Relativistic Treatment of Loosely Bound Systems in Scattering Theory*

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An approximate method for treating loosely bound systems is discussed, with the deuteron form factor in the impulse approximation treated as an example. Major advantages of the method are that it is simple and unambiguous, and leads to a quantitative estimate of its accuracy. The principal physical assumption behind the method is that the bound system can be regarded as coupled to only one channel at low energies. One can use this to obtain simple, relativistically invariant expressions for scattering amplitudes in which the bound state is described by a relativistic wave function. For the deuteron this wave function is related to the d-np vertex with one nucleon off the mass shell. A method by which this wave function can be determined from phenomenological nonrelativistic wave functions is discussed.

1. INTRODUCTION

THIS paper presents a new approach to the treatment of loosely bound systems in relativistic scattering theory. The deuteron is the most familiar example of such a system, and we will specifically discuss the deuteron in this paper, although the methods presented are quite general and can be applied to any loosely bound system. In fact, the methods are applicable to any scattering problem in which potentials can be employed.

The number of interesting scattering problems in which deuterons participate is large. There are, for example, elastic and inelastic electron-deuteron scattering, photodisintegration of the deuteron, and processes involving pions such as $\pi^+ + d \to p + p$ and $e + d \to \pi^- + d$. In all of these processes the structure of the deuteron plays an important role, and knowledge of this structure is essential to a theoretical understanding of these processes. In fact in many of these cases the deuteron is viewed solely as a source of target neutrons, and then very accurate knowledge of the deuteron structure is essential before unambiguous information about neutrons can be extracted from these experiments.

Until recently, the only systematic way in which deuteron structure was introduced into these problems was through the use of nonrelativistic wave functions. These wave functions are usually determined from a two-nucleon potential, which is in turn chosen to fit the nucleon-nucleon scattering data in the center-of-momentum system. Repeated use of this technique has given one confidence in its validity, and it has become fashionable, "because a fully relativistic theory of the two-nucleon interaction does not exist at the present time."

In spite of the absence of a precise theory of the strong interactions, one can still improve on the usual nonrelativistic approach to the deuteron; recent experimental data justify such an effort, as evidence from a number of sources has already produced discrepancies which an improved theory may be able to eliminate.

Two of these discrepancies occur in the results of recent measurements on the deuteron form factor. For a number of years the deuteron has been a source of information about the neutron-charge form factor at low momentum transfer. Recent measurements[1] suggest that $G_{EN}=0$ within experimental errors for momentum transfers from 0.3 to about 6 F^{-2}. Yet one of the best measured constants of nuclear physics is the slope of the neutron-charge form factor at $q^2=0$. This number comes from the scattering of thermal neutrons by bound electrons in atoms,[2] and the result is

$$-dG_{EN}(q^2)/dq^2=(0.021\pm0.001)\ F^{+2}. \quad (1.1)$$

Although it is not absolutely clear that this nonzero slope is inconsistent with the form-factor data, it is unlikely on the basis of our knowledge of the analyticity properties of $G_{EN}(q^2)$ that both experiments have been correctly interpreted. Furthermore, small corrections in the theory of the deuteron form factor could easily alter the interpretation of the form-factor experiments and bring these data more into line with (1.1). It is our hope that calculations to be done with the theory presented in this paper will improve matters.

Another interesting result from Stanford is that the deuteron magnetic form factor is 30% larger at 8 F^{-2} than expected from the nonrelativistic theory.[3] A possible explanation for this lies in the meson-exchange-current contribution.[4] In addition, this discrepancy could be partly due to other causes; calculations based on a relativistic version of the impulse approximation also suggest such an increase.[5]

Of course, in recent years it has been known that a D-state probability of about 7% seems to be necessary to reconcile a number of experiments involving the deuteron.[6] This number is inconsistent with the simple nonrelativistic theory of the magnetic moment, which implies a D-state probability of 4%. The explanation

* Supported in part by the U. S. Office of Naval Research.

[1] D. Benaksas, D. Drickey, and D. Frerejacque, Phys. Rev. Letters **13**, 353 (1964). Other references given here.
[2] See R. Hofstadter, *Nuclear and Nucleon Structure* (W. A. Benjamin Inc., New York, 1963), for reprints of a number of papers.
[3] C. D. Buchanan and M. R. Yearian, Phys. Rev. Letters **15**, 303 (1965).
[4] R. J. Adler and S. D. Drell, Phys. Rev. Letters **13**, 349 (1964).
[5] F. Gross, Phys. Rev. **136**, B140 (1964).
[6] See, for example, R. Wilson, *The Nucleon-Nucleon Interaction* (Interscience Publishers, Inc., New York, 1963).

for this undoubtedly lies in relativistic corrections to the deuteron form factor.[4]

While none of these discrepancies is large numerically, they are clearly beyond the experimental errors and hence are of interest. Furthermore, the possibility that they are only symptoms of more serious trouble cannot be disregarded until we have numerical estimates of the errors involved and how they depend on momentum. Finally, a systematic analysis of the deuteron may give us valuable clues about how to treat other strong-interaction problems.

This paper is divided into five sections. Section 2 is devoted to a discussion of the physical assumption which we retain in this theory, and how this assumption will be employed quantitatively in the remainder of the paper. Since these ideas are an extension of earlier work on the deuteron form factor,[5,7] they will be applied to the deuteron form factor as a specific example, and Sec. 3 is devoted to a derivation of the impulse approximation. In the next section we introduce relativistic wave functions, which are generalizations of the usual nonrelativistic wave functions, and we propose a method by which the relativistic wave function may be approximately determined from the phenomenological S- and D-state nonrelativistic wave functions. Finally, Sec. 5 is devoted to a summary of the paper and conclusions. In this section we discuss the physical origin of relativistic corrections to the deuteron form factor which emerges from this analysis. An attempt has been made to write Secs. 2 and 5 so that they may be read by one who is not interested in details.

2. THE ONE-CHANNEL COUPLING APPROXIMATION

The basic approximation we shall make in this paper is that the deuteron couples strongly to only one channel, the two-nucleon channel. It is this assumption (also a part of any Schrödinger description of the deuteron) which prevents our approach from being applicable at all energies. We shall refer to this assumption as one-channel coupling.

Most readers will probably immediately grant the reasonableness of this assumption. In this section we will explore the basis for its validity and indicate how one can formulate it in such a way as to exploit it.

First let us appeal to a crude but intuitive argument. The size of a loosely bound system depends on its asymptotic wave function, which goes as $r^{-1}\exp(-\lambda r)$ where $\lambda=(2M_RB)^{1/2}$, M_R being the reduced mass and B the binding energy. Hence we could argue that only for distances less than λ^{-1} is a particular channel important.

For the deuteron regarded as a bound state of two nucleons this characteristic distance is (μ is the pion mass)

$$R_0\approx(M\epsilon)^{-1/2}\equiv\alpha^{-1}\cong 3\mu^{-1}, \tag{2.1}$$

while if it is a bound state of a nucleon and an N-π pair of mass Λ we have

$$R_0'\approx\left[\frac{\Lambda+M}{2\Lambda M}\left(\frac{1}{M+\Lambda-M_D}\right)\right]^{1/2}\lesssim\frac{1}{(M\mu)^{1/2}}\equiv\frac{1}{\gamma}\approx 0.4\mu^{-1}. \tag{2.2}$$

Hence only at small distances ($\lesssim R_0'$) can we expect to detect the presence of the N-$N\pi$ channels.

Similarly, in momentum space the wave function is

$$4\pi/(\mathbf{p}^2+\lambda^2) \tag{2.3}$$

and the loosely bound channel dominates the more tightly bound channels by a factor

$$(\mathbf{p}^2+\gamma^2)/(\mathbf{p}^2+\alpha^2) \tag{2.4}$$

so that only for momenta of the order of

$$\mathbf{p}^2\gtrsim M\mu \tag{2.5}$$

will the presence of the additional channels become significant.

It is therefore reasonable to expect one-channel coupling to be valid at low energies. As Eq. (2.4) shows, the quantitative success of the assumption depends primarily on the ratio of the binding energies of the different channels, and can be expected to work best when one channel is very loosely bound while the others are tightly bound, as in the case of the deuteron. Furthermore, we see that at high momenta the approximation breaks down, since our interaction "feels" out the structure at small distances and thereby is sensitive to many channels.

Now the relativistic wave function of a two-body system depends upon the relative *four*-momenta of the two nucleons, and hence the above argument is not in a form suitable for incorporation into a relativistic theory. The first step is to find a way of stating the assumption more generally.

