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Anomalous Magnetic Moment of the Electron, Muon, and Nucleon*

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The anomalous magnetic moment of the electron, $\frac{1}{2}(g\text{-}2)$, is computed using dispersion theory. The analytic continuation is made in the mass of one of the external electron lines and only the one-electron one-photon states are retained in the absorptive amplitude. In this way we relate g-2 to the Compton amplitude which has a known *exact* threshold behavior. Our approximation is an expansion in the low-energy behavior rather than a perturbation expansion in powers of 1/137, and we are able to show that a major contribution to g-2 comes from the low-mass region of the electron-photon system near the threshold of the absorptive amplitude. First, in a purely nonrelativistic calculation, we find that a major part of the $\alpha/2\pi$ correction is accounted for by the Thomson limit. Further refining our calculation by including the exact residue of the pole terms in the Compton amplitude in accord with the low-energy theorem on Compton scattering, we find that electron-photon states below $2.5mc^2$ in the absorptive amplitude reproduce 90% of the $-0.328\alpha^2/\pi^2$ contribution and predict a value of $\sim+0.15\alpha^3/\pi^3$ for the sixth-order term. We also give a simple physical interpretation of the difference of the muon and electron g-2 values. Finally we calculate with this approach the anomalous magnetic moments of the proton and neutron, with the Kroll-Ruderman theorem on meson photoproduction providing the low-energy "anchor" in this case. Again retaining only the low-mass region of the absorptive amplitude, we obtain fair agreement with the magnitude and the isovector character of the moments, finding $\Delta\mu^P\approx0.7(\Delta\mu_{\text{expt}})$ and $\Delta\mu^N\approx0.9(\Delta\mu_{\text{expt}})$.

I. INTRODUCTION

In his report to the 12th Solvay Congress on "The Present Status of Quantum Electrodynamics" (QED), Feynman called for more insight and physical intuition in QED calculations. To quote from a particularly relevant passage[1]: "It seems that very little physical intuition has yet been developed in this subject. In nearly every case we are reduced to computing exactly the coefficient of some specific term. We have no way to get a general idea of the result to be expected. To make my view clearer, consider, for example, the anomalous electron moment, $[\frac{1}{2}(g-2)=\alpha/2\pi-0.328\alpha^2/\pi^2]$. We have no physical picture by which we can easily see that the correction is roughly $\alpha/2\pi$, in fact, we do not even know why the sign is positive (other than by computing it). In another field we would not be content with the calculation of the second-order term to three significant figures without enough understanding to get a rational estimate of the order of magnitude of the third. We have been computing terms like a blind man exploring a new room, but soon we must develop some concept of this room as a whole, and to have some general idea of what is contained in it. As a specific challenge, is there any method of computing the anomalous moment of the electron which, on first rough approximation, gives a fair approximation to the α term and a crude one to α^2; and when improved, increases the accuracy of the α^2 term, yielding a rough estimate to α^3 and beyond?"

This paper is our answer to this challenge. We will show that the Schwinger correction,[2] $\alpha/2\pi$, of the electron (or muon) magnetic moment can be calculated approximately and very simply in terms of the *exact* Thomson limit to Compton scattering of photons by electrons.[3] All that is needed by way of formalism are the nonrelativistic Pauli two-component theory of the electron and the analytic property of Feynman graphs in perturbation theory that allows us to write a dispersion relation for the electromagnetic interaction vertex.[4]

We can do better than this by keeping relativistic kinematics and by using the full content of the exact low-energy theorem on Compton scattering including the magnetic-moment contributions which are linear in the energy.[5] If we include the moment to order $\alpha/2\pi$ in the Compton amplitude, the correct sign and approximately correct magnitude are obtained for the α^2 correction of $-0.328\alpha^2/\pi^2$ as computed in fourth-order perturbation theory by Sommerfield and Petermann.[6] Including the α^2 moment term in the Compton amplitude leads to the prediction of $\approx+0.15\alpha^3/\pi^3$ to sixth order. We also give a simple physical interpretation and calculation of the difference between the muon and electron moments.

Finally, with this approach we calculate the anomalous magnetic moments of nucleons. In this case the Kroll-Ruderman theorem provides the low-energy "anchor," and fair agreement is obtained both with the magnitude and the isovector character of the moments.

