy, Phys. Rev. **102**, 568 (1956).

where $\mathbf{p}$ is $-i\nabla$; (iii) it should be charge-conjugation invariant; (iv) the time derivative should be the relativistic velocity, the ratio of the physical momentum to the energy; (v) it should be a polar vector with respect to rotations and space reflections and should be invariant under time reflections; (vi) the operator should be defined separately on the particle and antiparticle states; and (vii) it should be well defined in the rest system of the particle.

Newton and Wigner[4] have shown that, for given finite mass and spin of the particle, such a position operator exists. Physically speaking there is just one operator of this type since, as shown by Bargmann and Wigner,[5] there is just one theory of such a particle, the complete set of states in one formulation being related to the complete set in any other formulation by a unitary transformation. Nevertheless it is of interest to find explicit formulas for the operators and this is the problem considered in this paper. The operators given below are to some extent more general than those of Newton and Wigner since they apply to a system which includes both particle and antiparticle.

The operator satisfying all the requirements is given in the next section. It is produced by making the similarity transformation of Foldy's position operator, the coordinate $\mathbf{x}$ in his representation. For spin $\frac{1}{2}$ the operator is the Foldy-Wouthuysen[6] mean position.

In Sec. III a four-vector operator Y_μ is defined in such a way as to permit separating the generators of the homogeneous Lorentz group into an orbital part and a spin part that are each conserved tensors. The operator is closely related to Finkelstein's[7] position operator. It does not have many of the desirable properties of the three-vector operator. In Sec. IV it is shown that Y_μ is to be interpreted as the component of the position four-vector perpendicular to the world line of the particle.

As well as the authors mentioned above, Pryce[8] did much of the original work on position operators. Recently Berg[9] has discussed the three-vector for position in terms of Poincaré group generators. Jordan and Mukunda[10] have exhibited three-vector position operators for particles of arbitrary spin which satisfy many of the requirements listed above. They construct the operators by making a certain unitary transformation of the Foldy position operator. Their operators are different from those defined in this paper since they are designed to act on a different wave function. Fleming[11] recently has defined four-vector position operators starting from different considerations from those used here.

II. THREE-VECTOR POSITION OPERATOR

The notation used below is the same as in Ref. 1. Especially the $2(2s+1)$ spinor components of the wave function ψ are related to the Foldy wave function ϕ by

$$\psi(\mathbf{x},t)=m^s E^{-\frac{1}{2}} S\phi(\mathbf{x},t)\,, \tag{3}$$

where E is the positive square root of (m^2+p^2) and the operator S is defined and discussed in Ref. 1. There is a corresponding connection between an operator O_ψ that acts on the functions ψ and an operator O_ϕ that acts on the functions ϕ,

$$O_\psi = E^{-\frac{1}{2}} S O_\phi S^{-1} E^{1/2}. \tag{4}$$

The equation of motion is

$$H_\phi \phi = i\phi/\partial t\,, \tag{5}$$

where

$$H_\phi = \beta E. \tag{6}$$

The Hamiltonian for ψ is then found from Eq. (4) to be

$$H = S\beta E S^{-1}; \tag{7}$$

formulas for it are given in Ref. 1 for spins up to $\frac{3}{2}$. The inner product is defined by

$$(\psi^{(l)},\psi^{(n)}) = \int d\mathbf{x}\, \phi^{\dagger(l)}(x)\phi^{(n)}(x)\,, \tag{8}$$

and it is known to be Lorentz invariant. Operators O_ϕ that are Hermitian in the usual sense are related by Eq. (4) to operators O_ψ that are Hermitian in the sense that

$$(O_\psi\psi^{(l)},\psi^{(n)}) = (\psi^{(l)},O_\psi\psi^{(n)}). \tag{9}$$

The three-vector operator for position is defined by

$$\mathbf{X} = E^{-\frac{1}{2}} S \mathbf{x} S^{-1} E^{1/2}, \tag{10}$$

since then the desirable properties listed in the Introduction are satisfied. Since $\mathbf{x}$ is Hermitian in the usual sense, $\mathbf{X}$ is Hermitian in the sense of Eq. (9). Also for $\mathbf{x}$ it is true that

$$[x_i,x_j]=0\,, \tag{11}$$

$$[x_i,p_j]=i\delta_{ij}, \tag{12}$$

$$i[\beta E,\mathbf{x}] = (\mathbf{p}/E)\beta\,, \tag{13}$$

$$[\beta,\mathbf{x}]=0. \tag{14}$$

Equations (11) and (12) yield the commutation rules [Eqs. (1) and (2)] correctly. Equation (13) corresponds to

$$i[H,\mathbf{X}] = (\mathbf{p}/E)(H/E). \tag{15}$$

Since $\mathbf{p}(H/E)$ is to be identified as the physical momentum and E as the physical energy, this equation