A nucleon in the deuteron cannot really be distinguished from an $N\pi$ system, since the particle is surrounded by its pion cloud. However, if the square of its total rest energy u is close to the free value M^2 then one would expect it to be primarily a nucleon, for dissociation into an N-π pair involves a large change in energy and can occur only for times of the order of $\Delta t<\hbar/\Delta E\approx\hbar/\mu c$, i.e., at distances small compared with μ^{-1}. However, if u is close to $(M+\mu)^2$, then the nucleon becomes increasingly like an N-π system. Hence, a nucleon of mass $(M+\mu)^2$ cannot really be distinguished from an N-π system, and nucleons which are this far off the mass shell contribute in the same manner to the deuteron as the N-$N\pi$ channel, and are to be neglected.

But this consideration applies to both nucleons in the deuteron. If $p_1^2=u$ and $p_2^2=t$ are the masses of the two nucleons, then the above argument suggests that the

[7] F. Gross, Phys. Rev. **134**, B405 (1964).

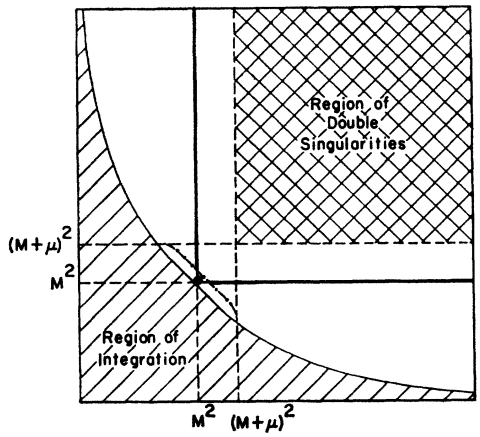

FIG. 1. A region of the real u, t plane showing singularities discussed in the text. The values of u and $t=M^2$ and $(M+\mu)^2$ are indicated by broken lines. The double singularity at $u=t=M^2$ is shown by a large dot, and it is very close to the region of integration bounded by $\mathbf{r}^2=0$ in the lower left corner. The cross-hatched region in the upper right corner is the region of double singularities corresponding to u and $t>(M+\mu)^2$. The dot-dashed line gives the boundary of anomalous singularities of Λ, and hence the region of double singularities corresponding to a singularity in Λ and u or $t=M^2$ are shown by the heavy solid lines.

one-channel coupling approximation is really two assumptions:

(a) The deuteron is well described at low energies by considering only its direct coupling to the two-nucleon channel.

(b) In treating this coupling, the principal contributions will arise from a small region about $u\approx t\approx M^2$.

Statement (b) follows because the truth of statement (a) does not depend on any detailed considerations of the interactions, but [as we have seen nonrelativistically in Eq. (2.4)] only on the relative binding energies of the channels. What we now must do is to establish that in the relativistic case it is again only the ratio of the binding energies which matters and at the same time develop a quantitative way to exploit statement (b) above.

First introduce the total and relative four-momenta of the two nucleons

$$\begin{aligned} D&=p_1+p_2\,,\\ r&=\tfrac{1}{2}(p_1-p_2)=(r_0,\mathbf{r})\,. \end{aligned} \tag{2.6}$$

Then we now can replace the variables u and t by the equivalent variables r_0 and $\mathbf{r}^2$. This is convenient, because in calculating scattering amplitudes in which deuterons are involved we integrate the relativistic wave function over r. This region of integration, defined by

$$\begin{aligned} -\infty<r_0<\infty\,,\\ 0<\mathbf{r}^2<\infty\,, \end{aligned}$$

can be mapped onto the u, t plane as shown in Fig. 1. The part of this region close to $u\approx t\approx M^2$ is now the region of small r_0 and $\mathbf{r}^2$. What remains to be discussed, then, is why the results should be dominated by the behavior of the wave function in this region.

This can be understood by examining the singularities in the Bethe-Salpeter amplitude,[8] which can be written in momentum space

$$\psi(p_1,p_2)=S_F(p_1)\Lambda^\alpha(p_1,p_2)CS_F{}^T(p_2)\xi_\alpha\,, \tag{2.7}$$

where $S_F(p)$ is the full Feynman propagator for a nucleon of momentum, p, $\Lambda(p_1,p_2)C$ is the proper d-np vertex function with both nucleons off the mass shell, ξ_α

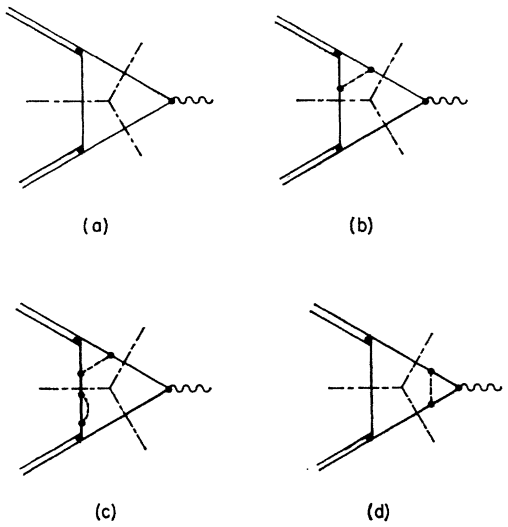

FIG. 2. Examples of diagrams included in the impulse approximation to the deuteron form factor. Double solid lines represent deuterons, single solid lines nucleons, dashed lines pions, and the wavy line a virtual photon. The dot-dashed lines are the three cuts separating each diagram into three distinct pieces by cutting nucleon lines only (see text). The same key is used in the following figures.

is the deuteron 4-polarization vector and C is the charge-conjugation matrix.[9] It is this amplitude which enters into all scattering diagrams involving deuterons. Our objective here is to eventually develop a simplified form of (2.7) which is more tractable and yet fully consistent with the one-channel coupling approximation.

Now the singularities in ψ come from both the propagators $S_F(p)$, and from the proper vertex function Λ. The singularities in $S_F(p)$ are well known[10] to include a pole at M^2 and a cut starting at $(M+\mu)^2$. The singularities in Λ are not restricted to the real u, t plane[11] but their location in the real u, t plane can be readily found.

[8] E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951); M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).

[9] Throughout this paper we use the metric $p^2=p_0{}^2-\mathbf{p}^2$ and the conventions for spinors and gamma matrices given in J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, Inc., New York, 1964).

[10] See G. Barton, *Introduction to Advanced Field Theory* (Interscience Publishers, Inc., New York, 1963).

[11] See, for example, M. Fowler, P. V. Landshoff, and R. W. Lardner, Nuovo Cimento 17, 956 (1960).

In the next section and in the Appendix we discuss this question in more detail. The singularities in ψ are shown in Fig. 1.

We now can see quite clearly why it is that for the deuteron the matrix elements involving (2.7) are dominated by the behavior of the wave function for small r_0 and $\mathbf{r}^2$, and why the one-channel coupling is valid. The wave function is clearly largest in regions close to simultaneous singularities in both u and t. The places where these double singularities occur are shown in Fig. 1. There is only one such singularity (the double pole at $u=t=M^2$) which is very close to the region of integration (its distance from the boundary is of the order of α^2), and the wave function is therefore largest in this region. The next nearest singularities arise from the anomalous thresholds in the d-np vertex (to be discussed in Sec. 3). The singularities characteristic of the N-$N\pi$ channel are a distance $\gamma^2 \cong M\mu$ away, and hence have a much smaller effect on the wave function. Hence once again we see that the ratio of the binding energies of the different channels is the significant parameter.

Our considerations with the relativistic wave function are really no different than those with the nonrelativistic wave function, except that the occurrence of the extra variable r_0 has made it a two-dimensional problem.

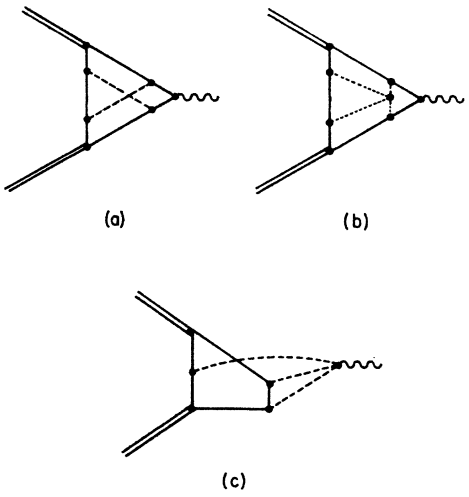

FIG. 3. Examples of diagrams *not* included in the impulse approximation. For key see caption to Fig. 2.

