General $SU(3)$ Crossing Matrices and the Projection Operators of 3×8*

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Expressions are obtained for the $SU(3)$ (u) and (t) crossing matrices in terms of isoscalar factors and Racah coefficients (6-j symbols). Careful attention is given to the question of phases and states. The cases of 3×8 and 8×8 are considered as examples. Methods for obtaining crossing matrices by means of projection operators are illustrated by examining the case of 3×8 in detail.

1. INTRODUCTION

WITH the success of the $SU(3)$ theory in explaining the symmetries of strong interactions,[1,2] attention has been given to deriving the crossing matrices of $SU(3)$. Various techniques are available for this. For example, one can use projection operators as did Cutkosky,[3] or one can use the $SU(3)$ Clebsch-Gordan coefficients as was done by de Swart.[4]

In this work, we will obtain a general expression for the u- and t-channel crossing matrices in terms of isoscalar factors and Racah coefficients. Techniques using projection operators will also be discussed. It will be seen that a judicious use of both of these methods will facilitate obtaining a crossing matrix.

Particular care is given to the touchy question of phases. It is hoped that this paper will illuminate the understanding of this aspect of the subject. Throughout this work, special attention is paid to the case of 3×8, which is important in discussing Sakata particle or quark-meson scattering. 8×8 is also worked out in detail.

In Sec. 2, we discuss the mathematical apparatus including the definition of crossing matrices, phases, states and fields, and $SU(3)$ direct product states. Our expression for the $SU(3)$ crossing matrices is derived in Sec. 3. The next section is devoted to the method of projection operators with reference to the case of 3×8. The projection operators of the s, u, and t channels for 3×8 are found. This gives a check to the results of Sec. 3. A discussion follows in Sec. 5.

2. MATHEMATICAL PRELIMINARIES

a. Crossing Matrices

In finding a crossing matrix in the literature, one must be sure to ascertain if the matrix is for the u or t channel, if it is for projection operators or for amplitudes, and to understand the relation between the projection operator and amplitude matrices.

In this paper, the s, u, and t channels will be defined by

$$\begin{aligned} &(s)\quad a+b\rightarrow c+d\,,\\ &(u)\quad a+\bar{d}\rightarrow c+\bar{b}\,,\\ &(t)\quad a+\bar{c}\rightarrow \bar{b}+d\,. \end{aligned} \tag{2.1}$$

The relationship between crossing matrices for the amplitudes and projection operators can be understood in terms of the t matrix of the pion-nucleon interaction.[5]

$$t_{pq}(z)=-v(q)v(p)\frac{4\pi}{(4\omega_p\omega_q)^{1/2}}\sum_{\alpha=1}^{4}P_\alpha(p,q)h_\alpha(z)\,, \tag{2.2}$$

where q and p are initial and final π momenta, $\alpha=(2T,2J)$ labels the (T,J) states, the P_α's are projection operators onto the α states, and the h_α's are related to the phase shifts and hence the amplitudes in the α state. Equation (2.2) is quite special to the Chew-Low theory, but it is adequate for our purposes, since we use it only to illustrate the connection between the crossing matrices.

Defining the u crossing matrix for projection operators to be

$$P_\alpha{}^u(q,p)=\sum_\beta A_{\alpha\beta}{}^{us}P_\beta{}^s(p,q)\,, \tag{2.3}$$

and using

$$t_{qp}(z)=t_{pq}(-z)\,, \tag{2.4}$$

allows us to discover that

$$h_\alpha{}^s(z)=\sum_{\beta=1}^{4}A_{\beta\alpha}{}^{us}h_\beta{}^u(-z)=\sum_{\beta=1}^{4}(B^{us})_{\alpha\beta}{}^{-1}h_\beta{}^u(-z)\,, \tag{2.5}$$

where B^{us} is defined to be the u-channel amplitude crossing matrix. Thus

$$B^{us}=[(A^{us})^T]^{-1}\,, \tag{2.6a}$$

where "T" means transpose. For the case of elastic scattering,

$$B^{us}{}_{\text{elas}}=(A^{us})^T{}_{\text{elas}}\,. \tag{2.6b}$$

Similarly, one can show that

$$B^{ts}=[(A^{ts})^T]^{-1}\,. \tag{2.7}$$

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[1] M. Gell-Mann (unpublished); Phys. Rev. **125**, 1067 (1962).
[2] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).
[3] R. E. Cutkosky, Ann. Phys. (N. Y.) **23**, 415 (1963).
[4] J. J. de Swart, Nuovo Cimento **31**, 420 (1964).
[5] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1955).

With Eqs. (2.6) and (2.7) one can relate the crossing matrices obtained by using either projection operators or amplitudes.

b. Phases

In this work, we will use the Condon and Shortley and de Swart phase conventions.[6a] Since we shall be using the triplet representation, this will require some discussion for de Swart defined his phases with respect to the octet model.

De Swart showed that the relation between eigenstates of the representations $\mathbf{N}$ and $\mathbf{N}^*$ is[6]

$$|\mathbf{N},T,T_z,Y\rangle=\eta(\mathbf{N})(-1)^{T_z+\frac{1}{2}Y}\times|\mathbf{N}^*,T,-T_z,-Y\rangle^*. \quad (2.8)$$

He realized that since every irreducible representation (IR) in the octet model has a state with $T_z=Y=0$, $\eta(N)$ could be defined by

$$|\mathbf{N},T,0,0\rangle=|\mathbf{N}^*,T,0,0\rangle^*, \quad (2.9)$$

$$\eta(\mathbf{N})=1. \quad (2.10)$$

However, for the quark model, this will not do, as there is no $T_z=Y=0$ state. We want to obtain a phase convention that will reduce to (2.9) in the octet case and yet will allow

$$|\mathbf{3},\tfrac{1}{2},\tfrac{1}{2},1\rangle=-|\mathbf{3},\tfrac{1}{2},-\tfrac{1}{2},-1\rangle^* \quad (2.11a)$$

in the Sakata model and

$$|\mathbf{3},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{3}\rangle=-|\mathbf{3}^*,\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{3}\rangle^* \quad (2.11b)$$

in the quark model. The question of the phases involved in states and creation operators will be discussed in the next subsection.

What we suggest is that

$$\eta(\mathbf{N})=(-1)\exp\{\tau(\mathbf{N})/3-\sum_{i\in\mathbf{N}}(T_{zi}+Y_i/2)/N\}, \quad (2.12)$$

where $\tau(\mathbf{N})$ is the triality[7,8] of the $\mathbf{N}$ representation, the sum extends over all states of the IR, and N is the dimension of the IR.

The octet IR trialities are all equal to 0, while that of the $\mathbf{3}(\mathbf{3}^*)$ is equal to $1(-1)$. This will allow (2.11b) to hold. The reason for the $\sum_{i\in\mathbf{N}}$ is a "centering" phase. All octet IR's and quark triplets are "centered" about the Y, T_z origin, i.e., the sum equals zero. But, for instance, the Sakata triplet is not. This extra factor will compensate for that and, with the triality contribution, will yield (2.11a). Thus, (2.12) gives us the desired relations (2.11a,b), and reduces to de Swart's convention in the octet model.