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[6] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
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implies that the velocity $d\mathbf{X}/dt$ is the momentum divided by the energy. Equation (14) implies that

$$[H/E,\mathbf{X}]=0, \tag{16}$$

so that the operator is defined separately on the positive and negative frequency states. Without this property it would be impossible to assign the position of a state at the same time as its particle-antiparticle character.

A general formula for $\mathbf{X}$, applying for all spins, can be derived from some previously established properties of the Lorentz group operators. For infinitesimal pure Lorentz transformations the generators are known to be[1]

$$\mathbf{G}_\psi=\mathbf{x}H-t\mathbf{p}-is\boldsymbol{\alpha}, \tag{17}$$

$$\mathbf{G}_\phi=\mathbf{x}E\beta-\tfrac{1}{2}i(\mathbf{p}/E)\beta-t\mathbf{p}+(m+E)^{-1}\mathbf{p}\times\mathbf{s}\beta. \tag{18}$$

These results may be substituted into Eq. (4) and Eq. (10) used to introduce $\mathbf{X}$ on the right. Also the quantity $S\beta\mathbf{s}S^{-1}$ that occurs on the right is the polarization operator $\mathbf{O}$ for which the formula

$$\mathbf{O}=m^{-1}(is\boldsymbol{\alpha}\times\mathbf{p}+\mathbf{s}H)-[mE(E+m)]^{-1}\mathbf{s}\cdot\mathbf{p}H\mathbf{p} \tag{19}$$

applies for all spins.[12] One can therefore solve for $\mathbf{X}$, obtaining

$$\mathbf{X}=\mathbf{x}+[m(E+m)]^{-1}\mathbf{s}\times\mathbf{p}-(is\boldsymbol{\alpha}H/mE)+i[mE^2(E+m)]^{-1}s\boldsymbol{\alpha}\cdot\mathbf{p}H\mathbf{p}+(i\mathbf{p}/2E^2). \tag{20}$$

This is closely related to a result of Berg's.[9] From this form one sees that the operator is well-defined in the rest frame in the sense that it has a definite limit at $\mathbf{p}=0$. It is clear that $\mathbf{X}$ is a three-space vector and, since $\boldsymbol{\alpha}$ is polar and $\mathbf{s}$ axial, that $\mathbf{X}$ is regular under space reflection. The invariance under time reflections and the charge-conjugation invariance follow from the facts that

$$\gamma_5\beta\mathbf{X}\beta\gamma_5=\mathbf{X}, \tag{21}$$

$$C^{-1}\mathbf{X}^*C=\mathbf{X}, \tag{22}$$

which are easily verified using the detailed properties of the matrices given in Ref. 1. It is necessary for a physical operator O to be charge-conjugation-invariant in the sense that

$$C^{-1}O^*C=O \tag{23}$$

in order that the identification of physically observable quantities with operators be independent of whether the particle or antiparticle is preferred. This does hold for the energy E, momentum $(H/E)\mathbf{p}$, and angular momentum $(H/E)(\mathbf{x}\times\mathbf{p}+\mathbf{s})$.

In terms of the position $\mathbf{X}$ of Eq. (20) and the polarization $\mathbf{O}$ of Eq. (19), the total angular momentum can be written as

$$(H/E)(\mathbf{x}\times\mathbf{p}+\mathbf{s})=\mathbf{X}\times\mathbf{p}(H/E)+\mathbf{O}. \tag{24}$$

The two terms on the right separately commute with H.