Note that although none of the double singularities lie in the region of integration single singularities in either u or t do. These can be regarded as singularities in r_0, whose position depends on $\mathbf{r}^2$.

Our simplified method for treating loosely bound states can now be precisely stated. We will first write down the general expression for any amplitude, employing the function (2.7). Then we will perform the integration over the relative energy r_0 by contour integration. We shall see that the singularities in r_0 all lie along the real axis, so this is easily done. However, it follows from our above considerations that the term arising from the singularity in u (or t) at M^2 will give the largest contribution, and hence we will implement statement (b) above by retaining only this term. This leaves an integration over $\mathbf{r}$ to be performed and gives a simple result which is a generalization of the usual nonrelativistic expressions. Furthermore, it will never be necessary to know (2.7); we will instead need only the residue of (2.7) at one of the poles u (or t) $=M^2$. But this is simply related to the d-np vertex with *one* particle off the mass shell, which can be more easily investigated.

In the next section we will illustrate this in detail for the impulse approximation to the deuteron form factor. Here we will show that the terms arising from more distant singularities are smaller by a factor of $\frac{1}{2}(\mathbf{r}^2/M\mu)^2$, and hence confirm the validity of the one-channel coupling approximation.

3. THE IMPULSE APPROXIMATION

As an example of a practical application of the ideas sketched in the previous section, we will derive a simple relativistic expression for the deuteron form factor in the impulse approximation. In the course of our discussion we will obtain an estimate of the accuracy of our approximation. The final expression we obtain for the form factor, besides being of interest itself, will motivate the definition of a relativistic wave function simpler than (2.7), which will be discussed in Sec. 4.

We define the impulse approximation to be the sum of all Feynman diagrams which can be partitioned into three distinct pieces by making three cuts, each of which cuts a single nucleon line on each leg. An example of a few diagrams which contribute to the impulse approximation is shown in Fig. 2. There are some of these diagrams [an example is Fig. 2(d)] which contribute to the structure of the nucleon form factor.

It should be pointed out that a great many Feynman diagrams are not included in the impulse approximation. A few of these are shown in Fig. 3. These are diagrams which cannot be partitioned into three pieces by cutting nucleon lines only. Included in this class is the interesting and important exchange-current contributions [Fig. 3(c) is an example] which will be discussed at some future date.

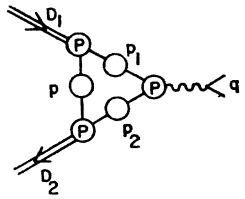

FIG. 4. Diagramatic representation of the integral (3.1). The bubbles are to be understood as the sum of all Feynman diagrams which could be inserted in place of them; those marked with a "P" are proper bubbles which exclude self-energy contributions to the external nucleon lines.

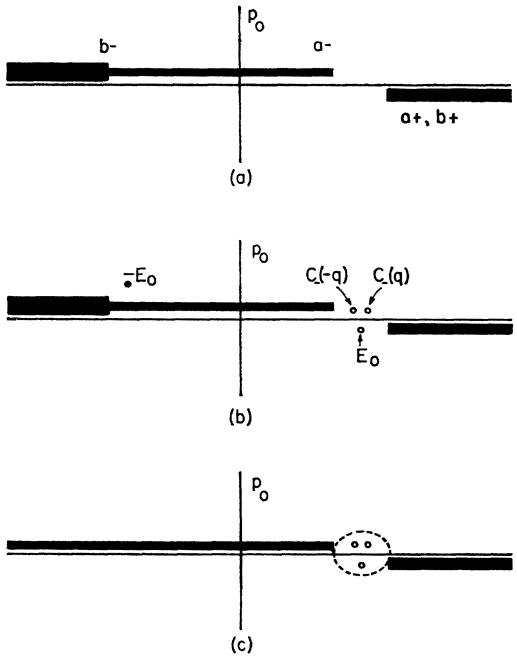

FIG. 5. Schematic representation of the singularities in the complex p_0 plane (for $\mathbf{p}^2$ small) of the integrand of (3.1). (a) Singularities in Λ, $\bar{\Lambda}$, and F alone, (b) singularities in the three S_F's alone, and (c) the combined singularities. The dotted oval locates the three singularities closest to $u\sim t\sim M^2$.

The sum of all the diagrams included in the impulse approximation can be represented symbolically as in Fig. 4. Here each bubble is taken to represent the sum of all diagrams which could be inserted in place of the bubble, and the proper bubbles (marked with a P) are this sum excluding all self-energy insertions on the external nucleon lines.

The sum of these diagrams can be represented by the integrals[12]

$$G_D{}^\mu(q)=K\int d^4p\ \mathrm{tr}\{C^{-1}\bar{\Lambda}^\beta(p,p_2)S_F(p_2)F^\mu(p_2,p_1)\times S_F(p_1)\Lambda^\alpha(p_1,p)CS_F{}^T(p)\}\xi_\alpha\eta_\beta{}^*$$
$$=K\int d^4p\ \mathrm{tr}\{\bar{\Lambda}^\beta(p,p_2)S_F(p_2)F^\mu(p_2,p_1)\times S_F(p_1)\Lambda^\alpha(p_1,p)S_F{}^C(p)\}\xi_\alpha\eta_\beta{}^*, \quad (3.1)$$

where $\Lambda^\alpha(p_n,p_p)C$ is the proper d-np vertex function with both nucleons off the mass shell, $S_F(p)$ is the full Feynman propagator for a nucleon of momentum p, $F^\mu(p_2,p_1)$ is the proper vertex function for the nucleon form factor, ξ^α and η^β are the deuteron polarization vectors and C is the charge-conjugation matrix. Both Λ and S_F were already introduced in Eq. (2.7), and it is through their contribution to sums of Feynman diagrams like (3.1) that the relativistic Bethe-Salpeter (B.S.) amplitude enters into all scattering problems. Also[13]

$$\bar{\Lambda}^\beta(p_1,p_2)=-C\gamma^5[\Lambda^\beta(p_2,p_1)]^T(C\gamma^5)^{-1}, \quad (3.2a)$$
$$S_F{}^C(p)=CS_F{}^T(p)C^{-1}. \quad (3.2b)$$

Finally, p_1 and p_2 are shorthand notation for

$$p_1=D_1-p,$$
$$p_2=D_2-p,$$

$q=D_2-D_1=p_2-p_1$, and K is a constant to be determined from the Feynman rules. In this paper it will not be necessary to know K.

The first step in evaluating (3.1) is to perform the integration over p_0. We specialize to the Breit system where $q=(0,\mathbf{q})$, $D_1=(D_0,-\frac{1}{2}\mathbf{q})$, $D_2=(D_0,\frac{1}{2}\mathbf{q})$ and perform the integral using the residue theorem. The p_0 singularities of the integrand can be located, and are shown in Fig. 5. They are compounded of the singularities from the propagators and from the proper vertex functions.

The determination of the position of singularities in p_0 for the proper vertex functions is straightforward but lengthy, and is discussed fully in the Appendix. It turns out that the singularities lie only along the real p_0 axis (it is for this reason that p_0 is a convenient variable to work with; as mentioned in Sec. 2 the singularities in the masses define a complex surface, and hence are not confined to the real axis[11]). They generate two cuts which run from $-\infty$ to a_- and from a_+ to ∞, where in the case of Λ and $\bar{\Lambda}$

$$a_+=[(M+\mu)^2+\mathbf{p}^2]^{1/2},$$
$$a_-=D_0-[(M+\mu)^2+(\mathbf{p}\pm\tfrac{1}{2}\mathbf{q})^2]^{1/2}. \quad (3.3)$$

In a_- the $+$ refers to Λ and the minus to $\bar{\Lambda}$. For F^μ the branch points are at

$$b_+=[(M+\mu)^2+\mathbf{p}_1{}^2]^{1/2}-\tfrac{1}{2}[M^2+\mathbf{p}_1{}^2]^{1/2}+\tfrac{1}{2}[M^2+\mathbf{p}_2{}^2]^{1/2},$$
$$b_-=-[(M+\mu)^2+\mathbf{p}_2{}^2]^{1/2}+\tfrac{1}{2}[M^2+\mathbf{p}_1{}^2]^{1/2}-\tfrac{1}{2}[M^2+\mathbf{p}_2{}^2]^{1/2}. \quad (3.4)$$

The location of these singularities is shown in Fig. 5(a).