With these phases we can now interchange quark and Sakata triplets. All our results, i.e., field operators and crossing matrices, are the same. One just has to change the labels. We shall arbitrarily use Sakata triplets for convenience, but we emphasize that our results are equally valid for quarks.

c. States and Fields

Due to the different phase conventions that have been used, some confusion exists over the phases that should be associated with the field operators versus the states. We desire, on the one hand, the simple convention of complex conjugation and, on the other hand, the Condon and Shortley and de Swart phases. This question is discussed by Carruthers.[9] We shall state the results pertinent for our purposes and refer the reader to the above reference for the details.

When we refer to a field operator, we will use the symbol of the particle, for example, π^+, but if we are referring to a state that is associated with a particle, we shall use $|\pi^+\rangle=|1,0,0\rangle=|T,T_z,Y\rangle$. (Note that, if there is no confusion, we shall omit the N symbol in the state.) No $(-)$ sign is associated here, as this will be identified with the fields operating on the vacuum. With this $|\pi^+\rangle$ we have that $T_-|\pi^+\rangle=+\sqrt{2}|\pi^0\rangle$, to agree with the Condon and Shortley phases.

The triplet field operators can be written

$$\begin{aligned}
\psi^{1*}|0\rangle&=|\psi^1\rangle\sim|p\rangle,\\
\psi^{2*}|0\rangle&=|\psi^2\rangle\sim|n\rangle,\\
\psi^{3*}|0\rangle&=|\psi^3\rangle\sim|\Lambda\rangle,\\
\psi^3|0\rangle=\varphi_3{}^*|0\rangle&=|\varphi_3\rangle\sim|\bar{\Lambda}\rangle,\\
\psi^2|0\rangle=\varphi_2{}^*|0\rangle&=|\varphi_2\rangle\sim|\bar{n}\rangle,\\
\psi^1|0\rangle=\varphi_1{}^*|0\rangle&=-|\varphi_1\rangle\sim-|\bar{p}\rangle.
\end{aligned} \quad (2.13)$$

The octet IR field operators can be written as

$$\begin{aligned}
&\psi^1\varphi_3=K^+,\\
&\psi^2\varphi_3=K^0,\\
&\psi^1\varphi_2=\pi^+,\\
&\psi^2\varphi_1=\pi^-=(\pi^+)^*,\\
&\psi^3\varphi_2=\bar{K}^0=(K^0)^*,\\
&\psi^3\varphi_1=K^-=(K^+)^*,\\
&(1/\sqrt{2})(\psi^2\varphi_2+\psi^1\varphi_1)=\pi^0,\\
&(1/\sqrt{6})(\psi^1\varphi_1-\psi^2\varphi_2+2\psi^3\varphi_3)=\eta^0.
\end{aligned} \quad (2.14)$$

[6] (a) E. V. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, England, 1935); J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963). (b) C. C. Shih (private communication) has verified the phase factor $(-1)\exp(T_z+\frac{1}{2}Y)$ by using a generalization of the methods of Sec. II of Ref. 14.

[7] M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (W. A. Benjamin, Inc., New York, 1964), Chap. 7.

[8] G. E. Baird and L. C. Biedenharn, *Proceedings of the 1964 Coral Gables Conference on Symmetry Principles at High Energy* (W. H. Freeman, San Francisco, 1964), p. 58.

[9] P. Carruthers, *Unitary Symmetry in Strong Interactions* (Interscience Publishers, Inc., New York, to be published). This book is a comprehensive treatment on the subject of $SU(3)$ designed especially for the reader who is learning the subject.

Their relation to the octet states is

$$\begin{bmatrix} |K^+\rangle \\ |K^0\rangle \\ |\pi^+\rangle \\ |\pi^0\rangle \\ |\pi^-\rangle \\ |\eta\rangle \\ |\bar{K}^0\rangle \\ |K^-\rangle \end{bmatrix} = \begin{bmatrix} K^+ \\ K^0 \\ -\pi^+ \\ \pi^0 \\ \pi^- \\ \eta \\ \bar{K}^0 \\ -K^- \end{bmatrix}^* |0\rangle . \qquad (2.15)$$

The states $|\alpha\rangle$, $\alpha=1,\cdots,8$, which will be used later are given by

$$\begin{bmatrix} \frac{1}{\sqrt{2}}(\pi^+ + \pi^-) \\ \frac{i}{\sqrt{2}}(\pi^+ - \pi^-) \\ \pi^0 \\ \frac{1}{\sqrt{2}}(K^+ + K^-) \\ \frac{i}{\sqrt{2}}(K^+ - K^-) \\ \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0) \\ \frac{i}{\sqrt{2}}(K^0 - \bar{K}^0) \\ \eta \end{bmatrix}^* |0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}\{-|1,0,0\rangle + |1,-1,0\rangle\} \\ \frac{i}{\sqrt{2}}\{|1,1,0\rangle + |1,-1,0\rangle\} \\ |1,0,0\rangle \\ \frac{1}{\sqrt{2}}\{|\frac{1}{2},\frac{1}{2},1\rangle - |\frac{1}{2},-\frac{1}{2},-1\rangle\} \\ -\frac{i}{\sqrt{2}}\{|\frac{1}{2},\frac{1}{2},1\rangle + |\frac{1}{2},-\frac{1}{2},1\rangle\} \\ \frac{1}{\sqrt{2}}\{|\frac{1}{2},-\frac{1}{2},1\rangle + |\frac{1}{2},\frac{1}{2},1\rangle\} \\ \frac{i}{\sqrt{2}}\{|\frac{1}{2},-\frac{1}{2},1\rangle - |\frac{1}{2},\frac{1}{2},-1\rangle\} \\ |0,0,0\rangle \end{bmatrix} \qquad (2.16)$$

and transforms as the regular representation with generators **F**. [See Eq. (3.23).]

d. Direct Product States

$SU(3)$ states Ψ can be given as a product of two IR's by

$$\Psi\begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ & & \nu \end{pmatrix} = \sum_{\nu_1\nu_2} \begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix} |\mathfrak{u}_1,\nu_1\rangle |\mathfrak{u}_2,\nu_1\rangle , \qquad (2.17)$$

where

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ \nu_1 & \nu_2 & \nu \end{pmatrix}$$

are the Clebsch-Gordan (CG) coefficients for $SU(3)$, $\mathfrak{u}$ denotes the IR, $\nu \equiv (T,T_z,Y)$ and the subscript γ differentiates between different IR's of the same dimension. (We are following de Swart's notation,[4,6] so here the Ψ's and χ's are states, not fields.)

