vantage of presenting a big difference between the Čerenkov/scintillation pulse ratio for fast charges and for fast monopoles. For example, if a Lucite Čerenkov counter is used, then the ratio for fast monopoles is 2.25 times that for fast charges. The ratio expected from fast charged monopoles is intermediate between the ratio for fast charges and that for fast monopoles. The range of unsaturated response of the scintillation counter will limit the pole strengths for which this type of experiment can be used. These considerations make this type of experiment particularly useful in searching for small-pole-strength monopoles or charged monopoles.

First-Method Experiment

This type of experiment compares the pulses from two Čerenkov counters with different refractive indices. The difference between the (Čerenkov counter *a*/Čerenkov counter *b*) pulse ratio for fast charges and fast monopoles depends on the permittivity ratio $(\epsilon_a'/\epsilon_b')$. For example, if a lucite/water combination is used then $(\epsilon_a'/\epsilon_b')=1.27$ and fast monopoles yield a (counter *a*/counter *b*) ratio that is 1.27 times that for fast charges. Fast charged monopoles would yield ratios intermediate between those for fast charges and fast monopoles. In the low-ionization region this type of experiment is limited by two considerations. First, a small pole strength yields a small number of Čerenkov-counter photons which results in poor statistics with which to determine the (counter *a*/counter *b*) ratio. Secondly, slow charges can give the same pulse ratio as fast monopoles and, at least for cosmic-ray work, there are a lot of slow charged particles in the low-ionization region. Here "slow" means a particle which is not fast for *both* counters *a* and *b*. As pointed out previously,[1] particles which are slow can be discriminated against by introducing a third Čerenkov counter using a gas.

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Model Hamiltonians for Local Quantum Field Theory*

KENNETH G. WILSON†
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
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We study the problem of renormalization for the interaction of a charged scalar meson field with a fixed two-level point source. Our especial interest is coupling-constant renormalization. We study in particular the problem of obtaining eigenstates and eigenvalues of the Hamiltonian for the fixed-source theory. We propose a model for the fixed-source theory, in which mesons exist only if their momenta (k) lie within an infinite set of intervals: $0<k<k_0$, $\frac{1}{2}\Lambda<k<\Lambda$, $\frac{1}{2}\Lambda^2<k<\Lambda^2$, etc., where k_0 is of the order of the meson mass, and Λ is much larger. We solve this model by treating the mesons in the nth interval (or lower) as a perturbation on mesons in the $(n+1)$st interval (and higher). This reduces the problem to the solution of two strongly cut-off Hamiltonians, one of which must be solved for an infinite sequence of coupling constants $\{g_n\}$, one for each momentum interval. We show that even if the low-momentum coupling constants g_1, g_2, etc., are small, the sequence goes to infinity as $n\to\infty$. We analyze the Lee model similarly; here the sequence is undefined above some finite value of n. We show a close analogy between our analysis and the analysis of quantum electrodynamics of Gell-Mann and Low. Then we analyze the full fixed-source Hamiltonian qualitatively. We expland the meson field in terms of a complete set of "wave-packet" states, the coefficients being discrete oscillator variables. The states are so chosen that the self-interactions of oscillators dominate the coupling between oscillators. For each order of magnitude for the meson momentum, there is one pair of oscillators coupled to the source; this coupling can be analyzed analogously to our model. The full fixed-source Hamiltonian is thereby reduced to the solution of a Hamiltonian for two oscillators coupled to a two-level source.

I. INTRODUCTION

THE Hamiltonian formulation of quantum mechanics has been essentially abandoned in investigations of the interactions of π mesons, nucleons, and strange particles. This is a pity. The Hamiltonian approach has several advantages over the kind of approach (using dispersion relations) presently in use. One advantage is that all properties of a system are uniquely determined, given the Hamiltonian of the system. A second advantage is the existence of many approximation methods for solving a given Hamiltonian. The principal approximation methods are perturbation theory, the variational method, and the W.K.B. approximation. A third advantage is that one can often analyze a Hamiltonian intuitively, using the uncertainty principle and classical arguments, to get order of magnitude estimates of its solution; from such

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an analysis one can often determine what approximation procedure will work best.

There are two reasons why the Hamiltonian approach was discarded in the study of strong interactions. One reason was that no one knew what Hamiltonian to use, or how to obtain the correct Hamiltonian. The other reason was the problem of renormalization: the problem that whenever one tried to solve a Hamiltonian for a Lorentz-invariant theory, particle self-energies and the like were infinite.

The problem of renormalization has several aspects. One is the practical aspect just mentioned, namely, that whenever one has tried to solve a Lorentz-invariant theory with a standard approximation procedure such as perturbation theory or the Tamm-Dancoff method, one gets infinities. Then there is a more fundamental difficulty. This difficulty is that any estimate of the order of magnitude of the interaction energy gives infinity, for a relativistic quantum-field theory. For example, the simplest relativistically invariant interaction Hamiltonian, for a neutral scalar field $\phi(x)$, is

$$H_I = g_0 \int \phi^3(\mathbf{x}) d^3x$$

where g_0 is a constant. We might estimate the order of magnitude of H_I by computing the square root of the matrix element $\langle\Omega|H_I^2|\Omega\rangle$ where $|\Omega\rangle$ is the vacuum state for a free field. This matrix element is infinite. This difficulty is a fundamental one, despite the crude nature of such estimates, because any sensible method for solving Hamiltonians requires that one first make just such crude estimates of the order of magnitude of the terms in the Hamiltonian.

A third aspect of the renormalization problem is that there is only one Hamiltonian known that can be solved but requires renormalization. This is the Hamiltonian of the Lee model.[1] Unfortunately, the Lee model (formulated so that the renormalization is infinite, i.e., without a "cutoff") cannot be solved without introducing a "ghost state"—a state with negative norm—which is undesirable. This example has made many people pessimistic about the possibility of resolving the renormalization problem.

It is not known whether relativistic Hamiltonians can be solved exactly without encountering the same infinities that occur in perturbation theory, and without introducing ghost states such as occur in the Lee model. It is clear that the appearance of infinities in perturbation theory does not imply infinities in the exact theory since one can use perturbation theory only when the interaction energy is small, and this is demonstrably not the case for field-theoretic interactions. There are other approximate methods for solving Hamiltonians besides perturbation theory, but they are equally useless so long as one cannot establish their validity even qualitatively. So to resolve this question one will have to learn to make qualitative analyses of relativistic Hamiltonians, using order-of-magnitude estimates, the uncertainty principle, etc.

We do not have the competence to make such an analysis at the present time. To develop the necessary techniques, it is useful to have model Hamiltonians which embody some but not all the difficulties presented by fully relativistic interactions. One such model Hamiltonian is presented in this paper. Our model is an offspring of the theory of scalar π mesons interacting with a fixed point source. Our model Hamiltonian involves an interaction Hamiltonian which by any order of magnitude estimate is infinite. In perturbation theory it gives infinities analogous to coupling constant and self-energy divergences in relativistic theories. However, we will be able to analyze it in a sensible way: not completely, but well enough to provide considerable insight into the real problems involved in solving Hamiltonians requiring renormalization (in particular, coupling constant renormalization).

Gell-Mann and Low[2] have considered the problem of what a solution of a relativistic theory would be like, if a solution exists. They examined quantum electrodynamics in particular. Their principle conclusion was that the "bare" coupling constant e_0 which appears in the unrenormalized Hamiltonian could not be arbitrary; it could take on only special values, or, in other words, it would have to be the solution of some kind of eigenvalue condition. To quote from their abstract:

> "Thus it is shown that the unrenormalized coupling constant $e_0^2/4\pi\hbar c$, which appears in perturbation theory as a power series in the renormalized coupling constant $e_1^2/4\pi\hbar c$ with divergent coefficients, may behave in either of two ways: (a) It may really be infinite as perturbation theory indicates; (b) It may be a finite number independent of $e_1^2/4\pi\hbar c$."

We shall discuss the ideas of Gell-Mann and Low in more detail in Sec. VI.

Now we give a brief outline of the contents of this paper. In Sec. II we formulate our model Hamiltonian. In Sec. III we replace the original model by a sequence of Hamiltonians H_n and analyze the individual Hamiltonians in the sequence. In Sec. IV, we show how to define our original model as a limit of the sequence considered in Sec. III. In Sec. V we analyze similarly a model derived from the Lee model. In Sec. VI we compare the results of the analysis of our models with Gell-Mann and Low's analysis of quantum electrodynamics. In Sec. VII we discuss a method for qualitatively analyzing field-theoretic Hamiltonians, which involves expanding the fields in a discrete orthonormal set of functions. We state our conclusions in Sec. VIII.