As we mentioned in Sec. 2, the propagators contain poles at $\pm E_0$ where $E_0=[M^2+\mathbf{p}^2]^{1/2}$, the positive energy pole being located in the lower half-plane. In addition, they contribute a cut starting at $p^2=(M+\mu)^2$, or $p_0=\pm[(M+\mu)^2+\mathbf{p}^2]^{1/2}$. These singularities are shown

[12] J. Tran Thanh Van, Nuovo Cimento **30**, 1100 (1963).

[13] An easy way to see (3.2a) is to note that $C\bar{\Lambda}$ is obtained from ΛC by simply interchanging the initial and final state, i.e., by interchanging p_1 and p_2 (since one of these can be regarded as playing the role of an incoming nucleon while the other is an outgoing antinucleon) and reversing the order of all γ matrices along each nucleon line in any Feynman diagram contributing to Λ. But this can simply be accomplished by taking the over-all transpose and recalling that $C\gamma^5\gamma^\mu(C\gamma^5)^{-1}=\gamma^{\mu T}$.

schematically in Fig. 5(b). The poles from $S_F(p_1)$ are at

$$C_\pm(\mathbf{q})=D_0\pm[M^2+(\mathbf{p}+\tfrac{1}{2}\mathbf{q})^2]^{1/2}, \tag{3.5}$$

while for $S_F(p_2)$ we have poles at $C_\pm(-q)$

$$C_\pm(-\mathbf{q})=D_0\pm[M^2+(\mathbf{p}-\tfrac{1}{2}\mathbf{q})^2]^{1/2}, \tag{3.6}$$

and the cuts are similarly placed. In what follows we will assume further that the singularities are such that the invariant functions of these propagators can be written in the form

$$I(p^2)=H(p^2)/(M^2-p^2-i\epsilon), \tag{3.7}$$

where $H(p^2)$ contains the cut starting at $(M+\mu)^2$ and satisfies unsubtracted dispersion relations.[14]

We now can proceed to do the p_0 integration by the method of residues, knowing the singularities in the integrand are placed as shown in Fig. 5(c). The contour could be closed in either half-plane yielding the same answer, and we will choose the lower half-plane because it eventually yields a simpler result. Now there is little interest in the full result; the considerations of Sec. 2 suggest that the largest contribution will come from the pole in p_0 which corresponds to u (or t) equal to M^2, and that furthermore this term is the only one consistent with the limitation to one-channel coupling (i.e., other terms can be expected to be of the same size as the contributions from other channels, which we are neglecting). But this pole is clearly the pole at E_0, and hence the largest term becomes:

$$G_D{}^\mu(q)=K\int\frac{d^3p}{2E_0}\,\mathrm{tr}\{\bar\Lambda^\beta(p,p_2)S_F(p_2)F^\mu(p_2,p_1) \times S_F(p_1)\Lambda^\alpha(p_1,p)(M-\not p)\}\xi_\alpha\eta_\beta{}^*, \tag{3.8}$$

where now everywhere that p_0 occurs we must replace it by E_0.

Equation (3.8) is our result for the impulse approximation. At this time it is instructive to make an estimate of its accuracy. To do this we make use of the fact that the integrand of (3.1) is a *product* of terms [here is where we make use of assumption (3.7)] and has the singularity structure of Fig. 5(c). Hence, a reasonable estimate can be obtained by replacing all of the cuts by poles, in which case we obtain for the integrand of (3.1) (neglecting spin):

$$I(p_0)\cong\frac{Af[(\mathbf{p}+\frac{1}{4}\mathbf{q})^2]f[(\mathbf{p}-\frac{1}{4}\mathbf{q})^2]}{(p_0-E_0+i\epsilon)[p_0-C(+)-i\epsilon][p_0-C(-)-i\epsilon](p_0-a_+ +i\epsilon)(p_0-a_--i\epsilon)}. \tag{3.9}$$

In Eq. (3.9) we have denoted $C_-(\pm q)$ by $C(\pm)$ and let A be a constant which represents the contributions of distant singularities of p_0 whose structure can be neglected in this order of magnitude argument. We have replaced the cuts starting at a_+ and a_- by poles at these points. The function $f(\mathbf{r}^2)$ has been introduced to account for the nontrivial structure of the residue of these poles at a_+ and a_- contributed by the vertex functions Λ and $\bar\Lambda$ due to the existence of the anomalous singularities (recall discussion of Sec. 2 and Fig. 1). These singularities do not appear in p_0, but will appear in $\mathbf{r}^2$, and will be important because they are so close to the physical region. Later we will choose the function f to simulate roughly the position of these singularities in the leading term, and use the same choice for the correction term. Although this will be quite rough, it should be good enough to give us a fairly precise estimate of the accuracy of the leading term.

One interesting thing about this estimate is that it does not depend on the size of A, and hence will not depend on the over-all magnitude of the residues of the poles at a_+ and a_-.

Performing the contour integration in the lower half-plane gives

$$\int_{-\infty}^{-\infty}dp_0I(p_0)=\frac{A'f[(\mathbf{p}+\frac{1}{4}\mathbf{q})^2]f[(\mathbf{p}-\frac{1}{4}\mathbf{q})^2]}{[E_0-C(+)][E_0-C(-)](E_0-a_+)(E_0-a_-)}+\frac{A'f[(\mathbf{p}+\frac{1}{4}\mathbf{q})^2]f[(\mathbf{p}-\frac{1}{4}\mathbf{q})^2]}{(a_+-E_0)[a_+-C(+)][a_+-C(-)](a_+-a_-)}. \tag{3.10}$$

Here the first term is the leading term we are retaining, while the second term is an (upper) estimate of the corrections neglected in this theory. From the discussion of Sec. 2 we know that the major contributions from (3.10) when it is integrated over $\mathbf{p}$ will come from the region where $\mathbf{p}$ is small. Hence, expanding the functions E_0, $C(\pm)$ using Eqs. (3.5) and (3.6) we have:

$$\int_{-\infty}^{-\infty}dp_0I(p_0)\cong\frac{-M^2A'f[(\mathbf{p}+\frac{1}{4}\mathbf{q})^2]f[(\mathbf{p}-\frac{1}{4}\mathbf{q})^2]}{\mu^2[\alpha^2+(\mathbf{p}-\frac{1}{4}\mathbf{q})^2][\alpha^2+(\mathbf{p}+\frac{1}{4}\mathbf{q})^2]}+\frac{+M^2A'f[(\mathbf{p}+\frac{1}{4}\mathbf{q})^2]f[(\mathbf{p}-\frac{1}{4}\mathbf{q})^2]}{2\mu^2[M\mu+(\mathbf{p}-\frac{1}{4}\mathbf{q})^2][M\mu+(\mathbf{p}+\frac{1}{4}\mathbf{q})^2]}, \tag{3.11}$$

[14] F. Selleri (to be published).

where we have taken $a_+\cong M+\mu$, $a_-\cong(M-\mu)$. Note the similarity between (3.11) and (2.4). The fractional error introduced by the second term is

$$\Delta=-\frac{1}{2}\frac{[\alpha^2+(\mathbf{p}+\tfrac{1}{4}\mathbf{q})^2][\alpha^2+(\mathbf{p}-\tfrac{1}{4}\mathbf{q})^2]}{[M\mu+(\mathbf{p}+\tfrac{1}{4}\mathbf{q})^2][M\mu+(\mathbf{p}-\tfrac{1}{4}\mathbf{q})^2]} \quad (3.12)$$

and hence we see that for small $\mathbf{q}$ and $\mathbf{p}$ the first term dominates the results. More precisely, if we expand[15] our results in powers of $\eta=\mathbf{p}^2/\mu M$ the errors arising from the neglect of the second term in (3.11) (higher singularities in p_0) are of second order, i.e., do not appear until terms of order η^2. And, in addition, the structure of (3.11) shows that the results are dominated by contributions from $\eta\ll 1$.

To examine this latter point more closely, we choose

$$f(\mathbf{r}^2)=\text{const}/(\mathbf{r}^2+\beta^2), \quad (3.13)$$

where β^2 is taken to be close to μ^2. Equation (3.13) is motivated by the fact that the first term in (3.11) corresponds to the contribution where the exchanged particle (Fig. 4) is on the mass shell ($p^2=M^2$) and hence f must simulate the behavior of the d-np vertex function with one particle on the mass shell [to be defined in Eq. (3.15) below]. But this function is known[7] to have an anomalous threshold starting at $M^2+2\mu(\mu+2\alpha)$, which corresponds roughly to the choice of $\beta^2=\mu^2$.