For spin zero Eq. (20) specializes to

$$\mathbf{X}=\mathbf{x}+(i\mathbf{p}/2E^2). \tag{25}$$

This operator is defined for all states, both positive and negative frequency. It is closely related to the Newton-Wigner operator, which they defined for positive-frequency states alone. Their operator is defined by

$$\mathbf{q}_{\mathrm{NW}}=i\nabla_{\mathbf{p}}-(i\mathbf{p}/2E^2), \tag{26}$$

operating on positive-frequency momentum-space functions $\Phi(\mathbf{p})$. The functions are related by

$$\phi_{\mathrm{KG}}=\frac{1}{(2\pi)^{3/2}}\int\frac{d\mathbf{p}}{E}\Phi(\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)}, \tag{27}$$

to coordinate-space wave functions ϕ_{KG} that are Lorentz scalars satisfying the Klein-Gordon equation. Applying $\mathbf{q}_{\mathrm{NW}}$ to the momentum space function $\Phi(\mathbf{p})$ is equivalent to applying the coordinate-space operator

$$\mathbf{X}_{\mathrm{NW}}=\mathbf{x}+(i\mathbf{p}/2E^2) \tag{28}$$

to ϕ_{KG}. The operator $\mathbf{X}$ acts on the two-component functions[1]

$$\psi=\frac{1}{\sqrt{2}}\begin{pmatrix}\phi_{\mathrm{KG}}\\ iE^{-1}\partial\phi_{\mathrm{KG}}/\partial t\end{pmatrix}. \tag{29}$$

If the discussion is restricted to the positive-frequency states, so that $iE^{-1}\partial/\partial t$ is unity, then $\mathbf{X}$ and $\mathbf{X}_{\mathrm{NW}}$ are identical.

For spin $\frac{1}{2}$, Eq. (20) leads to

$$\mathbf{X}=\mathbf{x}+(i\beta\boldsymbol{\alpha}/2E)-[2E^2(E+m)]^{-1}[i\beta\boldsymbol{\alpha}\cdot\mathbf{p}\mathbf{p}+\boldsymbol{\sigma}\times\mathbf{p}E]. \tag{30}$$

This operator was first studied by Pryce[8] and it is the Foldy-Wouthuysen[6] mean position operator. Also the positive-frequency projection of this operator coincides with the Newton-Wigner spin-$\frac{1}{2}$ position operator when their operator is converted into a coordinate-space operator by replacing $(i\nabla_{\mathbf{p}}-i\mathbf{p}/E^2)$ by $\mathbf{x}$.

For higher spins the explicit formulas become more complicated, for example for spin 1

$$\mathbf{X}=\mathbf{x}+(i\mathbf{p}/2E^2)+(i\beta\boldsymbol{\alpha}/m)-[mE(E+m)(2E^2-m^2)]^{-1}\times[(2E^2+2Em+m^2)i\beta\boldsymbol{\alpha}\cdot\mathbf{p}\mathbf{p}-2iE(\boldsymbol{\alpha}\cdot\mathbf{p})^2\mathbf{p}]+[m(E+m)]^{-1}\mathbf{s}\times\mathbf{p}-[m(2E^2-m^2)]^{-1}\times2[iE\boldsymbol{\alpha}\boldsymbol{\alpha}\cdot\mathbf{p}+\beta(\boldsymbol{\alpha}\times\mathbf{p})\mathbf{s}\cdot\mathbf{p}]. \tag{31}$$

Since $\mathbf{X}$, (H/E), and O_z all commute one can find functions that are simultaneously eigenstates of all of them. To get explicit formulas for such states, one can start from the states

$$\phi=v_{\epsilon,k}\delta(\mathbf{x}-\mathbf{x}_0),$$

which are eigenstates of β, βs_z, and $\mathbf{x}$ with eigenvalues ϵ, k, and $\mathbf{x}_0$. Then the functions

$$\psi=m^sE^{-s}Sv_{\epsilon,k}\delta(\mathbf{x}-\mathbf{x}_0) \tag{32}$$

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[12] A. Sankaranarayanan and R. H. Good, Jr., Nuovo Cimento 36, 1303 (1965).

are eigenfunctions of H/E, O_z, and $\mathbf{X}$ with the same eigenvalues. Here S for any spin can be found by the procedure given in Ref. 1.

III. FOUR-VECTOR OPERATOR

The operator $\mathbf{X}$ has complicated Lorentz transformation properties. However it is linearly related to the four-vector operator Y_μ which, as shown in Sec. IV, is the generalization of the classical notion of the component of the position four-vector normal to the world line of the particle.