II. FORMULATION OF MODEL

In this Section we formulate our model Hamiltonian. We begin by quoting the complete Hamiltonian for

[1] T. D. Lee, Phys. Rev. **95**, 1329 (1954).

[2] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

charged scalar mesons ($\pi^{\pm}$) interacting with a fixed point source with two quantum levels ("proton" and "neutron"):

$$H=\int_{\mathbf{k}}\omega_{\mathbf{k}}\{a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}+b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\}$$
$$+g_0\int_{\mathbf{k}}(2\omega_{\mathbf{k}})^{-1/2}\{(a_{\mathbf{k}}+b_{\mathbf{k}}^{\dagger})\tau^{+}+(a_{\mathbf{k}}^{\dagger}+b_{\mathbf{k}})\tau^{-}\} \quad (1)$$

where

$\int_{\mathbf{k}}$ means $(2\pi)^{-3}\int d^3k$,
$\omega_{\mathbf{k}}$ means $(\mathbf{k}^2+\mu^2)^{1/2}$,
μ is the meson mass, which we assume is of order **1**,
$a_{\mathbf{k}}^{\dagger}$ creates π^{+},
$b_{\mathbf{k}}^{\dagger}$ creates π^{-},
g_0 is the (unrenormalized) coupling constant,
τ^{+} takes the neutron state into the proton state,
τ^{-} is the Hermitian conjugate of τ^{+}.

The commutation relations of $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$ are

$$[a_{\mathbf{k}},a_{\mathbf{k}'}^{\dagger}]=[b_{\mathbf{k}},b_{\mathbf{k}'}^{\dagger}]=(2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}'). \quad (2)$$

This Hamiltonian when solved in perturbation theory leads to two divergences. The ground-state energy is linearly divergent; the renormalized coupling constant (related to a ground-state matrix element of τ^{+}) is logarithmically divergent. It also has the property that an estimate of the interaction energy gives infinity. For example let us estimate the order of magnitude of $\int_{\mathbf{k}}a_{\mathbf{k}}^{\dagger}(2\omega_{\mathbf{k}})^{-1/2}$; then we compute

$$\langle\Omega|\int_{\mathbf{k}}\int_{\mathbf{k}'}a_{\mathbf{k}}a_{\mathbf{k}'}^{\dagger}(2\omega_{\mathbf{k}})^{-1/2}(2\omega_{\mathbf{k}'})^{-1/2}|\Omega\rangle,$$

where $|\Omega\rangle$ is the meson vacuum state. This expression is easily computed and yields the quadratically divergent integral

$$\int_{\mathbf{k}}(2\omega_{\mathbf{k}})^{-1}.$$

Our model Hamiltonian is constructed as follows: We take the Hamiltonian of Eq. (1) and throw away most of it. Specifically, the integration over all $\mathbf{k}$ is replaced by an integral over an infinite set of intervals. The intervals are as follows (k is the magnitude of $\mathbf{k}$)

$$0<k<k_0,\ \tfrac{1}{2}\Lambda<k<\Lambda,\ \cdots,\ \tfrac{1}{2}\Lambda^n<k<\Lambda^n,\ \cdots,$$

where k_0 is an arbitrary number of the order of the meson mass μ and Λ is an arbitrary number much larger than either k_0 or μ. This is the only modification we make; the integrands in Eq. (1) are unchanged. What we have done is to chop out most of momentum space leaving an infinite set of well-separated intervals. We shall do this for both integrals in H, so that in our model, mesons with momenta outside our intervals do not exist.

We must ask immediately what are the advantages and disadvantages of our model compared with the original fixed source Hamiltonian. First, we want to make clear that our model Hamiltonian is *not* supposed to be an approximation to the complete fixed source Hamiltonian: We consider it as a completely new Hamiltonian. As to possible disadvantages, the important question is whether our model Hamiltonian has the same renormalization problems as the original fixed source theory. It is easy to see that it does. To prove this we look at the nature of the divergences for the original fixed source theory. In perturbation theory these divergences take the form of integrals over momentum. A linearly divergent integral will be of the form

$$\int_{\mathbf{k}}k^{-2}$$

and a logarithmically divergent integral will be of the form

$$\int_{\mathbf{k}}k^{-3}.$$

In each case the integrand behaves as k^{-2} or k^{-3} only if k is much larger than the meson mass μ. In our model the integrals over all k are replaced by integrals over our sequence of intervals. For example, the logarithmic divergence becomes

$$\sum_{n=1}^{\infty}\int_{\mathbf{k}}k^{-3}$$

with the integrals being over $\frac{1}{2}\Lambda^n<k<\Lambda^n$. These integrals are independent of n, and therefore the infinite sum diverges. So our model Hamiltonian has the divergence difficulties that we wish to investigate.

The advantage of our model Hamiltonian is that we can analyze it. This is because the mesons in different momentum intervals differ in their energies by orders of magnitude. For example a meson of momentum Λ^n has an energy Λ^n, while a meson of momentum Λ^{n-1} has energy Λ^{n-1}. This suggests that we can treat the mesons of momentum Λ^{n-1} or less as a perturbation with respect to mesons of momentum of order Λ^n. This idea is the basis of our analysis.

III. ANALYSIS OF MODEL HAMILTONIAN

In this section we shall present an analysis of our model Hamiltonian (defined in Sec. II). We use the method just proposed: mesons of momenta Λ^{n-1} or less are regarded as a perturbation relative to mesons of momentum of order Λ^n. Evidently, we cannot carry out this program unless we cut off our Hamiltonian at some finite value of n. Therefore, instead of working with the model Hamiltonian defined in Sec. II, we now define a sequence of Hamiltonians. We first define "H_{lab}" to be the Hamiltonian of Eq. (1) with the integral being only

over the interval $0<k<k_0$. We define H_1 to include integration over the first two intervals: $0<k<k_0$ and $\frac{1}{2}\Lambda<k<\Lambda$. The remaining Hamiltonians H_n are defined analogously: for H_n we integrate over all our intervals through $\frac{1}{2}\Lambda^n<k<\Lambda^n$. We shall analyze each of the Hamiltonians H_n, and then try to recover our original model Hamiltonian as a limit of the H_n as n becomes infinite.

We first study H_{lab}. It is equivalent to the complete fixed source Hamiltonian with a strong cutoff.[3] There are no problems of renormalization in H_{lab}. One can solve it when g_0 is small by perturbation theory. One can solve it when g_0 is very large by Wentzel's strong coupling approximation.[4] If g_0 is of order 1, one may use variational methods (for example, the Tomonaga approximation[5]). One can also estimate both the free-meson energy and the interaction energy by our crude methods and discover that both are of order μ.[6] So H_{lab} is a Hamiltonian which we understand and can solve to reasonable accuracy.

Secondly, we study the Hamiltonian H_1. The idea here is to divide H_1 into an unperturbed Hamiltonian H_0 and a perturbation H_I. The unperturbed Hamiltonian H_0 is simply the Hamiltonian H of Eq. (1) with the integrals restricted to the single interval $\frac{1}{2}\Lambda<k<\Lambda$. The perturbation is the same as H_{lab}. Note that our method of dividing H_1 into an unperturbed part and a perturbation has nothing to do with the size of the coupling constant g_0; both the free-meson energy and the interaction energy get split up in our procedure. Both terms contribute to the unperturbed Hamiltonian H_0, and both contribute to the perturbation H_I.