$SU(2)$ (isospin) wave functions can be given by

$$\chi\begin{pmatrix} \mu_1 & \mu_2 & & & \\ T_1Y_1 & T_2Y_2 & T & T_z & Y \end{pmatrix} = \sum_{T_{1z}T_{2z}} C(T_1T_2T;\, T_{1z}T_{2z}T_z) |\mathfrak{u}_1,\nu_1\rangle |\mathfrak{u}_2,\nu_2\rangle . \qquad (2.18)$$

Combining Eqs. (2.17) and (2.18) implies

$$\Psi\begin{pmatrix} \mu_1 & \mu_2 & \mu_\gamma \\ & & \nu \end{pmatrix} = \sum_{\substack{T_1Y_1 \\ T_2Y_2}} \left(\begin{matrix} \mu_1 & \mu_2 \\ T_1Y_1 & T_2Y_2 \end{matrix}\middle| \begin{matrix} \mu_\gamma \\ TY \end{matrix}\right) \times \chi\begin{pmatrix} \mu_1 & \mu_2 & & & \\ T_1Y_1 & T_2Y_2 & T & T_z & Y \end{pmatrix}, \qquad (2.19)$$

where

$$\left(\begin{matrix} \mu_1 & \mu_2 \\ T_1Y_1 & T_2Y_2 \end{matrix}\middle| \begin{matrix} \mu_\gamma \\ TY \end{matrix}\right)$$

TABLE I. Isoscalar factors for 3×8. Given are the isoscalar factors

$$\left(\begin{matrix} 3 & 8 \\ T_1Y_1 & T_2Y_2 \end{matrix}\middle| \begin{matrix} \mu_\gamma \\ TY \end{matrix}\right)$$

for the series $\mathbf{3}\times\mathbf{8}=\mathbf{3}+\mathbf{6^*}+\mathbf{15}$.

T_1	Y_1	T_2	Y_2	μ_γ: 15	6*	3
$Y=2,\ T=1$						
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1		
$Y=2,\ T=0$						
$\frac{1}{2}$	1	$\frac{1}{2}$	1		-1	
$Y=1,\ T=\frac{3}{2}$						
$\frac{1}{2}$	1	1	0	1		
$Y=-1,\ T=\frac{1}{2}$						
0	0	$\frac{1}{2}$	-1	1		
$Y=1,\ T=\frac{1}{2}$						
$\frac{1}{2}$	1	1	0	$\frac{1}{4}$	$-\frac{1}{4}\sqrt{6}$	$\frac{3}{4}$
$\frac{1}{2}$	1	0	0	$\frac{3}{4}$	$\frac{1}{4}\sqrt{6}$	$\frac{1}{4}$
0	0	$\frac{1}{2}$	1	$\frac{1}{4}\sqrt{6}$	$-\frac{1}{2}$	$-\frac{1}{4}\sqrt{6}$
$Y=0,\ T=1$						
0	0	1	0	$1/\sqrt{2}$	$-1/\sqrt{2}$	
$\frac{1}{2}$	1	$\frac{1}{2}$	-1	$1/\sqrt{2}$	$1/\sqrt{2}$	
$Y=0,\ T=0$						
$\frac{1}{2}$	1	$\frac{1}{2}$	-1	$\frac{1}{2}$		$\frac{1}{2}\sqrt{3}$
0	0	0	0	$\frac{1}{2}\sqrt{3}$		$-\frac{1}{2}$

are the isoscalar factors. Comparing Eqs. (2.17), (2.18), and (2.19) shows that

$$\begin{pmatrix}\mu_1 & \mu_2 & \mu_\gamma\\ \nu_1 & \nu_2 & \nu\end{pmatrix}=C(T_1T_2T;\,T_{1z}T_{2z}T)\times\left(\begin{matrix}\mu_1 & \mu_2\\ T_1Y_1 & T_2Y_2\end{matrix}\right|\left.\begin{matrix}\mu_\gamma\\ TY\end{matrix}\right). \tag{2.20}$$

Tables II–VI of Ref. 6 contain the isoscalar factors for **8×8**, **8×10**, **8×27**, **10×10** and **10×10***. Tables I and II of this paper give the isoscalar factors for **3×8** **=3+6*+15** and **3×3*=8+1**, which are given by Carruthers.[9] They were obtained by demanding that $T_\pm$ and $V_\pm$ obey Condon and Shortley phase conventions, i.e.,

$$T_\pm|T,T_z,Y\rangle=[(T\mp T_z)(T\pm T_z+1)]^{1/2}\times|T,T_z\pm1,Y\rangle. \tag{2.21}$$

The $SU(3)$ Clebsch-Gordan coefficients were taken to agree with de Swart's phases.[6] One can consult the literature for information on the procedure,[6,9,10] but the reader must take care. Many conventions have been used to define the operators $E_{\pm\alpha}$, $T_\pm$, $V_\pm$, $U_\pm$, and the phases that connect them to the states.

TABLE II. Isoscalar factors for 3×3. Given are the isoscalar factors

$$\left(\begin{matrix}3 & 3^*\\ T_1Y_1 & T_2Y_2\end{matrix}\right|\left.\begin{matrix}\mu_\gamma\\ TY\end{matrix}\right)$$

for the series **3×3*=1+8**.

T_1	Y_1	T_2	Y_2	8	1	μ_γ
		$Y=1,\ T=\frac{1}{2}$				
$\frac{1}{2}$	1	0	0	1		
		$Y=0,\ T=1$				
$\frac{1}{2}$	1	$\frac{1}{2}$	-1	1		
		$Y=-1,\ T=\frac{1}{2}$				
0	0	$\frac{1}{2}$	-1	1		
		$Y=0,\ T=0$				
$\frac{1}{2}$	1	$\frac{1}{2}$	-1	$1/\sqrt{3}$	$\sqrt{\frac{2}{3}}$	
0	0	0	0	$\sqrt{\frac{2}{3}}$	$-1/\sqrt{3}$	

3. $SU(3)$ CROSSING MATRICES

Using the orthogonality conditions given in Sec. 11 of Ref. 6, one obtains the relation

$$\left(\begin{matrix}\mu_3 & \mu_4\\ \nu_3 & \nu_4\end{matrix}\right|F\left|\begin{matrix}\mu_1 & \mu_2\\ \nu_1 & \nu_2\end{matrix}\right)=\sum_{\mu\nu\gamma\beta}\begin{pmatrix}\mu_3 & \mu_4 & \mu_\beta\\ \nu_3 & \nu_4 & \nu\end{pmatrix}F^s{}_{\mu\beta\gamma}\begin{pmatrix}\mu_1 & \mu_2 & \mu_\gamma\\ \nu_1 & \nu_2 & \nu\end{pmatrix}, \tag{3.1}$$

where F^s is the s-channel amplitude. Similar results for the u and t channels are

$$\left(\begin{matrix}\mu_3 & \mu_2^*\\ \nu_3 & \nu_2^*\end{matrix}\right|F\left|\begin{matrix}\mu_1 & \mu_4^*\\ \nu_1 & \nu_4^*\end{matrix}\right)=\sum_{\mu\nu\beta\gamma}\begin{pmatrix}\mu_3 & \mu_2^* & \mu_\beta\\ \nu_3 & \nu_2^* & \nu\end{pmatrix}F^u{}_{\mu\beta\gamma}\begin{pmatrix}\mu_1 & \mu_4^* & \mu_\gamma\\ \nu_1 & \nu_4^* & \nu\end{pmatrix}, \tag{3.2}$$

$$\left(\begin{matrix}\mu_2^* & \mu_4\\ \nu_2^* & \nu_4\end{matrix}\right|F\left|\begin{matrix}\mu_1 & \mu_3^*\\ \nu_1 & \nu_3^*\end{matrix}\right)=\sum_{\mu\nu\beta\gamma}\begin{pmatrix}\mu_2^* & \mu_4 & \mu_\beta\\ \nu_2^* & \nu_4 & \nu\end{pmatrix}F^t{}_{\mu\beta\gamma}\begin{pmatrix}\mu_1 & \mu_3^* & \mu_\gamma\\ \nu_1 & \nu_2^* & \nu\end{pmatrix}, \tag{3.3}$$

where $\nu^*\equiv(T,\,-T_z,\,-Y)$.