With this choice for f, the first term is seen to be simply a convolution of two relativistic Hulthèn wave functions.

We can now integrate (3.11) over p to get a rough numerical estimate of the dominance of the first term and hence the validity of the assumption that $\eta\ll 1$. One finds that for $\mathbf{q}=0$ the ratio of the second to first term is approximately

$$\delta=\alpha\beta^3/2(M\mu)^2=0.0035=0.35\%. \quad (3.14)$$

We return now to the discussion of our result, Eq. (3.8). We introduce the deuteron-nucleon vertex, with one particle on the mass shell as

$$\Gamma^\alpha(p_1)\equiv(M-\not{p}_1)S_F(p_1)\Lambda^\alpha(p_1, D-p_1), \quad (3.15)$$

where $(D-p_1)^2=M^2$, and hence Γ depends only on p_1. It is this quantity which completely describes the structure of the deuteron in the one-channel coupling approximation.

This vertex was introduced by Blankenbecler and Cook[16] and has been discussed by a number of other people,[5,17,18] but it may not have been observed how naturally (3.15) arises in all scattering processes in which the one-channel coupling approximation is exploited, and that subject to this approximation (3.15) gives us a complete description of deuteron structure.

In terms of Γ, Eq. (3.8) takes a very simple form:

$$G_D{}^\mu(q)=K\int d^4p\,\delta(p^2-M^2)\theta(p_0)\,\mathrm{tr}\left\{\bar{\Gamma}^\beta(p_2)\frac{M+\not{p}_2}{M^2-p_2{}^2}\times F^\mu(q)\frac{M+\not{p}_1}{M^2-p_1{}^2}\Gamma^\alpha(p_1)(M-\not{p})\right\}\xi_\alpha\eta_\beta{}^*, \quad (3.16)$$

where from (3.2) and (3.15)

$$\bar{\Gamma}^\beta(p_2)=-C\gamma^5\Gamma^T(p_2)(C\gamma^5)^{-1}=\bar{\Lambda}(D-p_2, p_2)S_F(p_2)(M-\not{p}_2). \quad (3.17)$$

In writing (3.16) we have replaced $F^\mu(p_1,p_2)$ by $F^\mu(q)$, the form factor with the nucleons on the mass shell. Since so little is known about the form factor with both nucleons off the mass shell, practical calculations will employ this approximation, but as one can see the derivation of (3.16) in no way is restricted by this assumption.

Equation (3.16) is a simple and clearly covariant expression for the form factor in the impulse approximation. It is no harder to work with than conventional nonrelativistic expressions, and is superior in that it gives us the correction terms up to order η unambiguously. In the next section we shall define a relativistic wave function which gives (3.16) an appearance formally identical to nonrelativistic theory.

It is amusing to observe that (3.16) is identical to the results one obtains from dispersion theory by limiting one's self to the anomalous region.[5] This can be verified by calculating the discontinuity in q^2 of the expression [using Cutkosky's rules and remembering that $\Gamma(p_1)$ and $\bar{\Gamma}(p_2)$ contribute to this discontinuity]. This is not at all surprising, as the effect of retaining only the one pole in the integration was to place the exchanged particle on the mass shell, which defines the anomalous region in dispersion theory. Since the anomalous region is known to dominate the problem, we have another way of seeing why this approximation is a good one. Finally, it would appear that (3.16) has considerable technical advantages to the dispersion theory expressions, as fewer integrals must be negotiated, the introduction of relativistic wave functions is more direct, and the method of deriving (3.16) is simpler and hence may be applied to more complicated cases.

4. WAVE FUNCTIONS

Now that we have a relativistic formulation of the impulse approximation, it is desirable to make a connection with nonrelativistic potential theory. The principal reason for doing this is to make use of the extensive accumulation of phenomenological information which 20 years' application of potential theory to the two-nucleon problem has produced.

[15] Note that the same expansion parameter was obtained in another related context by J. M. Charap and S. P. Fubini, Nuovo Cimento **14**, 540 (1959).

[16] R. Blankenbecler and L. F. Cook, Jr., Phys. Rev. **119**, 1745 (1960).

[17] I. J. McGee and L. Durand, III, Bull. Am. Phys. Soc. **10**, 62 (1965) and private communication.

[18] M. Gourdin, M. LeBellac, F. M. Renard, and J. Tran Thanh Van, Nuovo Cimento **37**, 524 (1964).

This connection is conveniently made by the introduction of a relativistic wave function, which reduces to the nonrelativistic wave function in suitable limits. Our proposed wave function for a deuteron of 3-momentum $\mathbf{d}$ and relative internal 3-momentum $\mathbf{r}$ is

$$\phi_d(\mathbf{r})\equiv\frac{K'^{1/2}[M+\frac{1}{2}D+r]\Gamma^\alpha(\frac{1}{2}D+r)[M-(\frac{1}{2}D-r)]C\xi_\alpha}{2M[M^2-(\frac{1}{2}D+r)^2]}, \tag{4.1}$$

where

$$\begin{aligned} &D=(D_0,\mathbf{d}), \quad D_0=[M_D{}^2+\mathbf{d}^2]^{1/2},\\ &r=(r_0,\mathbf{r}), \quad r_0=\tfrac{1}{2}D_0-[M^2+(\tfrac{1}{2}\mathbf{d}-\mathbf{r})^2]^{1/2},\\ &K'^{1/2}=(2K)^{1/2}(2\pi)^2. \end{aligned} \tag{4.2}$$

It should be pointed out immediately that this wave function is covariant and describes a deuteron of arbitrary momentum $\mathbf{d}$, and is therefore not restricted to a particular reference frame. Furthermore, (4.1) is simpler to use and to calculate than the conventional (B.S.) amplitude (2.7) because it depends on Γ, the *d-np* vertex with only one particle off the mass shell, and hence satisfies one-dimensional dispersion relations.

Any useful relativistic wave function must satisfy three conditions. These are: (1) The wave function must give a complete description of the composite character of the deuteron (subject, of course, to the one-channel coupling approximation), (2) we must know unambiguously how to insert this wave function into scattering amplitudes, and (3) we must produce a wave equation which this wave function satisfies, so that we may calculate it.

The third condition will be discussed in a later paper; we need not worry about it if we wish to treat the form factor with phenomenological wave functions. The ideas discussed in the two preceding sections have shown that the vertex function Γ and hence also the wave function (4.1) satisfies the first and second conditions, at least for the impulse approximation. Furthermore, it can also be shown that application of the same ideas leads to an expression for the exchange current contribution in which (4.1) again completely describes the deuteron structure.[19] It would seem that we have at our disposal a general technique for expressing relativistic results in terms of (4.1), and this is the main reason for its significance. In the usual nonrelativistic approach, one is never certain how to treat the relative time (or energy), or combine a nonrelativistic wave function expressed in terms of Pauli spinors with a relativistic interaction term written as a Feynmn diagram, and one of the major values of this formalism is that these ambiguities are completely eliminated.

In terms of this wave function (4.1) Eq. (3.16) takes a very elegant form:

$$G_D{}^\mu(q)=\frac{1}{2(2\pi)^4}\int d^3p\left(\frac{M}{E_0}\right)\times\mathrm{tr}\{\bar\phi_{\frac{1}{2}q}(\mathbf{p}+\tfrac{1}{4}\mathbf{q})F^\mu(q)\phi_{-\frac{1}{2}q}(\mathbf{p}-\tfrac{1}{4}\mathbf{q})\}, \tag{4.3}$$

where we have changed $\mathbf{p}$ to $-\mathbf{p}$ in the integration and

$$\bar\phi_d(\mathbf{r})=\frac{K'^{1/2}C^{-1}[M-(\frac{1}{2}D-r)]^{\ \beta}(\frac{1}{2}D+r)[M+\frac{1}{2}D+r]\eta_\beta{}^*}{2M[M^2-(\frac{1}{2}D+r)^2]}. \tag{4.4}$$

This is an immediate generalization of the usual nonrelativistic results. For scalar deuterons, for example, one writes for the form factor

$$\begin{aligned} G_D(q)&=F(q)\frac{1}{4\pi}\int\phi(\mathbf{r})^2\exp^{(i\frac{1}{2}\mathbf{q}\cdot\mathbf{r})}d^3r\\ &=F(q)\frac{1}{2(2\pi)^4}\int\phi(\mathbf{p})\phi(\mathbf{p}-\tfrac{1}{2}\mathbf{q})d^3p\\ &=F(q)\frac{1}{2(2\pi)^4}\int\phi(\mathbf{p}+\tfrac{1}{4}\mathbf{q})\phi(\mathbf{p}-\tfrac{1}{4}\mathbf{q})d^3p. \end{aligned} \tag{4.5}$$

Hence (4.3) is a natural extension of (4.5).