In Eq. (24) $\mathbf{X}$ was used to decompose the total angular-momentum operator into orbital and spin parts that are separately conserved. The operator Y_μ is found when a similar decomposition of the Lorentz angular-momentum tensor is made, separating it into an orbital part and a spin part that are separately conserved.

The generators of the Poincaré group for the wave function ψ are

$$p_\mu = -i\partial/\partial x_\mu, \tag{33}$$

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + N_{\mu\nu}, \tag{34}$$

where $N_{\mu\nu}$ is defined by

$$\begin{aligned} N_{ij} &= \epsilon_{ijk} s_k, \\ N_{i4} &= -N_{4i} = s\alpha_i, \\ N_{44} &= 0. \end{aligned} \tag{35}$$

The matrices $N_{\mu\nu}$ can be expressed in terms of the covariantly defined matrices of the type introduced by Barut, Muzinich, and Williams.[13] For spin zero $N_{\mu\nu}$ is zero; for spin $\frac{1}{2}$ it is

$$N_{\mu\nu} = -\tfrac{1}{4}i(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{36}$$

For spins 1 and $\frac{3}{2}$ the matrices are

$$N_{\mu\nu} = -\tfrac{1}{6}i(\gamma_{\mu\rho}\gamma_{\rho\nu} - \gamma_{\nu\rho}\gamma_{\rho\mu}), \tag{37}$$

$$N_{\mu\nu} = -\tfrac{3}{32}i(\gamma_{\mu\rho\sigma}\gamma_{\sigma\rho\nu} - \gamma_{\nu\rho\sigma}\gamma_{\sigma\rho\mu}), \tag{38}$$

where the $\gamma_{\mu\nu}$ are defined in Ref. 12 and the $\gamma_{\mu\nu\rho}$ are defined by Shay, Song, and Good.[14]

The commutation rules for the generators are

$$[p_\mu, p_\nu] = 0, \tag{39}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\delta_{\mu\rho}M_{\nu\sigma} + \delta_{\mu\sigma}M_{\rho\nu} - \delta_{\nu\rho}M_{\mu\sigma} - \delta_{\nu\sigma}M_{\rho\mu}), \tag{40}$$

$$[M_{\mu\nu}, p_\rho] = i(\delta_{\mu\rho}p_\nu - \delta_{\nu\rho}p_\mu). \tag{41}$$

Here $p_\mu p_\mu$ commutes with everything and may be replaced by $-m^2$.

The decomposition of the angular-momentum tensor is

$$M_{\mu\nu} = Y_\mu p_\nu - Y_\nu p_\mu + R_{\mu\nu}, \tag{42}$$

where the four-vector operator is defined by

$$Y_\mu = -(1/2m^2)[M_{\mu\rho}, p_\rho]_+, \tag{43}$$

and $R_{\mu\nu}$ is a polarization tensor given in terms of the generators by

$$R_{\mu\nu} = -m^{-2}(M_{\mu\nu}p_\rho + M_{\rho\mu}p_\nu + M_{\nu\rho}p_\mu)p_\rho. \tag{44}$$

Here Y_μ satisfies

$$Y_\mu p_\mu + p_\mu Y_\mu = 0, \tag{45}$$

and it is related to a type of position operator first discussed by Finkelstein[7] (it lacks his proper time term). A similar decomposition of the angular-momentum tensor was discussed by Bacry,[15] although the components of his position operator commute with each other. Equation (42) does not specify Y_μ uniquely and would still hold if a term proportional to p_μ were included in the definition of Y_μ. Shirokov's[16] operator $-(1/m^2)M_{\mu\rho}p_\rho$ differs from Y_μ by such a term. However such an additional term would give the Y_μ undesirable Hermitian or time-reversal properties, as discussed further below. The polarization tensor $R_{\mu\nu}$ is antisymmetric and $R_{\mu\nu}p_\nu$ is zero. The tensor in brackets on the right in Eq. (44) was discussed by Bargmann and Wigner.[5] Its components are the generators of the little group, and they lead to the Pauli-Lubánski[17] invariant. They are independent of x_μ and in fact, if Eq. (34) is used, then Eq. (44) simplifies to

$$R_{\mu\nu} = -m^{-2}(N_{\mu\nu}p_\rho + N_{\rho\mu}p_\nu + N_{\nu\rho}p_\mu)p_\rho. \tag{46}$$