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[1] R. P. Feynman, in *The Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1961).
[2] J. Schwinger, Phys. Rev. **73**, 416 (1948).
[3] W. Thirring, Phil. Mag. **41**, 1193 (1950).
[4] Y. Nambu, Nuovo Cimento **1**, 610 (1958); L. D. Landau, Nucl. Phys. **13**, 181 (1959); J. D. Bjorken, doctoral dissertation, Stanford University, 1959 (unpublished).
[5] F. Low, Phys. Rev. **96**, 1428 (1954); M. Gell-Mann and M. Goldberger, Phys. Rev. **96**, 1433 (1954).
[6] C. Sommerfield, Phys. Rev. **107**, 328 (1957); Ann. Phys. (N. Y.) **5**, 26 (1958); A. Petermann, Helv. Phys. Acta **30**, 407 (1957).

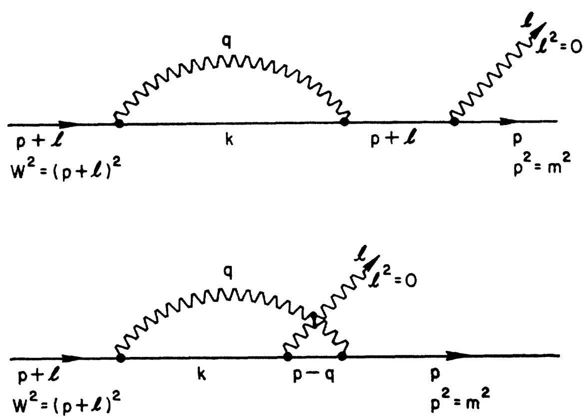


FIG. 1. Feynman graphs contributing to the first-order radiative corrections of the electron current.

II. NONRELATIVISTIC CALCULATION AND THE SCHWINGER CORRECTION

The Schwinger correction, $\frac{1}{2}(g-2)=\alpha/2\pi$, is found in lowest-order perturbation theory by evaluating the radiative correction to the electromagnetic vertex, Fig. 1. Instead of following this procedure we shall appeal to the familiar result that the Feynman amplitude for Fig. 1, studied as a function of the (mass)^2 of one of the external lines, with the other two on their mass shells, satisfies a dispersion relation in this variable. This property is valid to all finite orders in perturbation theory. In order to compute the anomalous moment $\frac{1}{2}(g-2)$ from the dispersion integral we must assume that the dispersion relation for the magnetic-moment part of the amplitude requires no subtractions; otherwise $g-2$, like the charge e, would be another parameter in the theory.

First we consider the analytic properties of the Feynman amplitude corresponding to Fig. 1 as a function of the invariant mass $W^2=(p+l)^2$ of the incoming electron[7] with the photon and outgoing electron set on their respective mass shells: $l^2=0$, $p^2=m^2$. The scalar functions multiplying the spinor factors, considered as functions of W^2, are analytic functions in the cut W^2 plane with a branch cut from m^2 to $+\infty$. The absorptive parts of these amplitudes are given by the discontinuity across this cut and are obtained by setting the internal photon and electron lines on their positive-energy mass shells. This is done by replacement of the propagators of these internal particles according to[8]

$$(q^2+i\epsilon)^{-1}(k^2-m^2+i\epsilon)^{-1}\to 2\pi^2\delta(k^2-m^2)\theta(k_0)\delta(q^2)\theta(q_0). \quad (1)$$

The absorptive amplitude will be given by

$$\mathrm{Abs}\left\{\int\frac{d^4q}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(q+k-p-l)\bar{u}(p)N(q,k,p)\frac{1}{q^2+i\epsilon}\,\frac{1}{k^2-m^2+i\epsilon}\right\}=\frac{W^2-m^2}{32\pi W^2}\int_{-1}^{1}dx\,\bar{u}(p)N(W^2,x), \quad (2)$$

where $x=\mathbf{q}\cdot\mathbf{l}/|\mathbf{l}|^2$ in the center-of-mass system with $p+l=(W,0)$ and $\mathbf{q}=-\mathbf{k}$. The quantity N is a numerator factor appearing in the amplitude, and is a polynomial which does not influence the analytic properties according to the Feynman rules which tie together the two ends of the internal photon line with[9] $g_{\mu\nu}$. The "cut" graphs corresponding to the absorptive amplitude of Eq. (2) are shown in Fig. 2 where we have indicated both the vertex and self-energy parts. These graphs illustrate how the absorptive part is obtained by multiplying the electromagnetic current by the Compton amplitude followed by an integration over the scattering angle x according to Eq. (2). This direct dependence of the absorptive amplitude on the Compton amplitude and on the current is the input that introduces physics via the factor N in the numerator.