Let us now examine in somewhat more detail the structure of (4.1). In preparation for this, observe that the off-mass-shell projection operator can be written

$$(M+\not p)/2M=\tfrac{1}{2}(1+\beta)u(\mathbf{p})\bar u(\mathbf{p})-\tfrac{1}{2}(1-\beta)v(-\mathbf{p})\bar v(-\mathbf{p}), \tag{4.6}$$

where $u(\mathbf{p})$ and $v(\mathbf{p})$ are the usual mass shell spinors[9] and

$$\beta=p_0/E_0, \quad E_0=[M^2+\mathbf{p}^2]^{1/2}. \tag{4.7}$$

This decomposition leads quite naturally to the view that the off-mass-shell projection operator describes a superposition of physical nucleon states; the joint presence of physical nucleons of momentum $\mathbf{p}$ and physical antinucleons of momentum $-\mathbf{p}$. This view is supported by the fact that the residue at the positive energy pole contains only the $u\bar u$ terms, while that at the negative energy pole contains only the $v\bar v$ terms. At energies intermediate between these poles, we have a superposition of states, or a sum of contributions from graphs involving different time orderings.

This can be clearly seen if we substitute (4.6) into (4.1) and rewrite as

$$\phi_d(\mathbf{r})=\phi^{++}(\mathbf{r},\mathbf{d})u(\tfrac{1}{2}\mathbf{d}+\mathbf{r})u^T(\tfrac{1}{2}\mathbf{d}-\mathbf{r})+\phi^{-+}(\mathbf{r},\mathbf{d})v(-\tfrac{1}{2}\mathbf{d}-\mathbf{r})u^T(\tfrac{1}{2}\mathbf{d}-\mathbf{r}), \tag{4.8}$$

[19] B. M. Casper (private communication).

where now

$$\phi^{++}(\mathbf{r},\mathbf{d})=\frac{K'^{1/2}M(1+\beta)\bar{u}(\frac{1}{2}\mathbf{d}+\mathbf{r})\Gamma^{\alpha}(\frac{1}{2}D+r)C\bar{u}^{T}(\frac{1}{2}\mathbf{d}-\mathbf{r})\xi_{\alpha}}{[M^2-(\frac{1}{2}D+r)^2]},$$

$$\phi^{-+}(\mathbf{r},\mathbf{d})=-\frac{K'^{1/2}M(1-\beta)\bar{v}(-\frac{1}{2}\mathbf{d}-\mathbf{r})\Gamma^{\alpha}(\frac{1}{2}D+r)C\bar{u}^{T}(\frac{1}{2}\mathbf{d}-\mathbf{r})\xi_{\alpha}}{[M^2-(\frac{1}{2}D+r)^2]}, \tag{4.9}$$

and ϕ^{++} is interpreted as the wave function of the deuteron in two positive energy states or two nucleons, and ϕ^{-+} is the wave function for one positive and one negative energy state, or using the usual hole interpretation, it is the wave function of an antinucleon and deuteron bound into a nucleon.[17] These two processes are shown diagramatically in Fig. 6(a) and (b), as well as the contributions they make to the deuteron form factor. The analogous processes ϕ^{+-} and ϕ^{--} do not contribute to (4.1) because one of the nucleons is on the mass shell.

The point is that Eq. (4.8) shows us that by virtue of the relativistic invariance, there is always a small contribution from ϕ^{-+} to scattering processes in which deuterons are involved, and this is a new contribution completely outside of the framework of conventional potential theory.

We now turn to the question of determining the wave functions ϕ. There are two possible approaches one can take. The first, a purely theoretical one, would proceed from this point by development of a relativistic wave equation for ϕ, and its solution in, say, the meson ladder approximation. It is our intention to pursue this at a later time. The other approach, and the one we will discuss now, is a phenomenological one.

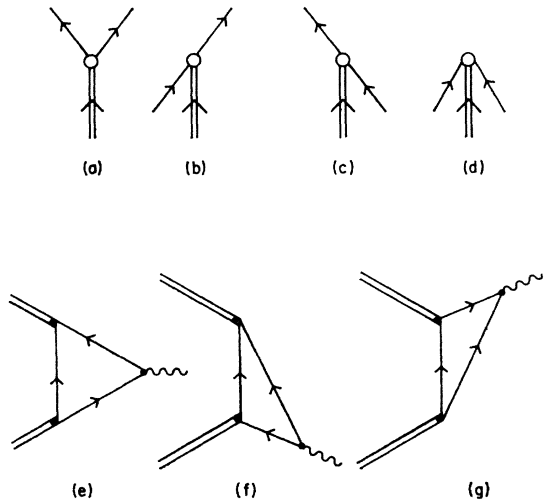

FIG. 6. Schematic of the different contributions to the wave function of the deuteron. Time increases from the bottom to the top of each diagram. (a) ϕ^{++}, (b) ϕ^{-+}, (c) ϕ^{+-}, (d) ϕ^{--}, (f) the contribution $(\phi^{++})^2$ to the deuteron form factor, (g) the contribution $\phi^{++}(\phi^{-+})$, (h) the contribution $(\phi^{-+})(\phi^{++})$. The contribution $(\phi^{-+})^2$ cannot be drawn easily and hence is not shown.

The expression (4.8) immediately suggests a procedure for identifying our relativistic wave function with nonrelativistic wave functions obtained from the analysis of nucleon-nucleon scattering. Use of these wave functions would seem to imply three assumptions: (1) We must restrict ourselves to the center-of-momentum system, for it is in this system only that the nonrelativistic analysis has been performed. (2) We must assume that there is no contribution from the negative energy states, for clearly nonrelativistic potential theory provides no mechanism for dealing with these. (3) We assume that the nonrelativistic wave functions were designed to be used with nonrelativistic expressions for the scattering amplitudes. This last assumption is necessary in order to define both the normalization and the over-all factor of momentum which should stand in front of the wave function.

Denoting the nonrelativistic wave function by $\phi^{\mathrm{NR}}(\mathbf{r})$, and employing the three assumptions above, we have

$$\phi^{++}(\mathbf{r},\mathbf{0})=(E_0/M)^{1/2}\phi^{\mathrm{NR}}(\mathbf{r}),$$
$$\phi^{-+}(\mathbf{r},\mathbf{0})=0. \tag{4.10}$$

The factor $(E_0/M)^{1/2}$ is necessary to comply with assumption (3) above, and reconciles Eq. (4.3) with (4.5). Any ambiguity in the identification (4.10) could be eliminated by a phenomenological analysis of nucleon-nucleon scattering using this formalism. [*Note added in proof.* Subsequent investigation suggests that a better procedure is to incorporate the factor $(M/E_0)^{1/2}$ into the current. This makes the normalization of the wave function consistent. It alters the results (4.17) for u_0 changing the $-5\mathbf{r}^2/12$ term to $-2\mathbf{r}^2/12$. See F. Gross (to be published).]

At first it might appear that these equations would not enable us to determine $\phi_d(\mathbf{r})$. However, as it turns out $\phi_d(\mathbf{r})$ depends on only four invariant functions which can be completely phenomenologically determined by (4.10). The rest of the results are kinematical, and hence we have at our disposal a systematic way to handle the nonrelativistic results in different reference frames.