Owing to (2.8) and (2.12), we have

$$\left(\begin{matrix}\mu_3 & \mu_4\\ \nu_3 & \nu_4\end{matrix}\right|F\left|\begin{matrix}\mu_1 & \mu_2\\ \nu_1 & \nu_2\end{matrix}\right)=(-)^{\pm\bar{\nu}_2\pm\bar{\nu}_4}\left(\begin{matrix}\mu_3 & \mu_2^*\\ \nu_3 & \nu_2^*\end{matrix}\right|F\left|\begin{matrix}\mu_1 & \mu_4^*\\ \nu_1 & \nu_4^*\end{matrix}\right), \tag{3.4}$$

$$\left(\begin{matrix}\mu_3 & \mu_4\\ \nu_3 & \nu_4\end{matrix}\right|F\left|\begin{matrix}\mu_1 & \mu_2\\ \nu_1 & \nu_2\end{matrix}\right)=(-1)^{\pm\bar{\nu}_2\pm\bar{\nu}_3}\left(\begin{matrix}\mu_2^* & \mu_4\\ \nu_2^* & \nu_4\end{matrix}\right|F\left|\begin{matrix}\mu_1 & \mu_3^*\\ \nu_1 & \nu_3^*\end{matrix}\right), \tag{3.5}$$

where, for us,

$$\bar{\nu}\equiv T_z+\tfrac{1}{2}Y+\tfrac{1}{3}\tau(\mathbf{N})-\sum_{i\epsilon\mathbf{N}}(T_{zi}+\tfrac{1}{2}Y_i)/N \tag{3.6}$$

$$\equiv T_z+\tfrac{1}{2}\mathcal{Y}. \tag{3.7}$$

The point to be made is that $\bar{\nu}$ is always equal to (±1) for our representations. Thus, we can use the $\pm$ at will, depending on what is convenient for sums that will later come into play.

[10] The book listed in Ref. 7 has a large collection of reprints of the basic papers in the field.

Combining the above and the Clebsch-Gordan orthogonality relations in (11.2a,b) of Ref. 6, yields, for the B^{us} and B^{ts} crossing matrices,

$$(\mu\beta\gamma|B^{us}|\mu'\beta'\gamma')=\sum_{\substack{\nu_1\nu_2{}^*\\ \nu_3\nu_4{}^*\\ \nu'}}(-1)^{\pm\bar{\nu}_2\pm\bar{\nu}_4}\begin{pmatrix}\mu_3 & \mu_2{}^* & \mu & \\ \nu_3 & \nu_2{}^* & \nu & \beta\end{pmatrix}\begin{pmatrix}\mu_1 & \mu_4{}^* & \mu & \\ \nu_1 & \nu_4{}^* & \nu & \gamma\end{pmatrix}\begin{pmatrix}\mu_3 & \mu_4 & \mu' & \\ \nu_3 & \nu_4 & \nu' & \beta'\end{pmatrix}\begin{pmatrix}\mu_1 & \mu_2 & \mu' & \\ \nu_1 & \nu_2 & \nu' & \gamma'\end{pmatrix}, \tag{3.8}$$

$$(\mu\beta\gamma|B^{ts}|\mu'\beta'\gamma')=\sum_{\substack{\nu_1\nu_3{}^*\\ \nu_2{}^*\nu_4\\ \nu'}}(-1)^{\pm\bar{\nu}_2\pm\bar{\nu}_3}\begin{pmatrix}\mu_2{}^* & \mu_4 & \mu & \\ \nu_4{}^* & \nu_4 & \nu & \beta\end{pmatrix}\begin{pmatrix}\mu_1 & \mu_3{}^* & \mu & \\ \nu_1 & \nu_3{}^* & \nu & \gamma\end{pmatrix}\begin{pmatrix}\mu_3 & \mu_4 & \mu' & \\ \nu_3 & \nu_4 & \nu' & \beta'\end{pmatrix}\begin{pmatrix}\mu_1 & \mu_2 & \mu' & \\ \nu_1 & \nu_2 & \nu' & \gamma'\end{pmatrix}, \tag{3.9}$$

where

$$F^u{}_{\mu\beta\gamma}=\sum_{\mu'\beta'\gamma'}(\mu\beta\gamma|B^{us}|\mu'\beta'\gamma')F^s{}_{\mu'\beta'\gamma'}\,, \tag{3.10}$$

$$F^t{}_{\mu\beta\gamma}=\sum_{\mu'\beta'\gamma'}(\mu\beta\gamma|B^{ts}|\mu'\beta'\gamma')F^s{}_{\mu'\beta'\gamma'}\,. \tag{3.11}$$

These are the inverses of de Swart's β_{III} and β_{II} matrices.[4]

It is possible to further reduce Eqs. (3.8) and (3.9). B^{us} can be written as

$$B^{us}=\sum_{\substack{T_1Y_1T_3Y_3\\ T_2Y_2{}^*T_4Y_4{}^*\\ T'Y'}}(-1)^{\pm\mathcal{Y}_2/2\pm\mathcal{Y}_4/4}\left(\begin{array}{cc|cc}\mu_3 & \mu_2{}^* & \mu & \\ T_3Y_3 & T_2Y_2{}^* & TY & \beta\end{array}\right)\left(\begin{array}{cc|cc}\mu_1 & \mu_4{}^* & \mu & \\ T_1Y_1 & T_4Y_4{}^* & TY & \gamma\end{array}\right)$$
$$\times Z^{us}(T_1T_2T_3T_4;\,T'T)\left(\begin{array}{cc|cc}\mu_3 & \mu_4 & \mu' & \\ T_3Y_3 & T_4Y_4 & T'Y' & \beta'\end{array}\right)\left(\begin{array}{cc|cc}\mu_1 & \mu_2 & \mu' & \\ T_1Y_1 & T_2Y_2 & T'Y' & \gamma'\end{array}\right), \tag{3.12}$$

where

$$Z^{us}=\frac{1}{2T+1}\sum_m\sum_{\substack{m_1m_3\\ m_2{}^*m_4{}^*\\ m'}}(-1)^{\pm m_2\pm m_3}C(T_3,\,T_2,\,T;\,m_3,\,-m_2,\,m)$$
$$\times C(T_1,\,T_4,\,T;\,m_1,\,-m_4,\,m)C(T_3,T_4,T';\,m_3,m_4,m')C(T_1,T_2,T';\,m_1,m_2,m)\,. \tag{3.12}$$

For clarity, we have set $m\equiv T_z$. The factor $(1/(2T+1)\sum_m)$ is equal to 1, since (3.8) is valid for any m.