To analyze H_0 it is convenient to introduce a scale transformation in order to eliminate (or more correctly, to determine) its dependence on Λ. First we make a change of variable in the integral, letting

$$\mathbf{k}=\Lambda\mathbf{p} \tag{3}$$

so that the range of $\mathbf{p}$ is $\frac{1}{2}<p<1$. Next we introduce new creation and destruction operators depending on $\mathbf{p}$ instead of $\mathbf{k}$:

$$a_{\mathbf{k}}^{\dagger}=\Lambda^{-3/2}A_{\mathbf{p}}^{\dagger},\quad a_{\mathbf{k}}=\Lambda^{-3/2}A_{\mathbf{p}}, \tag{4}$$

$$b_{\mathbf{k}}^{\dagger}=\Lambda^{-3/2}B_{\mathbf{p}}^{\dagger},\quad b_{\mathbf{k}}=\Lambda^{-3/2}B_{\mathbf{p}}. \tag{5}$$

The purpose of the factors $\Lambda^{-3/2}$ is to ensure that the commutation relations of the $A_{\mathbf{p}}$ etc. will not involve Λ. It follows from Eq. (2) that

$$[A_{\mathbf{p}},A_{\mathbf{p}'}^{\dagger}]=[B_{\mathbf{p}},B_{\mathbf{p}'}^{\dagger}]$$
$$=\Lambda^3(2\pi)^3\delta^3(\Lambda\mathbf{p}-\Lambda\mathbf{p}')=(2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}'). \tag{6}$$

Finally, we have

$$\omega_{\mathbf{k}}\approx k=\Lambda p. \tag{7}$$

Now one obtains

$$H_0=\Lambda H_s \tag{8}$$

with

$$H_s=\int_{\mathbf{p}} p\{A_{\mathbf{p}}^{\dagger}A_{\mathbf{p}}+B_{\mathbf{p}}^{\dagger}B\}+g_0\int_{\mathbf{p}}(2p)^{-1/2}$$
$$\times\{(A_{\mathbf{p}}+B_{\mathbf{p}}^{\dagger})\tau^{+}+(A_{\mathbf{p}}^{\dagger}+B_{\mathbf{p}})\tau^{-}\} \tag{9}$$

with $\frac{1}{2}<p<1$ in the integrals. The Hamiltonian H_s is very similar to H_{lab}; it also has no renormalization problems and can be solved at least roughly for any value of g_0. It is worth pointing out the importance of the finite lower limit $\frac{1}{2}$ on the momentum p in H_s. If this limit were replaced by 0, the charge renormalization integral would be

$$\int_0^1 p^{-3}d^3p$$

which diverges at the lower limit. There is no mass μ in H_s to provide a low-momentum cutoff for this integral.

The Hamiltonian H_s has two symmetries. First of all, it conserves charge, if we think of the proton source level as having charge 1 and the neutron level having charge zero. Secondly, it is invariant to the exchange $\pi^{+}\leftrightarrow\pi^{-}$, $p\leftrightarrow n$. There is no state of definite charge which is invariant to this transformation, for such a state must have charge $\frac{1}{2}$. Therefore the ground state of H_s must be at least doubly degenerate. We shall assume that the ground states have the same quantum numbers as the source levels. This is known to be true for very weak or very strong coupling. We can call these ground states $|P\rangle$ and $|N\rangle$; the source levels with no mesons present will be referred to as $|p\rangle$ and $|n\rangle$.

In addition to the ground states, the Hamiltonian H_s has excited states, which are either discrete levels usually called "isobars," or continuum levels which are meson-scattering states. We shall not need to know anything about these states except the energy spacing between the ground state and the first excited state. This spacing is of order 1 unless g_0 is very large, in which case it is of order g_0^{-2}.[4] The latter case will be ignored for the present.

We have now completed our discussion of H_0 and can consider the effect of the perturbation H_{lab}. Since the first excited state of H_0 is of order Λ above the ground state, we can consider H_{lab} as a perturbation provided it is much smaller than Λ. But according to our previous analysis of H_{lab} by itself, it is of order μ and therefore indeed much smaller than Λ. However H_0 has a degenerate ground state, so we must use degenerate perturbation theory. In fact, the ground state of H_0 is

[3] For an introduction to cutoff fixed source theories, see: E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill Book Company, Inc., New York, 1962).

[4] G. Wentzel, Helv, Phys. Acta **13**, 269 (1940); **14**, 633 (1941). For further references, see Ref. 3.

[5] S. Tomonaga, Progr. Theoret. Phys. (Kyoto) **2**, 6 (1947). See also Ref. 3.

[6] This estimate is wrong, if g_0 is large. If g_0 is large, the lowest eigenstates of H_{lab} involve a large number of mesons. Qualitatively, the interaction energy varies as the square root of the number (n) of mesons, while the free-meson energy is linear in n. The minimum total energy occurs for $n\sim g_0^2$; the minimum energy is of order μg_0^2.

highly degenerate, for to the ground states $|P\rangle$ and $|N\rangle$ of H_s we can add any number of "laboratory" mesons (mesons of momenta $<k_0$) without changing the eigenvalue of H_0.

Let us now examine the total Hamiltonian H_0+H_{lab} in the lowest order of degenerate perturbation theory. We shall consider only those states which reduce to the ground state of H_s in the absence of the perturbation. In lowest order, the energies and eigenstates are determined by an effective Hamiltonian "H_{eff}" obtained by restricting the full Hamiltonian to the subspace of states spanned by $|P\rangle$ and $|N\rangle$ plus arbitrary numbers of laboratory mesons. The laboratory meson creation and destruction operators are unaffected by this restriction. The unperturbed Hamiltonian H_0 becomes simply a constant ΛE_0 where E_0 is the ground-state energy of H_s. The τ matrices become operators which act only on the two levels $|P\rangle$ and $|N\rangle$, since they do not affect the laboratory mesons. In particular τ^+ has only one nonzero matrix element

$$\alpha=\langle P|\tau^+|N\rangle. \tag{10}$$

The other three matrix elements are 0 by charge conservation. We can choose the relative phases of $|P\rangle$ and $|N\rangle$ so that α is positive. We can restate this result by introducing a raising operator τ_R^+ for the states $|P\rangle$ and $|N\rangle$:

$$|P\rangle=\tau_R^+|N\rangle. \tag{11}$$

Then, in the restricted Hamiltonian H_{eff}, τ^+ becomes $\alpha\tau_R^+$, likewise τ^- becomes $\alpha\tau_R^-$. Note that if we had *not* restricted ourselves to the two ground states of H_s, but allowed excited states as well, then τ^+ would have been a much more complicated matrix connecting $|P\rangle$ and $|N\rangle$ to all the excited states of H_0 of appropriate charge.

The effective Hamiltonian now reads

$$H_{\text{eff}}=\Lambda E_0+\int_{\mathbf{k}}\omega_{\mathbf{k}}\{a_{\mathbf{k}}^\dagger a_{\mathbf{k}}+b_{\mathbf{k}}^\dagger b_{\mathbf{k}}\}+g_0\alpha\int_{\mathbf{k}}(2\omega_{\mathbf{k}})^{-1/2}\times\{(a_{\mathbf{k}}+b_{\mathbf{k}}^\dagger)\tau^+ +(a_{\mathbf{k}}^\dagger+b_{\mathbf{k}})\tau^-\} \tag{12}$$

with the integral over k restricted to $0<k<k_0$.

Evidently, H_{eff} is the same as the original H_{lab} except for an additive constant E_0 and, more importantly, a change in coupling constant from g_0 to $g_0\alpha$. This is hardly a surprise. What we have found is that when the mesons of momentum of order Λ ("Λ mesons") are added to H_{lab} to form H_1, the effect on the lowest energy levels (energies small compared with Λ) is simply what one would predict from renormalization theory: the ground-state energy is shifted and the coupling constant is modified.

We can now carry over our previous discussion of H_{lab} to show that the energy levels of H_{eff} differ from the ground-state energy of H_0 by an amount of order μ, as expected, and the perturbation analysis is justified. We must except from this remark the eigenstates of H_{eff} which involve a large number of free laboratory mesons and therefore a large energy; I presume that we can restrict our attention to states with not too many mesons.

It is now a trivial matter to analyze the Hamiltonian H_n. This Hamiltonian involves mesons in all intervals up to the interval $\frac{1}{2}\Lambda^n<k<\Lambda^n$. Now the mesons with momenta of order Λ^n, interacting with the source, form the unperturbed system, and the mesons of momenta Λ^{n-1} or less are treated as a perturbation. The unperturbed Hamiltonian can again be analyzed by means of a scale transformation, which reduces it to the Hamiltonian H_s apart from a factor Λ^n. Thus, the full Hamiltonian H_n becomes

$$H_n=\Lambda^n H_s+H_{n-1}. \tag{13}$$

Let us indicate explicitly that these Hamiltonians depend on a coupling constant g_0:

$$H_n(g_0)=\Lambda^n H_s(g_0)+H_{n-1}(g_0). \tag{14}$$

Now we use degenerate perturbation theory just as we did for H_1. Here we restrict H_n to the subspace of states spanned by $|P\rangle$ and $|N\rangle$ plus arbitrary numbers of mesons of momenta Λ^{n-1} or less. The result is an effective Hamiltonian of the form

$$H_{\text{eff}}=\Lambda^n E_0(g_0)+H_{n-1}[g_0\alpha(g_0)]. \tag{15}$$

The constant α is as before the matrix element $\langle P|\tau^+|N\rangle$; α depends on g_0 since the ground states $|P\rangle$ and $|N\rangle$ depend on g_0. The effective Hamiltonian describes only the ground state of H_n and those excited states with energy of order Λ^{n-1} or less above the ground state.