Using the established analytic properties of the Feynman amplitude we can now exploit the known exact low-energy behavior of the Compton amplitude to estimate the anomalous moment. Before presenting a more formal discussion that provides a basis for a systematic iteration scheme we shall give a "first rough

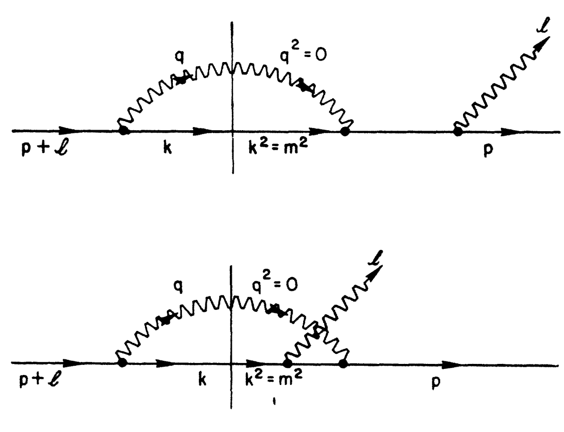


FIG. 2. Cut Feynman graphs contributing to the absorptive amplitude.

[7] A. M. Bincer, Phys. Rev. **118**, 855 (1960). Our notation in the following is: $p^2=p_0^2-\mathbf{p}^2=p_\mu p^\mu$; $g^{\mu\nu}=(1,-1,-1,-1)$; $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}$; $\sigma_{\mu\nu}=\frac{1}{2}i(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$; $\not{p}=p_\mu\gamma^\mu$, and $\hbar=c=1$.

[8] S. Mandelstam, Phys. Rev. **115**, 1741 (1959); R. Cutkosky, J. Math. Phys. **1**, 429 (1960).

[9] It is necessary to specify this particular gauge choice in computing the absorptive part because the amplitude for an off-mass-shell particle with arbitrary W^2 to come onto the mass shell upon radiation of a photon, $l^2=0$, is not in general gauge-invariant. By tying the internal photon line endings together with a $g_{\mu\nu}$ we ensure that the numerator $N(q,k,p)$ in Eq. (2) is purely a polynomial and does not influence the analytic properties of the Feynman amplitude which are determined by the denominator factors and for which there is a Nambu representation.

approximation" to $g-2$ using only the nonrelativistic Pauli two-component theory of the electron. The transition current for a Pauli electron with the gyromagnetic ratio g to emit a photon of momentum $\mathbf{l}\to 0$ is

$$\chi_f^*[e(\mathbf{l}/m)-ie(\boldsymbol{\sigma}\times\mathbf{l}/4m)g]\chi_i\,, \qquad (3)$$

where χ_f and χ_i denote the final and initial two-component spinors of the electron at rest and Eq. (3) is accurate to lowest order in v/c. In a relativistic dispersion study of the vertex as a function of electron mass W^2, the charge e and the normal moment $g=2$ associated with it in the Dirac theory are given by a subtraction constant as required by the Ward identity.[10] It is only the change in the g value $g-2$ arising from the radiative corrections contributing to the absorptive amplitude in Eq. (2) that we calculate from the dispersion relation

$$g-2=\frac{1}{\pi}\int_{m^2}^{\infty}\frac{\mathrm{Im}g(W^2)dW^2}{W^2-m^2}\,. \qquad (4)$$