Implicit in this expression are four Kronecker deltas: (i) $\delta(m',\,m_1+m_2)$, (ii) $\delta(m_1+m_2,\,m_3+m_4)$, (iii) $\delta(m,\,m_1-m_4)$, and (iv) $\delta(m_1-m_4,\,m_3-m_2)$, three of which are independent. By summing over m_3, m', and m, using the Clebsch-Gordan coefficient identities found in Chap. 3 of Rose,[11] and choosing $(+\bar{\nu}_2)$ and $(-\bar{\nu}_4)$ as the phases, one finds that $(\mu_3\equiv -m_1-m_2+m_4)$

$$Z^{us}=\frac{1}{2T+1}\left[\frac{(2T'+1)(2T+1)}{(2T_4+1)^2}\right]^{1/2}\sum_{m_4{}^*}\Big\{\sum_{m_1m_2{}^*}C(T_1T_2T';\,m_1,m_2)$$
$$\times C(T'T_3T_4;\,m_1+m_2,\,\mu_3)C(T_1TT_4;\,m_1,\,m_2-\mu_3)C(T_2T_3T;\,m_2,\mu_3)\Big\}(-1)^{-T_1+T_3}. \tag{3.14}$$

Combining (6.3) and (6.6a) of Rose[11] and the identity $\sum_{m_4{}^*}(1)=(2T_4+1)$, we have

$$Z^{us}=(-1)^{-T_1+T_3}(2T'+1)W(T_1T_2T_4T_3;\,T'T)\,, \tag{3.15}$$

where $W(abcd;\,ef)$ is the Racah coefficient. Thus

$$(\mu\beta\gamma|B^{us}|\mu'\beta'\gamma')=\sum_{\substack{T_1Y_1T_2Y_2\\ T_3Y_3T_4Y_4}}(-1)^{(\mathcal{Y}_2-\mathcal{Y}_4)/2}\left(\begin{array}{cc|cc}\mu_3 & \mu_2{}^* & \mu & \\ T_3Y_3 & T_2-Y_2 & TY & \beta\end{array}\right)\left(\begin{array}{cc|cc}\mu_1 & \mu_4{}^* & \mu & \\ T_1Y_1 & T_4-Y_4 & TY & \gamma\end{array}\right)$$
$$\times\sum_{T'Y'}Z^{us}(T_1T_2T_3T_4;\,T'T)\left(\begin{array}{cc|cc}\mu_3 & \mu_4 & \mu' & \\ T_3Y_3 & T_4T_4 & T'Y' & \beta'\end{array}\right)\left(\begin{array}{cc|cc}\mu_1 & \mu_2 & \mu' & \\ T_1Y_1 & T_2Y_2 & T'Y' & \gamma'\end{array}\right) \tag{3.16}$$

$$Z^{us}(T_1T_2T_3T_4;\,T'T)=(-1)^{-T_1+T_3}(2T'+1)W(T_1T_2T_4T_3;\,T'T) \tag{3.17}$$

$$=(-1)^{T_2+2T_3+T_4}(2T'+1)\begin{Bmatrix}T_1 & T_2 & T'\\ T_3 & T_4 & T\end{Bmatrix}\,. \tag{3.18}$$

$$\begin{Bmatrix}a & b & s\\ c & d & u\end{Bmatrix}$$

[11] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

is the 6-j symbol, for which tables are available.[12] The 6-j symbol has the advantage of easy manipulation of its indices. The reader is referred to the tables by Rotenburg *et al.*,[12] or Chap. 24 of Wigner[13] for further details.

A similar expression is obtained for the t-channel crossing matrix:

$$(\mu\beta\gamma|B^{ts}|\mu'\beta'\gamma')=\sum_{\substack{T_1Y_1T_2Y_2\\T_3Y_3T_4Y_4}}(-1)^{(-Y_2+Y_3)/2}\left(\begin{array}{cc|cc}\mu_2^* & \mu_4 & \mu & \beta\\ T_2-Y_2 & T_4Y_4 & TY & \end{array}\right)\left(\begin{array}{cc|cc}\mu_1 & \mu_3^* & \mu & \gamma\\ T_1Y_1 & T_3-Y_3 & TY & \end{array}\right)$$

$$\times\sum_{T'Y'}Z^{ts}(T_1T_2T_3T_4;T'T)\left(\begin{array}{cc|cc}\mu_3 & \mu_4 & \mu' & \beta'\\ T_3Y_3 & T_4Y_4 & T'Y' & \end{array}\right)\left(\begin{array}{cc|cc}\mu_1 & \mu_2 & \mu' & \gamma'\\ T_1Y_1 & T_2Y_2 & T'Y' & \end{array}\right), \quad (3.19)$$

$$Z^{ts}(T_1T_2T_3T_4;T'T)=(-1)^{T_1+T_4}(2T'+1)W(T_1T'TT_4;T_2T_3) \quad (3.20)$$

$$=(-1)^{-T-T'}(2T'+1)\begin{Bmatrix}T_1 & T' & T_2\\ T_4 & T & T_3\end{Bmatrix}. \quad (3.21)$$

Except for phase factors, the expressions Z^{us} and Z^{ts} are identical with the $SU(2)$ crossing matrices for arbitrary isospin that were obtained by Carruthers and Krisch.[14] However, since the phase factor varies depending on whether the crossed $SU(2)$ particles are pair-conjugate or self-conjugate, it is an unnecessary complication to express our $SU(3)$ crossing matrices in terms of $SU(2)$ crossing matrices. [We note here that the crossing matrices are similar in structure to the $SU(3)$ Racah coefficients given by Krammer.[15]]

The expressions (3.16) and (3.19) are independent of T and Y contained in the IR μ. This can be put to advantage, for a judicious choice of T and Y will reduce the work involved in calculating a matrix element. For example, if we take $(T,Y)=(0,2)$, then the $(6^*|B^{us}|3)$ element reduces to one term.

$$(6^*|B^{us}|3)=\left(\begin{array}{cc|c}3 & 8^* & 6^*\\ \frac{1}{2}\ 1 & \frac{1}{2}\ 1 & 0\ 2\end{array}\right)\left(\begin{array}{cc|c}3 & 8^* & 6^*\\ \frac{1}{2}\ 1 & \frac{1}{2}\ 1 & 0\ 2\end{array}\right)W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2};0\,0)\left(\begin{array}{cc|c}3 & 8 & 3\\ \frac{1}{2}\ 1 & \frac{1}{2}\ -1 & 0\ 0\end{array}\right)\left(\begin{array}{cc|c}3 & 8 & 3\\ \frac{1}{2}\ 1 & \frac{1}{2}\ -1 & 0\ 0\end{array}\right)$$

$$=-\tfrac{3}{8}.$$

The B^{us} and B^{ts} crossing matrices for $\mathbf{3\times8=3+6^*+15}$ are listed in Tables III and IV. The u crossing matrix agrees with the results previously found by Capps[16] and Cutkosky.[17] The u and t crossing matrices for $\mathbf{8\times8}$ are given in Table V. The top sign is the element for the B^{us} matrix, the bottom that of the B^{ts} matrix. The two matrices can be related by

$$(\mu\beta\gamma|B^{ts}|\mu'\beta'\gamma')=\xi_1(88\mu_\beta)\times\xi_1(88\mu'_{\beta'})(\mu\beta\gamma|B^{us}|\mu'\beta'\gamma'), \quad (3.22)$$

TABLE III. The elements of the crossing matrix $(\mu|B^{us}(3,8,3,8)|\mu')$ multiplied by 8.