What we now have is a recursion formula which we can state as follows: The lowest energy levels of the Hamiltonian $H_n(g_0)$ are equivalent to the energy levels of $\Lambda^n E_0(g_0)+H_{n-1}[f(g_0)]$, where

$$f(g_0)=\alpha(g_0)g_0. \tag{16}$$

By repeated use of this recursion formula, we see that we will generate a sequence of coupling constants:

$$\begin{aligned} g_n&=g_0,\\ g_{n-1}&=f(g_n),\\ g_{n-2}&=f(g_{n-1}),\\ &\vdots\\ g&=f(g_1). \end{aligned} \tag{17}$$

The ground state and lowest excited states of H_n are described by $H_{\text{lab}}(g)$ apart from a constant; H_{lab} describes those excited states with energies of order μ above the ground state. To get excited states with energies of order Λ one must solve $H_1(g_1)$. To get excited states of energy of order Λ^2 one must solve $H_2(g_2)$, and so forth.

We have now completed our analysis of the cutoff Hamiltonians H_n. We know the order of magnitudes of

the components of H_n: mesons of momenta of order $\Lambda^m (m \leq n)$ contribute a free energy and an interaction energy of order Λ^m. This fact allows us to solve H_n by perturbation techniques, which reduce the problem of solving H_n to the problem of solving H_s and H_{lab}, both of which can be handled by conventional methods. Our principal result is the appearance of a sequence of coupling constants g_m; we must be able to solve H_s for each of these coupling constants.

IV. LIMIT OF THE SEQUENCE OF HAMILTONIANS

Now we consider the problem of solving our original model Hamiltonian, which formally is the limit of H_n as $n \to \infty$. We shall solve this problem by redefining our original Hamiltonian. Namely, we require that the coupling constant g in H_{lab} stay fixed, as $n \to \infty$, and permit the original coupling constant g_0 to vary as necessary to achieve this. This procedure is a logical development from intuitive notions of coupling constant renormalization. To keep the ground-state energy finite we also must subtract from H_n its ground-state energy: call it E_n. Then what we propose, precisely, is the following: We define a sequence of coupling constants, starting with a given value of g.

$$\begin{aligned} g_1 &= f^{-1}(g), \\ g_2 &= f^{-1}(g_1), \\ &\vdots \\ g_n &= f^{-1}(g_{n-1}), \text{ etc.} \end{aligned} \tag{18}$$

where f^{-1} is the function inverse to f. We must think of g as being determined "experimentally," as if our Hamiltonian described some physical system. Since all the properties of the low-lying eigenstates are determined by H_{lab} which depends on g, it is no problem to determine g. It does not matter, for our purposes, if the function f^{-1} is multiple valued, for whenever one has several possibilities for a particular g_n, one can appeal to experiment to make the choice. We now define our model Hamiltonian H as

$$H = \operatorname*{Lim}_{n \to \infty} \{H_n(g_n) - E_n(g_n)\}. \tag{19}$$

The limit that we propose is a strange one, since the operators H_n do not all act in the same Hilbert space; instead, as n increases the number of eigenstates of H_n also increases. (Those who quibble because the number of eigenstates is infinite for any n and therefore cannot increase may observe instead that the number of meson degrees of freedom increases with n.) We should discuss what quantities actually have limits as $n \to \infty$. To simplify our discussion, suppose that H_n has a discrete spectrum with eigenvalues denoted by E_{mn} and states $|E_{mn}\rangle$. Then the following quantities have limits as $n \to \infty$:

(a) the eigenvalues E_{mn} (for fixed m);
(b) the matrix elements $\langle E_{mn} | E_{m'n} \rangle = \delta_{mm'}$;
(c) the matrix elements $\langle E_{mn} | a_{\mathbf{k}} | E_{m'n} \rangle$;
(d) the ratio of the matrix elements $\langle E_{mn} | \tau^+ | E_{m'n} \rangle$; to the matrix elements $\langle P; n | \tau^+ | N : n \rangle$;

where $|P; n\rangle$ and $|N; n\rangle$ are the two ground states of H_n. In fact, every quantity listed will be independent of n (for fixed m, m', and $\mathbf{k}$) if n is sufficiently large. This can be proved as a consequence of our analysis of the Hamiltonians H_n. One can easily discuss matrix elements of operators other than $a_{\mathbf{k}}$ or τ^+.

One is tempted, of course, to ask what happens if we keep g_0 fixed, in the limiting process, instead of g. But this is a pointless question, for, once we have determined g experimentally, we know that the Hamiltonian H *must* be given by the limit of Eq. (19), and no new information is gained by seeing whether another limiting process gives the same result.

Let us look in more detail at the sequence of coupling constants g, g_1, etc. First, we show that the quantity α is less than one. This is because

$$\tau^+ \tau^- + \tau^- \tau^+ = 1. \tag{20}$$

Let $|x\rangle$ be a complete set of eigenstates of H_s. Then

$$\sum_x \{\langle P | \tau^+ | x \rangle \langle x | \tau^- | P \rangle + \langle P | \tau^- | x \rangle \langle x | \tau^+ | P \rangle\} = 1. \tag{21}$$

Every term in the sum over x is positive, and one of the terms is α^2, namely the term with $|x\rangle = |N\rangle$. So α^2 is less than or equal to one. It is trivial to show that α cannot be equal to one (except when g_0 is 0).

Because α is less than one, we have

$$f(g_0) < g_0. \tag{22}$$

Secondly, we note that for $g_0 \gg 1$, $f(g_0) \approx \frac{1}{2} g_0$. This is a consequence of Wentzel's strong coupling theory[4] and will not be proved here. For small g_0 we have $f(g_0) = g_0 + \text{order } g_0^3$. If $f(g_0)$ is a continuous function of g_0, it follows from these results that the equation $g_{n-1} = f(g_n)$ has at least one solution g_n for any value of g_{n-1}. Hence there exists a complete sequence $\{g_n\}$ for any value of g. Furthermore, this sequence is increasing. The sequence cannot have a finite upper bound for if so it would approach a limit point, which we can call g_0; and we would have

$$g_0 = f(g_0)$$

which is impossible.

So whatever the value of g, the sequence of coupling constants g_n is an increasing sequence which approaches infinity as n becomes large.

This is unfortunate, for when n is so large that g_n^2 is of order Λ, our perturbation analysis ceases to be valid. This is because the first excited (isobar) state of H_s has an energy only g_n^{-2} above the ground state when g_n is large; when multiplied by Λ^n, this gives an energy of

order Λ^{n-1}, the same order as the terms we treated as a perturbation. So to solve H_n for very large g_n, the whole set of isobar states of H_s must be taken into account in the perturbation analysis, rather than just the two ground states. This is an involved task; we do not attempt it here. Because of this difficulty, the analysis of our model is incomplete. Our analysis would also be incorrect if, for values of g_0 of order 1, the ground states of H_s did not have the same charge as the two source levels.

V. AN OFFSPRING OF THE LEE MODEL

Another Hamiltonian that can be analyzed in the manner of Secs. III and IV is derived from the Lee model.[1] The Lee model Hamiltonian can be obtained from the fixed source Hamiltonian of Eq. (1) by dropping the π^- meson terms (i.e., we suppress b_k and $b_k^\dagger$). We can then construct a model Hamiltonian by replacing the full momentum integration by an integral over our sequences of intervals. We can then perform an analysis analogous to Secs. III and IV. We describe this analysis briefly.

First of all we have charge conservation, but we no longer have charge symmetry. To ensure that the physical proton state has the same energy as the physical neutron state, we let the bare proton state have an undetermined energy. This means adding a term $E\tau^{++}$ to the Hamiltonian, where τ^{++} has the properties:

$$\begin{aligned}\tau^{++}|p\rangle&=|p\rangle,\\ \tau^{++}|n\rangle&=0.\end{aligned}\tag{23}$$

($|p\rangle$ and $|n\rangle$ are the two source levels.)

Now we define, as before, a sequence of Hamiltonians H_n. The added term for H_n will be denoted by $E_n\tau^{++}$. We break H_n into an unperturbed and perturbed Hamiltonian, as before. The added term must likewise be split, say

$$E_n=\Lambda^n E_0+\epsilon_n.\tag{24}$$

Then after performing the scaling as in Sec. III, we obtain

$$H_n=\Lambda^n H_s+H_{n-1},\tag{25}$$

where the added term in H_{n-1} is $\epsilon_n\tau^{++}$, and

$$H_s=E_0\tau^{++}+\int_p pA_p^\dagger A_p+g_0\int_p (2p)^{-1/2}\times\{A_p\tau^+ + A_p^\dagger\tau^-\}\tag{26}$$

with $\frac{1}{2}<p<1$. We choose E_0 so that the ground states $|P\rangle$ and $|N\rangle$ of H_s are degenerate. These states and E_0 are easily derived; we obtain

$$|P\rangle=Z\left\{|p\rangle-g_0\int_p (2)^{-1/2}p^{-3/2}A_p^\dagger|N\rangle\right\},\tag{27}$$

$$|N\rangle=|n\rangle,\tag{28}$$

$$E_0=\tfrac{1}{2}g_0\int_p p^{-2},\tag{29}$$

$$Z^{-2}=1+\tfrac{1}{2}g_0^2\int_p p^{-3},\tag{30}$$

with $\frac{1}{2}<p<1$ in all integrals.