For $\mathrm{Im}g(W^2)$ in the Pauli approximation we return to Eq. (2), multiply on the right by χ_i for the incident electron, insert nonrelativistic expressions for N, and project out the spin-dependent amplitude from

$$\frac{W^2-m^2}{32\pi W^2}\int_{-1}^{1}dx\,\bar{u}_f(p)N(W^2,\mu)\chi_i \xrightarrow[(W\to m)]{} -\frac{(W^2-m^2)m}{16\pi W^2}\sum_{\text{spins}}\int_{-1}^{1}dx\{[\chi_f^*\sum_{r,s=1}^{3}\epsilon_r\mathcal{T}_{rs}\chi_m][\chi_m^*j_s\chi_i] -[\chi_f^*\sum_{r=1}^{3}\epsilon_r\mathcal{T}_{r0}\chi_m][\chi_m^*j_0\chi_i]\}\,. \qquad (5)$$

In Eq. (5) ϵ_r is the photon polarization vector; $\chi_f^*\mathcal{T}_{rs}\chi_m$ is a nonrelativistic approximation to the Compton amplitude. By the low-energy theorem the perturbation result in terms of the physical charge e is also the *exact* Thomson amplitude as $\mathbf{p}\to 0$ and $(W-m)\to 0$:

$$\chi_f^*\mathcal{T}_{rs}\chi_m \xrightarrow[(W\to m)]{} -(e^2/m)\delta_{rs}\chi_f^*\chi_m\,. \qquad (6)$$

For the time component $\mathcal{T}_{r0}$ we invoke current conservation,

$$\sum_{s=1}^{3}\mathcal{T}_{rs}q_s-\mathcal{T}_{r0}q=0\,, \qquad (7)$$

so that

$$\chi_f^*\mathcal{T}_{r0}\chi_m \xrightarrow[(W\to m)]{} -(q_r/q)(e^2/m)\chi_f^*\chi_m\,.$$

The components of the transition current in the Pauli theory are given by

$$\chi_m^*j_0\chi_i=e\chi_m^*\chi_i\,, \qquad (8)$$

$$\chi_m^*\mathbf{j}\chi_i=e\chi_m^*[(\mathbf{k}+\mathbf{p})/m+(i/2m)\boldsymbol{\sigma}\times(\mathbf{k}-\mathbf{p})]\chi_i\,, \qquad (9)$$

describing the Schrödinger plus the spin currents created when the initial electron of momentum $\mathbf{p}$ accelerates to the intermediate one with momentum $\mathbf{k}$.

Introducing Eqs. (6)–(9) into Eq. (5), performing the angular integral, and picking off the coefficient of $\boldsymbol{\sigma}$ gives simply

$$\mathrm{Im}g(W^2)=((W^2-m^2)/4\pi W^2)e^2=(\alpha/W^2)(W^2-m^2)\,. \qquad (10)$$

Near threshold, as $W\to m$, Eq. (10) may be compared with the absorptive amplitude obtained from a relativistic perturbation approach. As we shall see later it not only reproduces the absorptive amplitude in the $\alpha/2\pi$, or Schwinger, approximation but is, in fact, exact in this limit.[11] From the low-energy theorem we know that the perturbation calculation of the Compton amplitude, expressed in terms of the exact renormalized charge, reproduces the exact Thomson limit as $l=W-m\to 0$. However, the dispersion integral in Eq. (4) extends over the entire physical range of Compton scattering energies $m^2\leq W^2\leq\infty$ and, for large W, Eq. (10) will fail for purely kinematic reasons. In fact the dispersion integral of Eq. (4) for $g-2$ diverges logarithmically if we use Eq. (10). Introducing a cutoff we find

$$\frac{g-2}{2}=\mu=\frac{\alpha}{2\pi}\int_{m^2}^{\Lambda^2}\frac{dW^2}{W^2}=\frac{\alpha}{2\pi}\ln\frac{\Lambda^2}{m^2}\,, \qquad (11)$$

which shows that the major part of the Schwinger correction arises near threshold when $m\leq W\lesssim 1.7$ m.