We will sketch the results of the determination of the four invariant functions using (4.10). The invariant functions are contained in the *d-np* vertex

$$\Gamma^{\alpha}(p)\xi_{\alpha}=F(u)\gamma\cdot\xi-(G(u)/M)p\cdot\xi-((M-\not{p})/M)\times[H(u)\gamma\cdot\xi-(I(u)/M)p\cdot\xi], \tag{4.11}$$

where from (4.1)

$$p_0 = M_D - [M^2 + \mathbf{p}^2]^{1/2}, \qquad u = p^2. \tag{4.12}$$

Introducing the nonrelativistic wave function

$$\psi^{\mathrm{NR}}(\mathbf{x}) = (4\pi)^{-1/2}\chi_P\left[\frac{u(x)}{x}\boldsymbol{\sigma}\cdot\boldsymbol{\xi} - \frac{1}{\sqrt{2}}\frac{w(x)}{x} \times\left(\frac{3\boldsymbol{\sigma}\cdot\mathbf{x}(\mathbf{x}\cdot\boldsymbol{\xi})}{x^2} - \boldsymbol{\sigma}\cdot\boldsymbol{\xi}\right)\right]\frac{i\sigma^2}{\sqrt{2}}\chi_n, \tag{4.13}$$

where $u(x)$ and $w(x)$ are the usual S- and D-state wave functions, respectively, and $\boldsymbol{\xi}$ is the polarization vector of the deuteron,[5] the 3-space Fourier transform is

$$\phi^{\mathrm{NR}}(\mathbf{r}) = (4\pi)^{1/2}\chi_P\left[u_0(r)\boldsymbol{\sigma}\cdot\boldsymbol{\xi} + \frac{1}{\sqrt{2}}w_2(r) \times(3\sigma\cdot\mathbf{r}(\mathbf{r}\cdot\boldsymbol{\xi})/r^2 - \boldsymbol{\sigma}\cdot\boldsymbol{\xi})\right](i\sigma^2/\sqrt{2})\chi_n, \tag{4.14}$$

where $\mathbf{r}$ is the relative momentum,

$$u_0(r) = \int_0^\infty u(x) j_0(rx) x dx, \qquad w_2(r) = \int_0^\infty w(x) j_2(rx) x dx, \tag{4.15}$$

and

$$j_l(x) = (-x)^l\left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin x}{x} \tag{4.16}$$

are the usual spherical Bessel functions. Using the representation given in Ref. 9, we can now explicitly evaluate Eq. (4.10). If we retain only terms up to order η, we have

$$u_0(r) = \frac{N}{\mathbf{r}^2+\alpha^2}\left[F(u)\left(1 - \frac{5\mathbf{r}^2}{12M^2} - \frac{\alpha^2}{2M^2}\right) - \frac{H(u)}{M^2}(\mathbf{r}^2+\alpha^2) + \frac{\mathbf{r}^2}{3M^2}G(u)\right],$$

$$w_2(r) = \frac{\sqrt{2}N}{3(\mathbf{r}^2+\alpha^2)}\mathbf{r}^2\left(\frac{G(u)}{M^2} - \frac{F(u)}{2M^2}\right), \tag{4.17}$$

$$0 = H(u),$$

$$0 = \frac{G(u)-F(u)}{M}\left(1 - \frac{\mathbf{r}^2}{4M^2}\right) - \frac{2I(u)}{M}\left(1 + \frac{\mathbf{r}^2 - 2\alpha^2}{4M^2}\right).$$

Solving for the invariants we have

$$\frac{NF(u)}{\mathbf{r}^2+\alpha^2} = \left(u_0(r) - \frac{w_2(r)}{\sqrt{2}}\right)\left(1 + \frac{\mathbf{r}^2+2\alpha^2}{4M^2}\right),$$

$$\frac{NG(u)}{\mathbf{r}^2+\alpha^2} = \frac{3M^2}{\sqrt{2}\mathbf{r}^2}w_2(r) + \frac{1}{2}\left(u_0(r) - \frac{w_2(r)}{\sqrt{2}}\right)\left(1 + \frac{\mathbf{r}^2+2\alpha^2}{4M^2}\right), \tag{4.18}$$

$$H(u) = 0,$$

$$I(u) = \tfrac{1}{2}[G(u) - F(u)]\left(1 - (\mathbf{r}^2 - \alpha^2/2M^2)\right),$$

where, to first order

$$u = M^2 - 2(\mathbf{r}^2+\alpha^2); \quad N = K'^{1/2}/(2\pi)^{1/2}4M. \tag{4.19}$$

Hence the four invariants have been evaluated in terms of the S- and D-state wave functions[20] and we can now use (4.3) to calculate the relativistic form factor in terms of the usual S- and D-state wave functions. We will discuss these results in the next section.

5. SUMMARY AND CONCLUSIONS

We will summarize our method for treating loosely bound systems in scattering theory. First, one makes the assumption that the bound system is coupled to only one channel (one-channel coupling approximation). The validity of this assumption is discussed qualitatively in Sec. 2. Then, one writes down a general expression for the sum of all Feynman diagrams which one wishes to consider (impulse approximation or exchange-current contribution, for example) which is at the same time consistent with one-channel coupling. This means specifically that no diagrams are considered in which the bound state interacts directly with any channel other than the principal one. Then, the singularities in the integrand of the energy variable (or variables) are examined, and the integration over the energy(s) is performed by contour integration, retaining only the leading terms arising from singularities close to the region where both nucleons are on the mass shell (see Sec. 2). These leading terms always place one of the two bound-state particles on the mass shell, so that the bound-state vertex with only one particle off-the-mass shell is all that need be studied.

This process is illustrated in detail for the impulse approximation to the deuteron form factor in Sec. 3. As pointed out in this section, one of the advantages of this procedure is that it collects systematically the leading terms, and also admits easily a quantitative estimate of the errors involved. Besides, it is fully relativistic and unambiguous.

Contact with more conventional approaches is made by defining a relativistic bound-state wave function in

[20] Note that these results agree as well as can be expected with the identifications arrived by a different method in Ref. 5.

terms of the bound-state vertex function with one particle off-the-mass shell. For the deuteron this is done in Sec. 4, the wave function defined in Eq. (4.1). The major virtue of this wave function is that it is simpler than the B.S. amplitude, but just as valid in the domain of one-channel coupling. One may derive a relativistic wave equation for this wave function, or make some contact with phenomenology. A discussion of phenomenology is included in Sec. 4, with final identification of the four *d-np* invariants in terms of the *S*- and *D*-state wave functions (Eq. 4.18).

In conclusion, we can sum up three principal sources of error in nonrelativistic potential theory which this approach sheds some light on:

(1) Over all contributions due to relativistic kinematics [presence of the E_0 term in Eq. (4.3) and β in Eq. (4.9)];

(2) Contributions to the wave function due to motion of the deuteron;

(3) Contributions from antiparticle states, through the wave function ϕ^{-+} (Eq. 4.8).

Let us discuss these effects in turn. The first effect is the easiest to dispose of. It is not surprising that terms involving $(M^2+p^2)^{1,2}$ should appear in the correct expressions, due simply to the fact that the energies of the particles satisfy relativistic expressions. However, inasmuch as phenomenological wave functions have been calculated from potentials which have been chosen to fit the data using a nonrelativistic theory, such *over-all* factors should already be part of the wave function, and we cannot properly regard them as corrections. What they do tend to suggest is that an improved phenomenology (i.e., consistent with more experiments) should be possible with a relativistic formalism such as the one presented here.

The preceding remarks do not apply to the second source of error; correction terms arise from the fact that the wave function of a moving deuteron, $\phi^{++}(\mathbf{r},\mathbf{d})$, [defined in Eq. (4.8)] is not the same as the wave function of a stationary deuteron $\phi^{++}(\mathbf{r},\mathbf{0})$ which one identifies with the nonrelativistic wave functions. That this is true can be seen readily by expanding $\phi^{++}(\mathbf{r},\mathbf{d})$ in Pauli spinors. Besides the terms involving $\sigma\cdot\xi$ and $\sigma\cdot\mathbf{r}(\mathbf{r}\cdot\xi)$, we have other terms like $(\sigma\cdot\mathbf{r})(\mathbf{d}\cdot\xi)$, $\sigma\cdot\mathbf{d}(\mathbf{r}\cdot\xi)$, $\sigma\cdot\mathbf{d}(\mathbf{d}\cdot\xi)$, and $(\sigma\cdot\xi)\sigma\cdot\mathbf{d}(\sigma\cdot\mathbf{r})$, all of which are zero in the rest system of the deuteron. Hence, in the scattering amplitude where both deuterons cannot simultaneously be at rest (unless $q=0$) these terms will give small corrections arising from the fact that the *structure* of the wave function is altered when the deuteron is in motion.

Finally, the third source of error arises from neglect of contributions from ϕ^{-+}. These cannot be estimated without either a theoretical calculation of the deuteron nucleon invariants, or some phenomenological treatment of nucleon-nucleon scattering which includes these terms. It would appear that both approaches are possible, but for the time being the best assumption seems to be to assume that $\phi^{-+}(\mathbf{r},\mathbf{0})=0$. What we are saying is that the nonrelativistic phenomenological potentials have included as much as possible these negative energy contributions, by virtue of the fact that they fit the data.