We now solve H_n by degenerate perturbation theory. We restrict H_n to the subspace of states spanned by $|P\rangle$ and $|N\rangle$ plus mesons of momentum Λ^{n-1} or less. Under this restriction

$$\tau^+\longrightarrow\alpha\tau_R^+,\tag{31}$$

$$\tau^{++}\longrightarrow\beta\tau_R^{++},\tag{32}$$

where $\tau_R^{++}|P\rangle=|P\rangle$, $\tau_R^{++}|N\rangle=0$, and

$$\alpha=\langle P|\tau^+|N\rangle=Z,\tag{33}$$

$$\beta=\langle P|\tau^{++}|P\rangle=Z^2.\tag{34}$$

Thus, we obtain an effective Hamiltonian which is just $H_{n-1}(\alpha g_0)$; the added term is $E_{n-1}\tau_R^{++}$ with E_{n-1} set equal to $Z^2\epsilon_n$.

It is now easy to see that exactly as in Sec. IV, we want to define a sequence of coupling constants g_n. In the present problem we will have

$$g_n=f^{-1}(g_{n-1}),\tag{35}$$

where f^{-1} is the inverse function for the function f given by

$$f(g_0)=Z(g_0)g_0.\tag{36}$$

But now for large g_0, $Z(g_0)$ behaves as g_0^{-1}. Hence $f(g_0)$ has a finite upper bound. If g_{n-1} is larger than this upper bound, g_n does not exist. But since $Z(g_0)$ is always smaller than one, the sequence $\{g_n\}$ is increasing; there will inevitably be a finite value of n, above which g_n does not exist.

VI. COMPARISON WITH QUANTUM ELECTRODYNAMICS

We want to compare the results of our analysis of two model Hamiltonians with the analysis of quantum electrodynamics by Gell-Mann and Low. First we shall restate the ideas of Gell-Mann and Low in a form suitable for our purposes. For background the reader is referred to their paper.[7] Let e be the physical electron charge, and let m be the physical electron mass.[8] Let $d_C(k^2/m^2,e^2)$ be the photon propagator apart from a factor k^{-2}; we require the customary renormalization condition

$$d_C(0,e^2)=1.\tag{37}$$

[7] See Ref. 2. Some of the manipulations involved are well presented in N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), Chap. VIII; however, some of the important ideas are discussed only by Gell-Mann and Low.

[8] We use the same metric as Gell-Mann and Low: $(-1, 1, 1, 1)$.

Now define a one-parameter family of coupling constants e_λ as

$$e_\lambda^2 = e^2 d_C(\lambda^2/m^2, e^2). \tag{38}$$

These constants e_λ depend on e, but we shall not need to indicate this explicitly. Gell-Mann and Low introduce a fancy subtraction scheme for renormalizing perturbation theory, in which the subtractions are performed at momentum λ instead of momentum 0. The result of their scheme is

(1) their renormalized coupling constant is e_λ, not e;

(2) their photon propagator is normalized to 1 at $k^2=\lambda^2$ instead of 0;

(3) their photon propagator is a function which may be written $d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$;

(4) their photon propagator is related to the usual one according to the equation

$$e^2 d_C(k^2/m^2, e^2) = e_\lambda^2 d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2). \tag{39}$$

Alternatively, one can use Eq. (38) to express e^2 as a function of e_λ^2, and then *define* the function

$$d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$$

by Eq. (39).

(5) They argue that the function $d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$ does not depend on m when m^2/λ^2 is small.

This result is crucial to their discussion. They illustrate it with an example from fourth-order perturbation theory and propose a reason for it to be true to all orders [see, e.g., preceding Eq. (5.1) of Ref. 2]. Some people tend to regard this result skeptically, and to use this skepticism as an excuse to disregard the consequences which Gell-Mann and Low deduce from it. None of these people have succeeded (to my knowledge) in finding a specific counterexample to this claim of Gell-Mann and Low, although the paper it was made in is now over ten years old.

The specific equation we need is obtained as follows. We rewrite Eq. (39) as

$$e_k^2 = e_\lambda^2 d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2). \tag{40}$$

Differentiate with respect to k, holding λ and e_λ fixed (which means that e, on which e_k also depends, is fixed), and then set $k^2=\lambda^2$: we obtain

$$d(e_\lambda^2)/d\lambda = 2\lambda^{-1}\psi(m^2/\lambda^2, e_\lambda^2), \tag{41}$$

where

$$\psi\left(\frac{m^2}{\lambda^2}, e_\lambda^2\right) = e_\lambda^2 \frac{\partial d}{\partial x}\left(x, \frac{m^2}{\lambda^2}, e_\lambda^2\right)\Bigg|_{x=1}. \tag{42}$$

Now, according to result 5, for $\lambda \gg m$, the dependence of ψ on m^2/λ^2 is negligible, so that to a good approximation ψ depends only on e_λ^2. Thus we have

$$d(e_\lambda^2)/d\lambda = 2\lambda^{-1}\psi(e_\lambda^2) \tag{43}$$

for $\lambda \gg m$.

We can understand this equation by setting up an analogy with the results of the analysis of our model Hamiltonians. The function e_λ^2 is analogous to our sequences $\{g_n\}$, and has the same significance: The constant g_n determines the solution of the unperturbed part of the Hamiltonian H_n, which determines the essential features of the behavior of mesons of momentum Λ^n. The constant e_λ determines the essential features of the behavior of propagators and vertex functions when the momenta are of order λ; in particular these functions can be computed by a perturbation expansion in e_λ if e_λ is small.

We can illustrate this point by studying the photon propagator in perturbation theory. Let λ be very large, and let k be of order λ. Then $d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2)$ takes the form of a double power series in e_λ^2 and $\ln(k^2/\lambda^2)$.[2] For order of magnitude purposes, the coefficients can be set equal to one and we obtain

$$d(k^2/\lambda^2, m^2/\lambda^2, e_\lambda^2) = 1 + e_\lambda^2[\ln(k^2/\lambda^2)+1] + e_\lambda^4[\ln^2(k^2/\lambda^2)+\cdots]+\cdots. \tag{44}$$

Since $\ln(k^2/\lambda^2)$ is, by assumption, of order 1, this series in e_λ^2 converges rapidly if $e_\lambda^2 \ll 1$; we can compute d using the perturbation expansion in e_λ if $e_\lambda \ll 1$. This is not true of the function d_C expanded in powers of e; for large k we have (in order of magnitude)

$$d_C(k^2/m^2, e^2) = 1 + e^2[\ln(k^2/m^2)+1] + e^4[\ln^2(k^2/m^2)+\cdots]+\cdots. \tag{45}$$

Clearly, this power series converges rapidly only if $e^2 \ln(k^2/m^2) \ll 1$. So the criterion for whether the photon propagator can be computed in perturbation theory, when k is of order λ, is not $e \ll 1$ but $e_\lambda \ll 1$.

Equation (43) for e_λ is analogous to our recursion formula for g_n:

$$g_{n-1} = f(g_n). \tag{17}$$

This equation determines how g_n changes with n, given g_n, while Eq. (43) for e_λ determines the rate of change of e_λ, given e_λ. We note that there is no question that the function $f(g_n)$ is independent of the meson mass μ; this mass disappeared when we approximated ω_k by k, for k large. Thus, when we go from quantum electrodynamics to a theory which we understand, the question of mass dependence becomes trivial. I suspect the same will be true of quantum electrodynamics when a decent formulation of it becomes available.