This result provides an interpretation of the Schwinger correction in terms of the Pauli current and the exact classical low-energy Thomson amplitude, which determines, moreover, the sign of the moment correction. Beyond this it suggests a program for computing the electron magnetic moment to higher accuracy by making full use of all the information in the low-energy theorem for scattering of light by a spin-$\frac{1}{2}$ particle. In addition to the exact Thomson limit at zero energy, Gell-Mann, Goldberger, and Low[5] proved that the terms in the Compton amplitude linear in photon energy can be expressed exactly in terms of the experimental magnetic moment of the electron (fermion). We may include these along with the Thomson limit in

[10] F. Low, Phys. Rev. **110**, 974 (1948). This corresponds to the fact that the absorptive amplitude for $F_1(W^2)$, when $l^2=0$, vanishes identically because real transverse photons cannot be radiated or absorbed in zero-zero transitions, or without flipping helicity for a Dirac particle of spin $\frac{1}{2}$. The charge thus appears as a subtraction constant.

[11] In Eq. (23) we find that the relativistic perturbation calculation differs from Eq. (10) simply by a kinematic factor m^2/W^2.

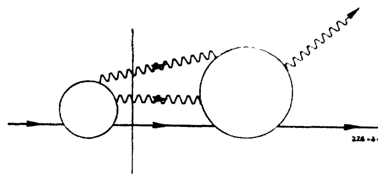
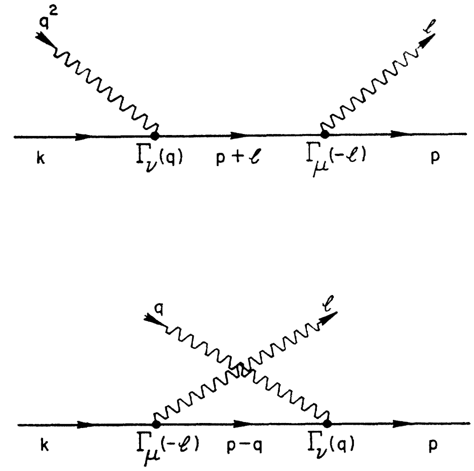


FIG. 3. Three-body intermediate state contributing to the absorptive amplitude to order α^2.



$$\Gamma_\mu(\ell) \equiv \gamma_\mu + \frac{(g-2)}{8m}\left[\gamma_\mu, \not{\ell}\right]$$

FIG. 4. Pole-term contribution to the Compton amplitude.

Eqs. (6) and (5), obtaining in this way an algebraic relation between the moment we wish to compute on the left-hand side of Eq. (4) and a quadratic form in μ on the right-hand side. In the spirit of a perturbation expansion in α, an input on the right of the anomalous moment accurate to order α^n gives an output to order α^{n+1} since the terms in $\mathrm{Im}g(W^2)$ are proportional to α, $\alpha\mu$, and $\alpha\mu^2$ depending on whether the charge or moment currents appear at the vertices of the Compton amplitude.

The expansion is not equivalent to the straight perturbation expansion in powers of α. In a complete calculation to order α^2 for instance, it is necessary to include radiative corrections both to the vertex and to the Compton scattering parts as in Fig. 3; this means three-body intermediate states, containing an electron plus two photons in the absorptive amplitude. However, the success of Eqs. (10) and (11) in approximately reproducing the Schwinger correction motivates us to retain nothing more than the full contents of the low-energy theorem as the major contributor to the absorptive amplitude. We do this now with the full relativistic kinematics in order to avoid an improper and exaggerated emphasis on the large W region and sensitivity to the cutoff. This means keeping the pole diagrams with physical charge plus magnetic moment vertices as in Fig. 4 in the Compton calculation. We want to see how well this approach reproduces the computed and measured α^2 contribution to the moment[6,12]:

$$\tfrac{1}{2}(g-2)_{\mathrm{theory}}=\alpha/2\pi-0.328\alpha^2/\pi^2\,,$$
$$\tfrac{1}{2}(g-2)_{\mathrm{experiment}}=\alpha/2\pi-[0.327\pm0.005](\alpha^2/\pi^2)\,. \quad (12)$$

In that we find that both the sign and approximate magnitude of $-0.3\alpha^2/\pi^2$ are obtained with this approach, we are encouraged to draw the conclusion that the α^3 result which emerges, $\approx+0.15\alpha^3/\pi^3$, does indeed have something to do with the full accurate result that awaits a very major calculation at this time.