$\mu \backslash \mu'$	3	6*	15
3	−1	−6	15
6*	−3	6	5
15	3	2	3

[12] M. Rotenburg, R. Bivins, N. Metropolis, and J. K. Wooten, Jr., *The 3-j and 6-j Symbols* (The Technology Press, MIT, Cambridge, Massachusetts, 1959).

[13] E. P. Wigner, *Group Theory and its Applications to the Quantum Mechanics of Atomic Structure* (Academic Press Inc., New York, 1959).

[14] P. Carruthers and J. P. Krisch, Ann. Phys. (N. Y.) **33**, 1 (1965).

[15] M. Krammer, Acta Phys. Austriaca, Suppl. **1**, 183 (1964).

[16] R. H. Capps, Nuovo Cimento **34**, 932 (1964).

[17] R. E. Cutkosky, Phys. Rev. Letters **12**, 530 and Errata 572 (1964).

where the ξ_1's are given in Table I of Ref. 6. The $\mathbf{8\times8}$ matrices agree with those of Cutkosky[3] and de Swart,[4] respectively.

It should here be stated that in the literature $8_A=8_2=8'$ is that octet which transforms according to the regular representation (R.R.) and which has been formed by $V^*\mathbf{F}V$ where V is a vector that belongs to the R.R. The octet $8_S=8_1=8$ is that octet which transforms according to the R.R. that has been formed by $V^*\mathbf{D}V$. The matrices $\mathbf{F}$ and $\mathbf{D}$ are the two sets of eight 8×8 matrices that are formed from the antisymmetric structure constants f_{ijk} and the symmetric constants d_{ijk} by

$$(F^k)_{ij}\equiv -if_{ijk}, \quad (3.23)$$

$$(D^k)_{ij}\equiv d_{ijk}; \quad (3.24)$$

8_{12} means $8_2\to 8_1$.

Our conventions for the listing of the crossing matrices are the same as those in Ref. 14 and have the following physical interpretation. Suppose there is scat-

TABLE IV. The elements of the crossing matrix $(\mu|B^{ts}(3,8,3,8)|\mu')$ divided by $\sqrt{6}$.

$\mu_{\beta\gamma} \backslash \mu'$	3	6*	15
1	$\frac{1}{4}$	$\frac{1}{2}$	5/4
8_28	$\frac{3}{16}$	$\frac{1}{8}$	$-\frac{5}{16}$
8_18	$-\frac{1}{16}\sqrt{5}$	$\frac{1}{8}\sqrt{5}$	$-\frac{1}{16}\sqrt{5}$

TABLE V. The elements of the crossing matrices $(\mu\beta\gamma|B^{us}(8,8,8,8)|\mu'\beta'\gamma')$ and $(\mu\beta\gamma|B^{ts}(8,8,8,8)|\mu'\beta'\gamma')$. The upper sign refers to the u matrix and the lower sign to the t matrix.

$\mu\beta\gamma$ \ $\mu'\beta'\gamma'$	27	10	10*	8_{11}	8_{12}	8_{21}	8_{22}	1
27	7/40	$\pm\frac{1}{12}$	$\pm\frac{1}{12}$	$\frac{1}{5}$	0	0	$\pm\frac{1}{3}$	$\frac{1}{8}$
10	$\pm 9/40$	$\frac{1}{4}$	$\frac{1}{4}$	$\pm\frac{2}{5}$	$\pm 1/\sqrt{5}$	$1/\sqrt{5}$	0	$\mp\frac{1}{8}$
10*	$\pm 9/40$	$\frac{1}{4}$	$\frac{1}{4}$	$\pm\frac{2}{5}$	$\mp 1/\sqrt{5}$	$-1/\sqrt{5}$	0	$\mp\frac{1}{8}$
8_{11}	27/40	$\pm\frac{1}{2}$	$\pm\frac{1}{2}$	$-\frac{3}{10}$	0	0	$\mp\frac{1}{2}$	$\frac{1}{8}$
8_{12}	0	$\pm\frac{1}{4}\sqrt{5}$	$\mp\frac{1}{4}\sqrt{5}$	0	$-\frac{1}{2}$	$\pm\frac{1}{2}$	0	0
8_{21}	0	$\frac{1}{4}\sqrt{5}$	$-\frac{1}{4}\sqrt{5}$	0	$\pm\frac{1}{2}$	$-\frac{1}{2}$	0	0
8_{22}	$\pm 9/8$	0	0	$\mp\frac{1}{2}$	0	0	$\frac{1}{2}$	$\mp\frac{1}{8}$
1	27/8	$\mp 5/4$	$\mp 5/4$	1	0	0	∓ 1	$\frac{1}{8}$

tering in the s channel in the μ_s multiplet state. In the t channel, the crossed scattering has the physical interpretation of being the exchange of a state in the multiplet μ_s which gives rise to forces in allowed multiplets μ_t. To find the magnitudes of the forces, one reads down the column μ_s. The force in the μ_t state is proportional to the entry in the row labeled by μ_t.

4. PROJECTION OPERATORS

In this section, we will obtain the projection operators for the direct and crossed channels of $\mathbf{3}\times\mathbf{8}$. We will use this and other methods to obtain the crossing matrices as illustrations of tensorial techniques that can be used to find crossing symmetry. Cutkosky[3] and Neville[18] have discussed the $\mathbf{8}\times\mathbf{8}$ case. However, since Neville was scattering bosons, he did not encounter the phase difficulties of the 8_{12} and 8_{21} matrix elements.

When discussing the $\mathbf{3}\times\mathbf{8}$ system we will label the channels by

$$\begin{aligned}(s)\quad & B_j+P_\beta\rightarrow B_i+P_\alpha\,,\\ (u)\quad & B_j+\bar{P}_\alpha\rightarrow B_i+\bar{P}_\beta\,,\\ (t)\quad & B_j+\bar{B}_i\rightarrow \bar{P}_\beta+P_\alpha\,.\end{aligned}\tag{4.1}$$

The projection operators will act on states S^jA^β, where S^j and A^β are the three-component and eight-component number states given by (2.13) and (2.16). $(P_3)_{\alpha\beta;ij}$ can be found immediately. It is analogous to

$$(P_{T=1/2})_{\alpha\beta;ij}=\frac{\tau^{(\alpha)i}{}_n\tau^{(\beta)n}{}_j}{\tau^2}=\tfrac{1}{3}\tau^{(\alpha)i}{}_n\tau^{(\beta)n}{}_j\,,\tag{4.2}$$

that is valid for isospin.[19] [The $\tau^{(\alpha)}$ are Pauli spin matrices.] The result here is

$$\begin{aligned}(P_3)_{\alpha\beta;ij}&=\frac{\lambda^{(\alpha)i}{}_n\lambda^{(\beta)n}{}_j}{\lambda^2}=\tfrac{3}{16}\lambda^{(\alpha)i}{}_n\lambda^{(\beta)n}{}_j\\ &=\tfrac{1}{8}\delta_{\alpha\beta}\delta^i{}_j+\tfrac{3}{16}d_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_j+\tfrac{3}{16}if_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_j\,,\end{aligned}\tag{4.3}$$

To proceed further, one generalizes to realize that we are considering tensors of the type

$$D^{(\alpha)ki}{}_{jm}\sim N^kM^{(\alpha)i}{}_m\,,\tag{4.4}$$

where N^k and $M^{(\alpha)i}{}_m$ come from the $\mathbf{3}$ and $\mathbf{8}\sim\mathbf{3}\times\mathbf{3}^*$ tensors, respectively. The fourth symbol is to be contracted with the j of the state S^j.