Gell-Mann and Low discussed the possible behavior of e_λ for very large λ, using Eq. (43). There is a close analogy between this question and the problem of how the sequence $\{g_n\}$ behaves for large n. First let us discuss how the sequence g_n behaves, in terms of possible types of functions f. We show in Fig. 1 three types of functions f. The curve labeled A is the function that results from our Lee-model-type Hamiltonian discussed in Sec. V. The curve labeled B results from our fixed source model. The curve labeled C has not been derived

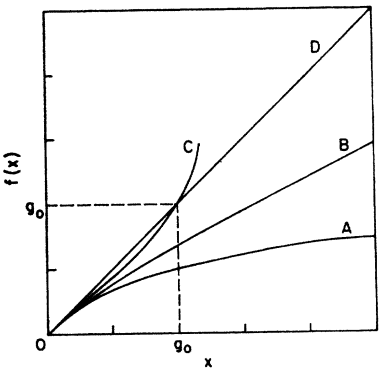

FIG. 1. Plots of the function $f(x)$ versus x. Three cases are shown: the curves labeled A, B, and C. The line D is the line $f(x)=x$.

from any model but it is instructive to discuss the behavior of the sequence $\{g_n\}$ for such a function. We have discussed curves A and B previously. If f is given by curve A, then the sequence $\{g_n\}$ stops after a finite number of terms. If f is given by curve B, then the sequence $\{g_n\}$ exists for all n, it is increasing, and it approaches infinity as $n\to\infty$. Now consider the curve C. We have not shown the whole of this curve because of the enormous number of possibilities for the remaining part of the curve. The part shown has the following properties:

(a) The inverse function $f^{-1}(y)$ is uniquely defined for not too large y.
(b) The equation $y=f^{-1}(y)$ has a solution $y=g_0$.
(c) If $y<g_0$, then $y<f^{-1}(y)<g_0$; if $y>g_0$, then $y>f^{-1}(y)>g_0$.

From these properties it follows that if the low-momentum coupling constant g is smaller than g_0, then the sequence $\{g_n\}$ will be increasing, and approach g_0 in the limit $n\to\infty$. If g is greater than g_0, but not above the range of the curve C, then the sequence $\{g_n\}$ will be decreasing and again approach the limit g_0 as $n\to\infty$.

For both the curves B and C, we see that the sequence $\{g_n\}$ approaches a limit as $n\to\infty$ which is *independent* of the low-momentum coupling constant g that we start with. For the curve B, the limit of g_n is infinite; for curve C the limit is the finite number g_0 which solves the equation

$$g_0=f(g_0). \tag{46}$$

We can invent possible forms for the function $\psi(x)$ which are analogous to the three curves A, B, and C for f. These are shown in Fig. 2. To set up the analogy, we note that g_n-g_{n-1} constitutes the change in g_n as the momentum changes by a factor Λ; therefore g_n-g_{n-1} is roughly analogous to the derivative of e_λ^2 with respect to $\ln\lambda$, which is just the function ψ. Now for curve A for f, the difference g_n-g_{n-1} becomes very large as $g_n\to\infty$. By analogy we obtain the curve A of Fig. 2. For the curve B for f, the difference g_n-g_{n-1} is just a constant times g_n when g_n is large. Thus we obtain the curve B of Fig. 2 which is linear in x for large x. Finally, when f is of type C, the difference g_n-g_{n-1} passes through 0 as g_n passes through g_0; hence the curve C of Fig. 2.

To find e_λ^2 as a function of λ from Eq. (43), we simply integrate this equation. To simplify matters let ψ be independent of m if $\lambda>m$, and let e_λ be e_m at $\lambda=m$. Then

$$\ln(\lambda^2/m^2)=\int_{e_m^2}^{e_\lambda^2} dx/\psi(x). \tag{47}$$

For curve A, the integral is finite for $e_\lambda^2\to\infty$ which means we get a finite upper bound for λ, beyond which e_λ is undefined. For curve B, λ increases as e_λ^2 increases and $\lambda\to\infty$ as $e_\lambda^2\to\infty$. In this case we can invert our equation to get e_λ as a function of λ; we see that e_λ^2 increases as λ increases and $e_\lambda^2\to\infty$ as $\lambda\to\infty$. For curve C, the integral diverges as $e_\lambda^2\to e_0^2$. For $e_\lambda^2<e_0^2$, λ increases as e_λ^2 increases. Hence, if $e_m^2<e_0^2$, the function e_λ^2 will be increasing, and approach the limit e_0^2 as $\lambda\to\infty$.

The curve A will never occur in practice. For if it did occur then e_λ would not exist for large λ. But because of our definition of e_λ [Eq. (38)], the photon propagator also would not exist for large photon momentum k. But quantum electrodynamics cannot exist without a photon propagator, and in particular the function ψ will then not exist either. So, instead of the curve A being a possibility, we have the possibility that no consistent quantum electrodynamics exists.

The curves B and C were the specific alternatives considered by Gell-Mann and Low, and led to their alternatives (a) and (b) which we quoted in the Introduction.

VII. QUALITATIVE ANALYSIS OF COMPLETE FIXED SOURCE HAMILTONIAN

In the Introduction we emphasized the importance of making qualitative estimates of various terms in a Hamiltonian before trying to solve it quantitatively. For this purpose, the creation and destruction operators a_k and $a_k^\dagger$ are a nuisance. In particular, the operator $a_k^\dagger$, since it creates a meson in a plane wave state, creates states of infinite norm. Thus, if one tries to estimate the "order of magnitude" of $a_k^\dagger$ in the manner

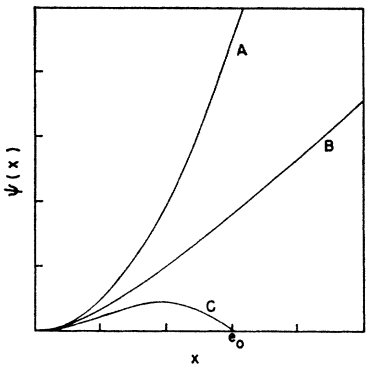

FIG. 2. Plots of the function $\psi(x)$ versus x. Three cases are shown: the curves labeled A, B, and C.

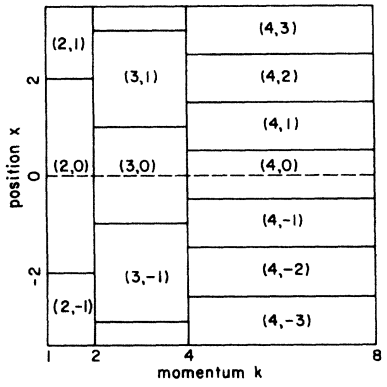

FIG. 3. Our division of phase space into cells of unit volume shown in two dimensions (one coordinate and one momentum dimension). Each cell is labeled with its indices (n,m).

we have used, which is to calculate the norm of the state created by $a_{\mathbf{k}}^{\dagger}$, we get infinity. So far we have not had to estimate $a_{\mathbf{k}}^{\dagger}$ itself, but only integrals involving $a_{\mathbf{k}}^{\dagger}$ of the form $\int a_{\mathbf{k}}^{\dagger}u(\mathbf{k})d^3k$. If the function $u(\mathbf{k})$ in the integral is normalizable, then the resulting operator creates a normalizable state and estimates are possible. However, in relativistic Hamiltonians, the operators analogous to $a_{\mathbf{k}}^{\dagger}$ are integrated with δ functions (to ensure momentum conservation) rather than normalizable functions.

What we shall propose here is that the creation and destruction operators, which depend on the continuous variable $\mathbf{k}$, be expanded in terms of a discrete, complete, orthonormal set of functions $u_l(\mathbf{k})$. The coefficients of this expansion are themselves creation and destruction operators for mesons in normalizable "wave-packet" states. We obtain

$$a_{\mathbf{k}}=\sum_l u_l(\mathbf{k})a_l, \tag{48}$$

$$a_{\mathbf{k}}^{\dagger}=\sum_l u_l^{*}(\mathbf{k})a_l^{\dagger}. \tag{49}$$

The operators a_l and $a_l^{\dagger}$ are operators with the same commutation rules as an infinite set of harmonic oscillator variables:

$$\begin{aligned}[a_l,a_m]&=0,\\ [a_l,a_m^{\dagger}]&=\delta_{lm},\\ [a_l^{\dagger},a_m^{\dagger}]&=0,\end{aligned} \tag{50}$$

where δ_{lm} is the Kronecker delta.

When a Hamiltonian involving $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$ is re-expressed in terms of the operators a_l and $a_l^{\dagger}$, we obtain a Hamiltonian for an infinite set of coupled, perhaps anharmonic, oscillators. It seems to be easier to analyze such Hamiltonians qualitatively than to analyze Hamiltonians containing the operators $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^{\dagger}$. We shall illustrate this shortly by discussing the full fixed source Hamiltonian of Eq. (1).

There are very many complete orthonormal sets of functions $u_l(\mathbf{k})$. However, the complete sets of interest to us will have to satisfy some restrictions, which we shall now discuss. The restrictions will be imposed with the aim of making the self-interactions of the individual oscillators more important than interactions between different oscillators. We shall not try to demonstrate that our choice of restrictions accomplishes this until we discuss a particular Hamiltonian.