An interesting, somewhat paradoxical result emerges from this procedure. The condition $\phi^{-+}(\mathbf{r},\mathbf{0})=0$ places two restrictions on the *d-np* vertex invariants [defined in Eq. (4.11)]:

$$H=0,\qquad I=\tfrac{1}{2}(G-F),\tag{5.1}$$

which is a result contrary to theoretical estimates of these quantities from one-pion-exchange calculations.[5] In these calculations it was found that H was a large and important term.

The resolution of this apparent paradox may have quite exciting implications about the two-nucleon system. Although the situation is not yet clear, it may not be out of place to hazard a few preliminary remarks about it now.

Examination of one-pion exchange suggests that F is small and H is large. This should be expected, however, because the γ^5 interaction would tend to favor the negative energy part of the *d-np* vertex (H and I). This suggests that the two pion contribution should have the converse effect, and indeed this is exactly what one finds. The two-pion exchange enhances F considerably and depresses H (slightly). Hence, F is larger than it first appeared, being enhanced by exchange of even numbers of pions, while H is enhanced by odd numbers of pions. However, it turns out that F and H have opposite signs, and it appears that they nearly cancel for large momenta. Since the S state is described by a sum of F and H[5] (Eq. 4.17), this would therefore show the symptoms of a repulsive core, where *neither* F nor H *alone* shows any such symptoms. Hence it appears that neglecting H may be acceptable even though it is large, provided we replace F with a phenomenological hard-core wave function, and that one should not be surprised if a calculation of F (alone) shows no sign of a hard core.

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APPENDIX

In this Appendix we determine the location of the singularities of the relative energy variable r_0 for the diagram shown in Fig. 7, which is the simplest of the proper contributions to the deuteron nucleon vertex. It is reasonable to suppose that all other proper diagrams will contribute no singularities not already included by this diagram. This remark is amplified at the end of the Appendix.

We will limit ourselves to scalar particles. Then the diagram, in Feynman parameters, becomes

$$\Lambda=\frac{1}{\pi^2 i}\int d^4q\{[q^2-\mu^2+i\epsilon][(\tfrac{1}{2}D+r+q)^2-M^2+i\epsilon][(\tfrac{1}{2}D-r-q)^2-M^2+i\epsilon]\}^{-1}$$

$$=2\int_0^1 d\alpha d\beta d\gamma\frac{\delta(\alpha+\beta+\gamma-1)}{[M_D{}^2\alpha\beta+(\tfrac{1}{2}D+r)^2\alpha\gamma+(\tfrac{1}{2}D-r)^2\beta\gamma-M^2(\alpha+\beta)-\mu^2\gamma+i\epsilon]}, \tag{A1}$$

where $D=p_1+p_2$ and $r=\frac{1}{2}(p_1-p_2)$ (see Fig. 7).

Taking as new variables of integration $x=\gamma$ and $y=(\alpha-\beta)/(1-\gamma)$ we have

$$\Lambda=\int_0^1 dx\int_{-1}^{+1}\frac{dy}{x}\frac{1}{(r_0-r_0{}^+{+}i\epsilon)(r_0-r_0{}^-{-}i\epsilon)}, \tag{A2}$$

where

$$r_0{}^\pm=\frac{-D_0 y}{2}\pm\left(\frac{\alpha^2}{x}+\frac{\mu^2}{1-x}+(\mathbf{r}+\tfrac{1}{2}y\mathbf{d})^2+\frac{M_D{}^2}{4}\frac{y^2}{x}\right)^{1/2}, \tag{A3}$$

$$\alpha^2=M^2-\tfrac{1}{4}M_D{}^2.$$

Note that it follows from (A2) that if $r\to -r$, then Λ is unchanged. Also the singularities in r_0 are along the real r_0 axis. It remains only to locate the branch points. These will occur at the minimum value of $r_0{}^+$ and the maximum value of $r_0{}^-$ which occur in the region of integration.

To find the extremes in x we require

$$\partial r_0{}^\pm/\partial x=0=-x^{-1}(\alpha^2+\tfrac{1}{4}M_D{}^2y^2)+\mu^2/(1+x)^2,$$
$$x=c/\mu+c;\quad c=[\alpha^2+\tfrac{1}{4}M_D{}^2y^2]^{1/2}.$$

Hence the extremum in x of $r_0{}^\pm$, $R_0{}^\pm$, is

$$R_0{}^\pm=-\tfrac{1}{2}D_0y\pm[(\mu+c)^2+(\mathbf{r}+\tfrac{1}{2}y\mathbf{d})^2]^{1/2}. \tag{A4}$$

Now we will prove that the y derivative is never zero.

$$\frac{\partial R_0{}^\pm}{\partial y}=-\tfrac{1}{2}D_0\pm\frac{(\mu+c)(M_D{}^2y/4c)+\frac{1}{2}(\mathbf{r}+\frac{1}{2}y\mathbf{d})\cdot\mathbf{d}}{[(\mu+c)^2+(\mathbf{r}+\frac{1}{2}y\mathbf{d})^2]^{1/2}},$$

but we can show that

$$D_0{}^2[(\mu+c)^2+(\mathbf{r}+\tfrac{1}{2}y\mathbf{d})^2]$$
$$>[((\mu+c)/c)\tfrac{1}{2}M_D{}^2y+(\mathbf{r}+\tfrac{1}{2}y\mathbf{d})\cdot\mathbf{d}]^2. \tag{A5}$$

This follows immediately from the three inequalities

$$M_D{}^2(\mu+c)^2>\tfrac{1}{4}M_D{}^4y^2((\mu+c)/c)^2,$$
$$\mathbf{d}^2(\mu+c)^2+M_D{}^2(\mathbf{r}+\tfrac{1}{2}y\mathbf{d})^2 \tag{A6}$$
$$>M_D{}^2y((\mu+c)/c)(\mathbf{r}+\tfrac{1}{2}y\mathbf{d})\cdot\mathbf{d},$$
$$\mathbf{d}^2(\mathbf{r}+\tfrac{1}{2}y\mathbf{d})^2\geqslant[(\mathbf{r}+\tfrac{1}{2}y\mathbf{d})\cdot\mathbf{d}]^2,$$

which in turn follow from

$$1>M_D{}^2y^2/(4\alpha^2+M_D{}^2y^2), \tag{A7}$$

which is true for all $-1<y<1$. To obtain (A5) we simply add the three inequalities (A6) and use

$$D_0{}^2=M_D{}^2+\mathbf{d}^2.$$

Hence

$$\partial R_0{}^\pm/\partial y<0 \tag{A8}$$

and the branch points are at

$$a_\pm=\pm[(\mu+M)^2+(\mathbf{r}\pm\tfrac{1}{2}\mathbf{d})^2]^{1/2}\mp\tfrac{1}{2}D_0$$
$$\approx\pm\mu. \tag{A9}$$

Note that as μ increases, these branch points recede away from the origin, and hence the branch points for the sum of all possible processes will be determined by that process for which μ is a minimum, and this is just the one-pion exchange diagram we have considered.

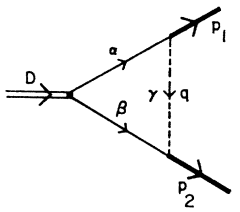

FIG. 7. The one-pion-exchange contribution to the d-np vertex, discussed in the Appendix.

Our derivation did not depend on the size of the deuteron mass; there were no "anomalous" thresholds in the relative energy r_0. The reason for this is that we restricted ourselves to $\mathbf{r}^2>0$, and it is well known that only for complex 3-momenta do anomalous thresholds occur. This can be graphically seen by examining Fig. 1. The "region of integration" in the figure is the region covered by $\mathbf{r}^2>0$, and it can be seen that no line of constant $\mathbf{r}^2>0$ (parabolas similar to the boundary) will intersect the dot-dashed line which is the line of anomalous singularities of the vertex function. For $\mathbf{r}^2<0$ we would encounter these singularities, which at $u=M^2$ (for example) are located along a cut beginning at $t=M^2+2\mu(\mu+2\alpha)$.

If we choose p_1 and D as our independent variables, then $r=p_1-\frac{1}{2}D$, and we obtain for the branch points in the $p_1{}^0$ plane

$$a_+=[(\mu+M)^2+\mathbf{p}_1{}^2]^{1/2},$$
$$a_-=D_0-[(\mu+M)^2+(\mathbf{p}_1-\mathbf{d})^2]^{1/2}. \tag{A10}$$