The projection operators will have forms such as

$$P\sim\lambda^{(\beta)m}{}_kD^{(\alpha)ki}{}_{jm}\,.\tag{4.5}$$

Our representations will come from the traceless tensors that are symmetric and antisymmetric in the upper indices. Thus

$$D\begin{pmatrix}15\\6^*\end{pmatrix}=\begin{pmatrix}r\\t\end{pmatrix}\Big[\delta^k{}_j\lambda^{(\alpha)i}{}_m\pm\lambda^{(\alpha)k}{}_m\delta^i{}_j-\begin{pmatrix}s\\a\end{pmatrix}\lambda^{(\alpha)k}{}_j\delta^i{}_m-\begin{pmatrix}s'\\a'\end{pmatrix}\delta^k{}_m\lambda^{(\alpha)i}{}_j\Big]\,,\tag{4.6}$$

where r and t are normalization constants, and s, s', a, and a' are constants to be determined to make $D\begin{pmatrix}15\\6^*\end{pmatrix}$ traceless.

Taking the symmetric case and demanding that

$$\operatorname*{Tr}_{m=i}D(15)=\operatorname*{Tr}_{m=k}D(15)=0\tag{4.7}$$

yields two equations in s and s' which have the solution $s'=s=\frac{1}{4}$. Putting this into (4.5) and (4.6),

$$(P_{15})_{\alpha\beta;ij}=r\big[\lambda^{(\alpha)i}{}_m\lambda^{(\beta)m}{}_j+2\delta_{\alpha\beta}\delta^i{}_j-\tfrac{1}{4}\lambda^{(\beta)i}{}_m\lambda^{(\alpha)m}{}_j-0\big]\,.\tag{4.8}$$

Since the last term is zero, what we are doing is subtracting the third term from the symmetric combination of $D^{ki}{}_m$ and $D^{ik}{}_m$. But the third term is the P_3 projection operator and came from contracting i and m in $D^{ki}{}_m$, which then obviously is a three-component tensor. Thus, we are subtracting the appropriate amount of the P_3 projection operator from the symmetric and antisymmetric tensors to obtain P_{15} and P_{6^*}. (Edmonds[20] uses a similar method to prove tensorially that $\mathbf{3}\times\mathbf{8}=\mathbf{3}+\mathbf{6}^*+\mathbf{15}$.)

Writing (4.8) in terms of the d's and f's,

$$(P_{15})_{\alpha\beta;ij}=r\big[\tfrac{5}{4}\delta_{\alpha\beta}\delta^i{}_j+\tfrac{3}{4}d_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_j-\tfrac{5}{4}if_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_j\big]\,.\tag{4.9}$$

r can be determined by using the fact that $(P_{15})^2=P_{15}$.

[18] D. E. Neville, Phys. Rev. **132**, 844 (1963).

[19] P. Carruthers, Ann. Phys. (N. Y.) **14**, 229 (1961), Appendix A.

[20] A. R. Edmonds, Proc. Roy. Soc. (London) **A268**, 567 (1962).

However, it is easier to use a trick that is valid in this case[21]:

$$\operatorname*{Tr}_{i=j} \sum_{\alpha,\beta} (P_N)_{\alpha\beta;ij} = N. \qquad (4.10)$$

Here $N=15$, if $r=\frac{1}{4}$, so that

$$(P_{15})_{\alpha\beta;ij} = \tfrac{5}{16}\delta_{\alpha\beta}\delta^i{}_j + \tfrac{3}{16}d_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_j - \tfrac{5}{16}if_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_j. \qquad (4.11)$$

Similarly,

$$(P_{6^*})_{\alpha\beta;ij} = \tfrac{1}{4}\delta_{\alpha\beta}\delta^i{}_j - \tfrac{3}{8}d_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_m - \tfrac{1}{8}if_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_j. \qquad (4.12)$$

Since $f_{\alpha\beta\gamma} = -f_{\beta\alpha\gamma}$ and $d_{\alpha\beta\gamma} = d_{\beta\alpha\gamma}$, it is a matter of algebra to find the u crossing matrix. The result is

$$(P_u)_{\beta\alpha;ij} = A_{us}(P_s)_{\alpha\beta;ij} = (B_{us})^T(P_s)_{\alpha\beta;ij}, \qquad (4.13)$$

where B^{us} is given by Table III, in agreement with (2.6).

Obtaining the t crossing matrix is now relatively simple. We can say immediately, because of transformation properties, that

$$\begin{aligned} P_1 &\sim \delta_{\alpha\beta}, \\ P_{8_18} &\sim d_{\alpha\beta\gamma}\lambda^{(\gamma)}, \\ P_{8_28} &\sim if_{\alpha\beta\gamma}\lambda^{(\gamma)}, \end{aligned} \qquad (4.14)$$

meaning that

$$\begin{bmatrix} P_1 \\ P_{8_28} \\ P_{8_18} \end{bmatrix} = \begin{bmatrix} a & a & a \\ 3b & b & -b \\ c & -c & \frac{1}{5}c \end{bmatrix} \begin{bmatrix} P_3 \\ P_{6^*} \\ P_{15} \end{bmatrix}.$$

The determination of three matrix elements will give us the three normalizations and hence the entire matrix. We could use the results of Sec. 3, but there exist methods using projection operators. For example,

$$\begin{aligned} F &\equiv \langle P_2\bar{P}_3 | P_{8_28} | B^j\bar{B}_i\rangle \\ &= \langle P_2\bar{P}_3 | \{ \sum_{T,T_z,Y\in 8} | 8_2,T,T_z,Y\rangle\langle 8_2T,T_z,Y| \} | B^j\bar{B}_i\rangle. \end{aligned}$$

Using the states given in Sec. 2 and the $SU(3)$ Clebsch-Gordan coefficients in the tables by McNamee and Chilton,[22] we have

$$\begin{aligned} F = \frac{-i}{\sqrt{2}} &\{\langle 1,1,0| + \langle 1,-1,0|\}\langle 1,0,0| \\ &\times\left\{ -\frac{1}{\sqrt{3}}|1,0,0\rangle|1,1,0\rangle\langle 8,1,1,0| \right. \\ &\left. +\frac{1}{\sqrt{3}}|1,0,0\rangle|1,-1,0\rangle\langle 8,1,-1,0| \right\} |B^j\rangle|\bar{B}_i\rangle \end{aligned}$$

$$= \frac{-i}{\sqrt{6}}\{-\langle\varphi_2|\langle\psi^1| + \langle\varphi_1|\langle\psi^2|\}|B^j\rangle|\bar{B}_i\rangle$$

$$= \frac{i}{\sqrt{6}}\left[\begin{bmatrix} 0&0&0\\1&0&0\\0&0&0\end{bmatrix} - \begin{bmatrix} 0&-1&0\\0&0&0\\0&0&0\end{bmatrix}\right]$$

$$= \frac{i}{\sqrt{6}} f_{231}\lambda^{(1)}.$$

Therefore,

$$(P_{8_28})_{\alpha\beta;ij} = \frac{i}{\sqrt{6}} f_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_j. \qquad (4.15)$$