The basic assumption is that the major contribution of the higher radiative corrections are contained in the pole terms of the Compton amplitude that are responsible for the low-energy theorem. We thereby relegate to a minor role the additional radiative corrections to the vertex and scattering amplitudes, due both to their virtual internal photons as well as the real ones exchanged between the vertex and scattering amplitudes.[13] With this approach a close agreement to the known α^2 moment is found, again with emphasis on the low-energy scattering region and with a logarithmic cutoff dependence for the dispersion integral. It is upon this result that we base our confidence in the α^3 prediction.

Before turning to the formal calculation we may review Feynman's remarks and ask what we have accomplished. A physical picture of the $\alpha/2\pi$ contribution has been given based formally on the existence of a disperson relation but with the simple elementary physical input of the Pauli current and Thomson amplitude. Correcting this calculation by making the kinematics relativistic and including the entire low-energy Compton scattering amplitude via the pole terms with physical vertices we obtain a good approximation to the α^2 result and offer an estimate of the α^3 term. Moreover we can systematically improve our calculations by keeping higher terms in the dispersion theory expansion over real intermediate states.

III. RELATIVISTIC CALCULATION

We return now to a covariant formulation in order both to give a systematic reduction to the Pauli two-component result of the last section and to initiate an iteration procedure for achieving higher accuracy in the calculation of $g-2$.

The most general expression for the electromagnetic current with the emerging electron and photon on their mass shells $p^2=m^2$ and $l^2=0$ takes the form[7]

$$e\bar{u}(p)\Gamma_\mu(p,p+l)=e\bar{u}(p)[\{F_1{}^+(W^2)\gamma_\mu-F_2{}^+(W^2)(i\sigma_{\mu\nu}l^\nu/2m)+F_3{}^+(W^2)l_\mu\}(\not{p}+\not{l}+m)/2m$$
$$+\{F_1{}^-(W^2)\gamma_\mu-F_2{}^-(W^2)(i\sigma_{\mu\nu}l^\nu/2m)+F_3{}^-(W^2)l_\mu\}(-\not{p}-\not{l}+m)/2m]\,. \quad (13)$$

[12] D. T. Wilkinson and H. R. Crane, Phys. Rev. **130**, 852 (1963).
[13] These corrections must always be included to the same accuracy in a perturbation expansion to avoid infrared difficulties.

The $F_i^{\pm}(W^2)$ are invariant functions of the scalar $W^2=(p+l)^2$ and are defined in the cut W^2 plane with a branch point at $W^2=m^2$ as discussed in the preceding section and illustrated in Fig. 5; for $W^2>m^2$, $F_i^{\pm}(W^2)=\lim_{\epsilon\to 0^+}F_i^{\pm}(W^2+i\epsilon)$. A further restriction on Eq. (13) comes from the Ward-Takahashi identity[14] which fixes

$$\bar{u}(p)\Gamma_\mu(p,\, p+l)l^\mu=\bar{u}(p)\boldsymbol{l}.$$

Hence in Eq. (13), $F_1^+(W^2)=F_1^-(W^2)=1$, a constant independent of W^2, and evidently the normal Dirac moment $g=2$ together with the charge appear *entirely* as a subtraction constant in the dispersion approach in the W^2 plane.[7,10] Equation (13) now simplifies to

$$e\bar{u}(p)\Gamma_\mu(p,\, p+l)=e\bar{u}(p)[\gamma_\mu+\{(-i\sigma_{\mu\nu}l^\nu/2m)F_2^+(W^2)+l_\mu F_3^+(W^2)\}\{(\boldsymbol{p}+\boldsymbol{l}+m)/2m\} +\{(-i\sigma_{\mu\nu}l^\nu/2m)F_2^-(W^2)+l_\mu F_3^-(W^2)\}\{(-\boldsymbol{p}-\boldsymbol{l}+m)/2m\}]. \quad (14)$$

F_2^+ is recognized as the anomalous magnetic moment as we go on to the mass shell,[15] $W^2=m^2$

$$e\bar{u}(p)\Gamma_\mu(p,\, p+l)u(p+l)=\bar{u}(p)[e\gamma_\mu-(ie\sigma_{\mu\nu}l^\nu/2m)\tfrac{1}{2}(g-2)]u(p+l). \quad (15)$$