Our first requirement is that *all* the functions $u_l(\mathbf{k})$ be good wave packet functions. By this we mean that, if Δk_l is the momentum width of the function $u_l(\mathbf{k})$, and Δx_l is the width of the Fourier transform $v_l(\mathbf{x})$ of u_l, then the product $\Delta x_l\Delta k_l$ is near the lower limit set by the Heisenberg uncertainty principle

$$\Delta x_l\Delta k_l\approx\tfrac{1}{2}. \tag{51}$$

For order of magnitude purposes, each function u_l can now be thought of as occupying a "cell" in phase space, of unit volume. If the set of functions u_l is complete, then the total volume occupied by the functions u_l must fill all of phase space. The orthogonality of the different u_l means that regions occupied by different u_l should not overlap.

These conditions are supposed to be understood qualitatively. That is, the function $u_l(\mathbf{k})$ and $v_l(\mathbf{x})$ are certainly not 0 outside their assigned regions in momentum space and position space, respectively but, hopefully, their "tails" outside these regions are small and go rapidly to zero as one goes away from the assigned region. I have been unable to construct a specific example of a complete orthonormal set of functions satisfying these requirements: the functions I construct always have long tails in position space and no tails in momentum space, or vice versa. But the conditions proposed represent, I think, the physics of a complete orthonormal set of wave functions, and if so it is reasonable to proceed as if such a set of functions did exist.

We still have to specify how phase space is to be divided into unit cells; we shall presume that we may do this arbitrarily. The particular division that we shall adopt is one in which momentum space is divided evenly on a logarithmic scale. To be precise, we divide momentum space into an infinite number of nested spherical shells, the nth shell being

$$2^{n-1}<k<2^n. \tag{52}$$

Now for each shell separately we divide position space linearly into cubes of the appropriate size. For the nth shell the volume in momentum space is of order 2^{3n} so the volume of a cell in position space will be of order 2^{-3n}. So in order of magnitude the cubes may be defined as

$$m_i-\tfrac{1}{2}\leq 2^n x_i<m_i+\tfrac{1}{2}, \tag{53}$$

where the x_i are position space coordinates and m_1, m_2, and m_3 are arbitrary integers. Our division of phase space is illustrated in Fig. 3.

Now we shall assume that there exists a complete orthonnormal set of functions $u_l(\mathbf{k})$ corresponding to the division of phase space just described. To be precise, $u_l(\mathbf{k})$ will be labeled by four indices (n,m_1,m_2,m_3) which specify the location of the corresponding phase space cell; let us use the single symbol l to stand for

these four indices. From the normalization requirement

$$\int_{\mathbf{k}} |u_l(\mathbf{k})|^2 = \int |v_l(\mathbf{x})|^2 d^3x = 1 \tag{54}$$

we can compute the order of magnitude of u_l and v_l. The volume of the nth shell in momentum space is of order 2^{3n}, and the volume of the corresponding cell in position space is of order 2^{-3n}. Hence, inside these regions we have

$$\begin{aligned} |u_l(\mathbf{k})| &\sim 2^{-3n/2}, \\ |v_l(\mathbf{x})| &\sim 2^{+3n/2}. \end{aligned} \tag{55}$$

Outside these regions, u_l and v_l are smaller.

To illustrate the usefulness of expanding the creation and destruction operators in terms of oscillator variables, we shall examine the Hamiltonian of Eq. (1) (with no limitations on k in the integrals). First we must express H in terms of the operators a_l and $a_l^\dagger$. We must introduce corresponding oscillator variables also for $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^\dagger$. Then H has the form

$$\begin{aligned} H = \sum_j \sum_l C_{jl}(a_j^\dagger a_l + b_j^\dagger b_l) + g_0 \sum_l D_l(a_l \tau^+ + b_l \tau^-) \\ + g_0 \sum_l D_l^*(b_l^\dagger \tau^+ + a_l^\dagger \tau^-), \end{aligned} \tag{56}$$

where

$$C_{jl} = \int_{\mathbf{k}} \omega_{\mathbf{k}} u_j^*(\mathbf{k}) u_l(\mathbf{k}), \tag{57}$$

$$D_l = \int_{\mathbf{k}} (2\omega_{\mathbf{k}})^{-1/2} u_l(\mathbf{k}). \tag{58}$$

The Hamiltonian H is now the Hamiltonian of an infinite number of oscillators coupled to each other and to a two-level source.

To simplify our Hamiltonian, we examine the coefficients C_{jl} and D_l. Except for the factor $\omega_{\mathbf{k}}$, the matrix C would be diagonal because the functions u_l are orthogonal. Even with the factor $\omega_{\mathbf{k}}$, the off-diagonal terms should be small: If the momentum shells n_j and n_l are distinct, then the functions u_j and u_l do not overlap very much. If u_j and u_l are in the same momentum shell but different spatial cells, this fact will be reflected in a rapid variation in the phase of $u_j^* u_l$ as a function of $\mathbf{k}$, which, again, makes the integral small. So, for a first approximation, we keep only the diagonal terms C_{ll} of the matrix C. These we can estimate to be a mean energy times the normalization integral. In the nth shell the mean energy is either about μ, or about 2^n, whichever is larger. Let μ be 1 for convenience. So

$$\begin{aligned} C_{ll} &\sim 2^n \quad (n>0) \\ &\sim 1 \quad (n<0). \end{aligned} \tag{59}$$

Now consider D_l. The function $u_l(\mathbf{k})$ carries a $\mathbf{k}$-dependent phase factor associated with the center of the appropriate cell in position space, unless this center is at the origin. Thus D_l will be small unless the m_i in l are all 0: $l=(n,0,0,0)$. These latter D_l we denote by D_n. They are easily estimated from the known volume of the nth shell and the known order of magnitude of $u_l(\mathbf{k})$; we get

$$\begin{aligned} D_n &\sim 2^n \quad (n>0) \\ &\sim 2^{3n/2} \quad (n<0). \end{aligned} \tag{60}$$

We have now as a rough approximation

$$\begin{aligned} H \sim \sum_l C_{ll}(a_l^\dagger a_l + b_l^\dagger b_l) + g_0 \sum_n D_n(a_n \tau^+ + b_n \tau^-) \\ + g_0 \sum_n D_n^*(b_n^\dagger \tau^+ + a_n^\dagger \tau^-), \end{aligned} \tag{61}$$

where the operators a_n, etc. are the operators a_l, etc. with $m_i=0$. In this approximation the free-meson field has been replaced by independent harmonic oscillators for each phase space cell, with a frequency depending only on the mean momentum of the cell. The interaction of the meson field with the source has been replaced by an interaction of those oscillators located at the origin (where the source is) with the source. The remaining terms of the original Hamiltonian are to be considered as a perturbation.

To get orders of magnitude, we first neglect the interaction with the source, in Eq. (61). Then we have independent harmonic oscillators. In this case the orders of magnitude are evidently

$$a_l \sim 1, \quad a_l^\dagger \sim 1 \tag{62}$$

for all l. The energy contributed by the lth oscillator is of order 2^n or 1 whichever is larger. As in our model, the oscillators associated with large momentum in phase space contribute most to the energy. Now suppose the order of magnitudes of a_l and $a_l^\dagger$ are unchanged by the presence of the source. The matrices τ^+ and τ^- are of order 1 at most [this follows from the sum rule, Eq. (20)]. The energy due to the interaction of nth shell mesons with the source is therefore of order 2^n, if $n>0$, or $2^{3n/2}$ if $n<0$. This is of the same order as the free-meson energy for $n>0$, and much smaller for $n<0$. Since the interaction energy of a meson is not greater than its free energy, we do not expect the interaction to change the order of magnitude of a_l and $a_l^\dagger$. [This discussion fails if g_0 is very large, for then the interaction energy is large and the orders of magnitude do change (compare Ref. 6).]

Let us consider now the problem of finding the ground state of H. First, we ignore the oscillators not located at the origin since in our rough approximation these are uncoupled. For the oscillators coupled to the source we use the same method as we used on our model of Sec. II. We cut off the Hamiltonian at a large finite value of n, say N. Then the oscillators associated with shells within the Nth shell are treated as a perturbation on the Nth shell oscillators. Thus, our unperturbed Hamiltonian H_0 is

$$\begin{aligned} H_0 = \omega_N(a_N^\dagger a_N + b_N^\dagger b_N) + g_0 D_N(a_N \tau^+ + b_N \tau^-) \\ + g_0 D_N^*(b_N^\dagger \tau^+ + a_N^\dagger \tau^-), \end{aligned} \tag{63}$$

where ω_N is C_{ll} for $l=(N,0,0,0)$. The perturbation consists of the analogous terms for $n<N$.