Similarly,

$$(P_{8_18})_{\alpha\beta;ij} = -(\tfrac{3}{10})^{1/2} d_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_j, \qquad (4.16)$$

$$(P_1)_{\alpha\beta;ij} = \frac{1}{2\sqrt{6}}\delta_{\alpha\beta}\delta^i{}_j. \qquad (4.17)$$

This gives us $A_{ts}(3\times 8)$, and it agrees with $(B_{ts}{}^T)^{-1}$ of Table IV.

We mention another method that will give the values of P_{8_28} and P_1. If

$$(P_{8_28})_{\alpha\beta;ij} = Cf_{\alpha\beta\gamma}\lambda^{(\gamma)i}{}_j,$$

then

$$\begin{aligned} (P_{88})_{\alpha\beta;\gamma\delta} &= (P_{8_28}{}^\dagger P_{8_28})_{\alpha\beta;\gamma\delta} \\ &= |C|^2 f_{\gamma\delta\epsilon} f_{\alpha\beta\varphi} \operatorname{Tr}\{\lambda^{(\epsilon)}\lambda^{(\varphi)}\} \\ &= 2|C|^2 f_{\gamma\delta\epsilon} f_{\alpha\beta\epsilon}. \end{aligned} \qquad (4.18)$$

Using

$$(P_{88})_{\alpha\beta;\gamma\delta} = (P_{88})_{\alpha\beta;\sigma\rho}(P_{88})_{\sigma\rho;\gamma\delta}$$

gives us

$$\begin{aligned} 2|C|^2 f_{\gamma\delta\epsilon} f_{\alpha\beta\epsilon} &= 4|C|^4 f_{\gamma\delta\epsilon} f_{\sigma\rho\epsilon} f_{\sigma\rho\varphi} f_{\alpha\beta\varphi} \\ &= 12|C|^4 f_{\gamma\delta\epsilon} f_{\alpha\beta\epsilon}, \end{aligned} \qquad (4.19)$$

where the equations

$$if_{abc} \equiv -F^c{}_{ab}, \qquad (4.20)$$

$$\operatorname{Tr}[F^aF^b] = 3\delta^{ab} \qquad (4.21)$$

have been used. The F's are the $SU(3)$ regular representation matrices.

Equation (4.19) shows us that $C = \pm i/\sqrt{6}$, so that

$$(P_{8_28})_{\alpha\beta} = \frac{\pm i}{\sqrt{6}} f_{\alpha\beta\gamma}\lambda^{(\gamma)}.$$

To obtain the phase, one just must realize that what we have done here is entirely analogous to the $SU(2)$ projection operator[19]

$$(P^{N\bar{N}\to\pi\pi}{}_{T=1})_{\alpha\beta;ij} = \tfrac{1}{2}i\epsilon_{\alpha\beta\gamma}\tau^{(\gamma)i}{}_j. \qquad (4.22)$$

The phase is the same, for the π states are just the first three $|\alpha\rangle$ states, the Pauli matrices $\tau^{(\gamma)}$ are contained in the upper left-hand corner of λ,[1–3] and the ϵ's are

[21] K. Barnes (private communication).

[22] P. McNamee, S. J. and F. Chilton, Rev. Mod. Phys. 36, 1005 (1964).

the $SU(2)$ antisymmetric structure constants analogous to the f's of $SU(3)$. Therefore, we know the sign to be $(+)$.

In a similar manner, by considering $(P^{N\bar{N}\to\pi\pi}{}_{T=0})$, one can obtain the phase and normalization for P_1.

5. DISCUSSION

At this juncture we wish to discuss points of interest in the expressions for the crossing matrices (3.16) and (3.19). For the case of elastic scattering $(\mu_1=\mu_3; \mu_2=\mu_4)$, crossing symmetry is especially important because it determines the sign and magnitude of the forces in the direct (s) channel due to the crossed (t) and (u) channels. A bound on these forces can be given by finding a bound on the respective element of the crossing matrix. Capps[23] has shown that if $SU(n)$ invariance is assumed, then the limit on the force exerted in a u or t state of multiplicity N_u or N_t due to the exchange of an s multiplet of multiplicity N_s is limited by

$$|B^{us}| \leqslant (N_s/N_u)^{1/2}, \tag{5.1}$$

$$|B^{ts}| \leqslant (N_s/N_t)^{1/2}. \tag{5.2}$$

These formulas agree with the explicitly obtained result for $SU(2)$.[14]

The relations (5.1) and (5.2) tell us that a multiplet with large multiplicity leads to a large crossing matrix element for multiplets in crossed channels that have low multiplicities, but not vice versa.

Because of this, Capps[23] argues that since the crossing matrix element is the limit on the forces, it is natural that the representations higher than **8** and **10** have not been found thus far to be taken by the lighter particles. The fact that there are more octets than singlets is also not in violation with this outlook, because often a direct product will contain more octets than singlets.

However, one must take this argument with mixed feelings, for, as Capps points out, this does not explain why, if one uses a triplet model, octets are found but triplets are not. One must beware of mistaking group theory for physics. The solution to this puzzle is likely to lie in the dynamics of the triplets, and a study of it, perhaps to see if triplets can bootstrap properly, may be a fruitful path to follow. Capps[16] and Cutkosky[17] have discussed this question.

For the u channel, if $\mu_4=\mu_2^*$, i.e., if the crossed particles belong to representations that are the complex conjugate of each other, then the orthogonality theorems found in Refs. 4 and 6 for $SU(3)$ can be applied to show that $(B^{us})^2=I$. The corresponding result is true for B^{ts} if $\mu_3=\mu_2^*$. Also, because of the relationship of the crossed and uncrossed multiplets, many matrix elements can be figured out from elements that have already been calculated by using the formulas given in Ref. 6 that connect different isoscalar and/or $SU(3)$ Clebsch-Gordan coefficients. These points are discussed in detail in Refs. 4 and 6. A striking example that we have already observed is the connection between the u and t matrices if $\mu_1=\mu_2=\mu_3=\mu_4=8$.

The method given in Sec. 3 for obtaining crossing matrices is convenient because it removes thinking from the process. One just has to crank away. However, for those who like elegance, saving time, finding projection operators, or assurance, the use of projection operators or a combination of the two methods may well provide them. At worst, the operator methods provide a valuable check. However, if no tricks are to be found, one can use (3.16) and (3.19) and calculate matrix elements to one's heart's content.

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[23] R. H. Capps, *Proceedings of the 12th Annual International Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965).