Some properties of the polarization tensor for arbitrary spin of the particle were derived recently by Sankaranarayanan and Good[12] and by Hilgevoord and deKerf.[18]

The commutation rules for Y_μ are found to be

$$[Y_\mu, Y_\nu] = im^{-2}M_{\mu\nu}, \tag{47}$$

$$[Y_\mu, p_\nu] = i(\delta_{\mu\nu} + m^{-2}p_\mu p_\nu), \tag{48}$$

$$[M_{\mu\nu}, Y_\rho] = -i\delta_{\nu\rho}Y_\mu + i\delta_{\mu\rho}Y_\nu. \tag{49}$$

From Eqs. (40), (47), and (49) it is seen that $M_{\mu\nu}$ and Y_ρ satisfy the commutation rules for the generators of the deSitter group.[19]

For some purposes it is appropriate to introduce the

[13] A. O. Barut, I. Muzinich, and D. N. Williams, Phys. Rev. **130**, 442 (1963).

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[18] J. Hilgevoord and E. A. deKerf., University of Amsterdam report (unpublished).

[19] See, for example, F. Gürsey, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, edited by F. Gürsey (Gordon and Breach Science Publishers, Inc., New York, 1964), p. 365.

operators $\bar{p}_\mu$ and $\bar{M}_{\mu\nu}$

$$\bar{p}_\mu = (\mathbf{p}, iH), \tag{50}$$

$$\bar{M}_{ij} = M_{ij} = x_i p_j - x_j p_i + \epsilon_{ijk} s_k, \tag{51}$$

$$\bar{M}_{j4} = -\bar{M}_{4j} = iG_{\psi j} = ix_j H - itp_j + s\alpha_j. \tag{52}$$

These are produced when $i\partial/\partial t$, as it occurs in the generators, is moved to the right and replaced by H. Consequently, when acting on solutions ψ with the correct time dependence the barred operators are equivalent to the unbarred. Carrying on this idea, one defines

$$\bar{Y}_\mu = -(1/2m^2)[\bar{M}_{\mu\rho}, \bar{p}_\rho]_+, \tag{53}$$

$$\bar{R}_{\mu\nu} = -m^{-2}(\bar{M}_{\mu\nu}\bar{p}_\rho + \bar{M}_{\rho\mu}\bar{p}_\nu + \bar{M}_{\nu\rho}\bar{p}_\mu)\bar{p}_\rho. \tag{54}$$

The barred quantities satisfy relations similar to those of the unbarred; especially Eqs. (39) to (42) and (45) to (49) apply to the barred operators as well as to the unbarred. The $\bar{R}_{\mu\nu}$ above coincides with $(H/E)R_{\mu\nu}$ of Ref. 12.

The generators $\bar{M}_{\mu\nu}$ and $\bar{p}_\mu$ are integrals of the motion in the sense that

$$d\bar{M}_{\mu\nu}/dt = i[H, \bar{M}_{\mu\nu}] + (\partial/\partial t)\bar{M}_{\mu\nu} = 0, \tag{55}$$
$$d\bar{p}_\mu/dt = 0.$$

Consequently any operator built from $\bar{p}_\mu$ and $\bar{M}_{\mu\nu}$ also is conserved, expecially

$$d\bar{Y}_\mu/dt = 0. \tag{56}$$

This result emphasizes that $\bar{Y}_\mu$ should not itself be interpreted as the operator for the position of a particle.

It is known that the operators $\mathbf{p}$, H, $\mathbf{J}$, $\mathbf{G}_\psi$ are Hermitian in the sense of Eq. (9). Consequently $\bar{Y}_i$ is Hermitian and $\bar{Y}_4$ is anti-Hermitian. The operators $\bar{Y}_\mu$ have the same time-reversal properties as a position four-vector:

$$\bar{Y}_\mu(t) = \beta\gamma_5 C^{-1}\bar{Y}_\mu^*(-t)C\gamma_5\beta.$$

One can see now that, if an extra term ap_μ were included in the definition of Y_μ, [Eq. (43)] the Hermitian property of $\bar{Y}_\mu$ would be destroyed if a were imaginary and the time-reversal property would be destroyed if a were real.