The unperturbed Hamiltonian H_0 is the Hamiltonian of two harmonic oscillators coupled to a two-level source, which is a problem in ordinary quantum mechanics. It cannot be solved exactly, but otherwise it is a trivial problem to analyze, for any value of g_0. It should have a doubly degenerate ground state, as did the unperturbed Hamiltonian of our model.

The perturbation has to be analyzed by degenerate perturbation theory. The states which are degenerate are the two ground states of H_0, and with each of these states the oscillators of the lower shells ($n<N$) can be in arbitrary states. As in the analysis of our model, the perturbation, in lowest order, leads to an effective Hamiltonian of the form

$$H_{\text{eff}}=E_0+\sum_{n<N}\omega_n(a_n^\dagger a_n+b_n^\dagger b_n)+f(g_0)\sum_{n<N}D_n(a_n\tau^+ +b_n\tau^-)+f(g_0)\sum_{n<N}D_n^*(b_n^\dagger\tau^+ +a_n^\dagger\tau^-), \quad (64)$$

where $f(g_0)$ is g_0 times the ground-state matrix element of τ^+. As in our model, we see that the effective Hamiltonian is similar to the original Hamiltonian cutoff at $N-1$, except for an over-all energy shift and a modification of the coupling constant.

It is unnecessary for our purposes to continue this analysis. There is one point that must be discussed, however. In our rough approximation, we disregarded the coupling terms between oscillators (involving the constants C_{jl} with $j\neq l$). We also disregarded the coupling to the source of oscillators not located at the origin. These terms were dismissed as being a perturbation, but in our approximate Hamiltonian the terms with $n<N$ were also a perturbation, yet were taken into account. We can justify our procedure as follows. If we take into account the oscillators not located at the origin, then we can choose the unperturbed Hamiltonian to involve the self-energies of all Nth-level oscillators. The ground states of the unperturbed system consist of either of the two ground states of H_0, with the other Nth-level oscillators in their ground states. The oscillators of the lower shells can be in any state. The perturbations now consist of

(a) the coupling terms between oscillators of the Nth shell,
(b) the coupling to the source of oscillators of the Nth shell not at the origin,
(c) the coupling of oscillators in lower shells to oscillators of the Nth shell or lower shells,
(d) the coupling of oscillators of lower shells not located at the origin to the source,
(e) the self-energies of oscillators of lower shells, and
(f) the coupling of oscillators of lower shells at the origin to the source.

In lowest order perturbation theory we are interested only in matrix elements of the perturbation between ground states of the unperturbed Hamiltonian. It is easily seen, by explicit calculation, that perturbations (a) and (b) do not contribute to these matrix elements, nor do those terms of (c) which involve coupling to Nth shell oscillators. But now the dominant terms in the perturbation are (e) and (f), in particular those terms from the $N-1$st shell. This is the justification for taking only terms (e) and (f) into account.

Our rough analysis reduces the problem of solving the Hamiltonian of Eq. (1) to the solution of Hamiltonians like H_0 [Eq. (63)], which couple two oscillators to a two-level source. As in our model, the Hamiltonian H_0 has to be solved for an infinite sequence of coupling constants, one for each shell in momentum space. To arrive at this simplification we made some very crude approximations, but this is always what one has to do with complicated Hamiltonians.

To conclude this section, we shall use our analysis just given to explain why it has been so difficult to obtain good solutions of the fixed source theory of the pion-nucleon interaction.[9] In this theory the pions are pseudoscalar not scalar, and there are three pions, not two. The cutoff, experimentally determined, is about six times the pion mass. The standard method for solving this theory is the Tomonaga intermediate coupling method,[5] which in essence replaces the meson field by nine oscillators. The nine oscillators are associated with the three charge states and three spin states of p-wave pions; all nine oscillators are associated with pions of the same mean energy. The wave function in momentum space associated with these oscillators is varied as part of a variational calculation of the ground state.

In our analysis we found that the most important oscillators, as far as the interaction energy was concerned, were the oscillators located at the origin with mean energy of the order of the cutoff energy. Hence we would expect the variational method to determine mainly the behavior of mesons near the cutoff energy. But practically speaking, we are interested in the behavior of low-energy pions, which are described by oscillators with low-momentum wave functions. These do not contribute very importantly to the energy; in our analysis they were treated by perturbation theory. In a variational approach one would want to allow more oscillators than considered by Tomonaga and require the wave functions of the extra oscillators to have a low mean momentum.

VIII. CONCLUSION

In this paper we analyzed in detail a model Hamiltonian obtained from the theory of charged scalar pions interacting with a point source. To construct our Hamiltonian we suppressed the mesons with momenta outside an infinite sequence of intervals. By keeping some of the very large momentum mesons, we ensured

[9] See Ref. 3, p. 179.

that our Hamiltonian had the same renormalization problems as the full fixed source theory. We were able to solve our model Hamiltonian to the extent of reducing it to the solution of two strongly cut-off fixed-source Hamiltonians (H_{lab} and H_s).

We also discussed a corresponding model derived from the Lee model (Sec. V). We compared the results of the analysis of these models with Gell-Mann and Low's analysis of quantum electrodynamics (Sec. VI).

In Sec. VII we showed how one could perform a qualitative analysis of a complete fixed-source Hamiltonian. The basic idea was to expand the meson field in a complete orthonormal set of "wave packet" functions. The coefficients in the expansion can be thought of as the quantum variables for an infinite set of oscillators, each associated with a cell in phase space of unit volume. The division of phase space was chosen so that, for a given momentum range, the self-interactions of the oscillators were more important than the coupling between oscillators; and so that the mean energies associated with different momentum ranges were sufficiently different to allow the use of perturbation theory.

There are a number of ideas involved in the analysis of our models which probably are important in the analysis of any Hamiltonian which requires coupling constant renormalization to give finite results in perturbation theory. Let us discuss these ideas briefly.

First, we note the result of our model that there is not just one coupling constant, but an infinite number of coupling constants, each relevant to a particular momentum range. In a strongly cut-off theory one has a single coupling constant, and one uses perturbation theory, or variational methods, or strong-coupling theory, depending on whether the coupling constant is small, or of order one, or large. In a theory involving renormalization, some of the coupling constants may be small and others large in which case one would have to use both perturbation methods and strong-coupling methods to obtain a complete solution of the theory. In any case, it invariably turns out in practice that if the low-momentum coupling constant is small, the coupling constants increase as the momentum increases until perturbation theory is invalid.

Secondly, we note that after breaking down the full Hamiltonian for our static model into contributions from the various momentum intervals, the individual contributions were finite and could easily be estimated. The Hamiltonian as a whole was finite only when cut off, and the eigenstates of the Hamiltonian had to be defined as a limit with the cutoff going to infinity. In the Hamiltonian of a fully relativistic theory, it is no longer sufficient to break up the Hamiltonian into contributions from various momentum intervals, because in this case the momentum conservation δ functions cause trouble. One must introduce discrete oscillator variables, as was done in Sec. VII, to replace the field variables, and then divide the Hamiltonian into contributions from various cells in phase space. Then the individual terms are finite and can be estimated.

Thirdly, we found that the disparity in energy of mesons of different momentum was the key fact that made possible our analysis of the model Hamiltonian. We were able to regard the mesons of low momentum as a perturbation with respect to the mesons of large momentum. Renormalization resulted from the perturbation analysis: low-momentum mesons were described by an effective Hamiltonian containing renormalized constants. We can expect the energy disparity to be of fundamental importance in working out a theory of renormalization for Hamiltonians of relativistic theories.

Fourthly, we found that qualitatively the full fixed-source Hamiltonian, involving a meson field coupling to a source, could be reduced to a Hamiltonian involving two oscillators (associated with a single phase-space cell) coupled to the source. This Hamiltonian has to be solved for an infinite set of coupling constants, one for each momentum range. (For low momenta the Hamiltonian is modified to take into account the meson mass. To save space we have not discussed this problem.) Analogously, one would hope that a qualitative analysis of a Hamiltonian for a relativistic quantum field would reduce it to the Hamiltonian of a small number of coupled anharmonic oscillators, whose parameters would depend on the momentum range being considered.

To conclude, we note that in the past our understanding of renormalization problems has depended on models that can be solved exactly, such as the Lee model. There is scant hope that we will ever understand the Hamiltonians of relativistic theories if we await an exact solution of them. Therefore we have tried to emphasize methods for understanding the renormalization of Hamiltonians which do not require an exact solution. To analyze our model we partially solved it, but were left with two Hamiltonians H_{lab} and H_s, which are not exactly soluble, to my knowledge. In analyzing the full fixed-source Hamiltonian, we relied entirely on qualitative arguments.

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