The relation between $\bar{Y}_\mu$ and X_i is

$$\bar{Y}_i = X_i + (2m^2)^{-1}[(X_j p_j - tH)p_i + p_i(p_j X_j - tH)] - [m(E+m)]^{-1}(H/E)\epsilon_{ijk}p_j O_k, \tag{57}$$

$$\bar{Y}_4 = it + (2m^2)^{-1}[(X_j p_j - tH)iH + iH(p_j X_j - tH)]. \tag{58}$$

One can verify this result by expressing $\bar{Y}_\mu$ in terms of $\mathbf{G}_\psi$, $\mathbf{X}$, H, and $\mathbf{O}$, using Eqs. (53), (51), (52), and (24), and then operating from the left with $S^{-1}E^{\frac{1}{2}}$, from the right with $E^{-\frac{1}{2}}S$. The similarity transformation converts the four operators above into $\mathbf{G}_\phi$, $\mathbf{x}$, βE, and $\beta\mathbf{s}$. Actually $\bar{Y}_\mu$ depends on $\mathbf{X}$ and t only by way of the time-independent operator

$$\mathbf{X}^0 = \mathbf{X} - (\mathbf{p}/E)(H/E)t;$$

in detail one finds that

$$\bar{Y}_i = X_i^0 + (2m^2)^{-1}[X_j^0 p_j p_i + p_i p_j X_j^0] - [m(E+m)]^{-1}(H/E)\epsilon_{ijk}p_j O_k,$$
$$\bar{Y}_4 = (2m^2)^{-1}[X_j^0 p_j iH + iHp_j X_j^0].$$

Thus the value of $\bar{Y}_\mu$ depends on the starting value of $\mathbf{X}$.

IV. DISCUSSION

The interpretations of the operators $\mathbf{X}$ and Y_μ are clarified by considering the properties of a free spinless particle in classical (nonquantum) relativistic mechanics.

Let $X_{\mathrm{cl}\mu}$ be the four-vector $(\mathbf{X}_{\mathrm{cl}}, it)$ where $\mathbf{X}_{\mathrm{cl}}(t)$ is the position of the particle as a function of time. The motion satisfies

$$d\mathbf{X}_{\mathrm{cl}}/dt = \mathbf{q}/E, \tag{59}$$

where $\mathbf{q}$ and E are the physical momentum and energy constants of the motion. Here $\mathbf{q}$ and iE are components of a four-vector q_μ with

$$q_\mu q_\mu = -m^2. \tag{60}$$

The vector $Y_{\mathrm{cl}\mu}$ defined by

$$Y_{\mathrm{cl}\mu} = X_{\mathrm{cl}\mu} + m^{-2}(X_{\mathrm{cl}\nu}q_\nu)q_\mu \tag{61}$$

is orthogonal to q_μ,

$$Y_{\mathrm{cl}\mu}q_\mu = 0, \tag{62}$$

and is to be interpreted as the component of $X_{\mathrm{cl}\mu}$ normal to the world line of the particle's motion. Since the world line is straight it is clear geometrically that

$$dY_\mu/dt = 0. \tag{63}$$

In the quantum theory $\mathbf{X}$ has all the properties described in Sec. I and is identified as the operator corresponding to $\mathbf{X}_{\mathrm{cl}}$. The velocity as given in Eq. (15) agrees with Eq. (59) since $\mathbf{p}(H/E)$ is the operator for the physical momentum $\mathbf{q}$. However, unlike $(\mathbf{X}_{\mathrm{cl}}, it)$, $(\mathbf{X}, it)$ is not a Lorentz four-vector. The operator Y_μ, defined by Eq. (43) in terms of the Poincaré generators, corresponds to the classical concept $Y_{\mathrm{cl}\mu}$. They are each four-vectors, they each have orthogonality conditions with q_μ, Eqs. (45) and (62), and they are each constant in time, Eqs. (56) and (63). Also the relation between $\bar{Y}_\mu$ and $\mathbf{X}$, [Eqs. (57) and (58)], coincides with the relation between $Y_{\mathrm{cl}\mu}$ and $\mathbf{X}_{\mathrm{cl}}$, Eq. (61), except for commutators and a contribution from the spin.

ACKNOWLEDGMENT

The authors thank Professor D. L. Pursey for several valuable discussions.