This is the quantity of interest to us and we project it out of Eq. (14) with the operator[7]

$$\nu_\mu^{(2)}(p,l,s)=-(m/2(W^2-m^2)^2)[(\boldsymbol{p}+\boldsymbol{l}+m)(-i\sigma_{\mu\tau}l^\tau)+3l_\mu(\boldsymbol{p}+\boldsymbol{l}-m)]u(p,s), \quad (16)$$

which gives

$$\mathrm{Tr}\sum_{\pm s} e\bar{u}(p,s)\Gamma^\mu(p,\, p+l)\nu_\mu^{(2)}(p,l,s)=-(e/2m)F_2^+(W^2).$$

The dispersion relation for $F_2^+(W^2)$ is taken to be an unsubtracted one

$$F_2^+(m^2)\equiv\frac{g-2}{2}=\frac{1}{\pi}\int_{m^2}^{\infty}\frac{dW^2}{W^2-m^2}\,\mathrm{Im}F_2^+(W^2), \quad (17)$$

i.e., the charge e is the only coupling parameter introduced into quantum electrodynamics.[16] The anomalous moment is calculated from the radiative corrections.

We must now make a dynamical assumption in introducing $\mathrm{Im}F_2^+(W^2)$ on the right-hand side of Eq. (17) and this we do by retaining only the two-particle intermediate state of one electron plus one photon as in Fig. 2. We can now write this approximation using Eq. (2), and projection operator of Eq. (16):

$$\mathrm{Im}F_2^+(W^2)=-(m^2/8\pi)\rho(W^2)\times\sum_{\text{spins}}\int_{-1}^{1}dx\,\bar{u}(p)T^{\mu\nu}u(k)\bar{u}(k)\gamma_\nu\nu_\mu{}^2, \quad (18)$$

where $\rho(W^2)=(W^2-m^2)/W^2$ is a purely kinematic phase space factor. Here $\epsilon_{(l)}{}^\mu\epsilon_{(q)}{}^\nu\bar{u}(p)T_{\mu\nu}u(k)$ is the Compton amplitude for an initial photon of momentum q and polarization $\epsilon_{(q)}{}^\nu$, and an electron momentum k to scatter to a final state of a photon of momentum l and polarization $\epsilon_{(l)}{}^\mu$ and an electron of momentum $p=k+q-l$. In the center-of-mass of the scattering process $\mathbf{p}=-\mathbf{l}, \mathbf{q}=-\mathbf{k}$, $p_0+l_0=W$, $x=\mathbf{q}\cdot\mathbf{l}/|\mathbf{q}||\mathbf{l}|$. Equation (18) is the relativistic generalization of Eq. (5).

The exact Compton amplitude, and any satisfactory approximation to it, will satisfy the requirement of current conservation, i.e.,

$$l_\mu\bar{u}(p)T^{\mu\nu}u(k)=0\,,\qquad q_\nu\bar{u}(p)T^{\mu\nu}u(k)=0\,. \quad (19)$$

The first of these relations in particular permits us to

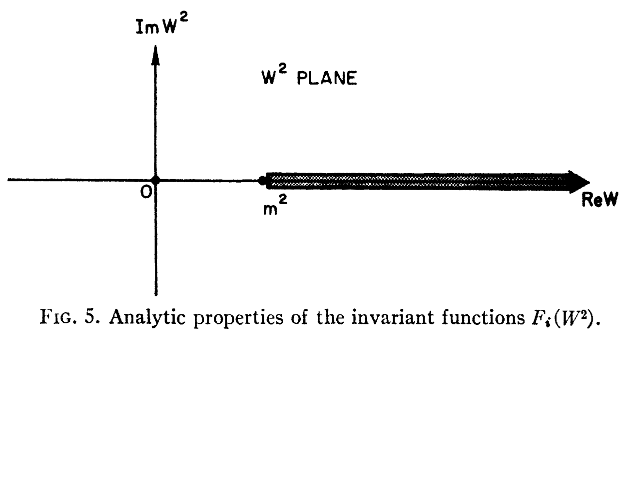


FIG. 5. Analytic properties of the invariant functions $F_i(W^2)$.

[14] J. C. Ward, Phys. Rev. **78**, 1821 (1950); Y. Takahashi, Nuovo Cimento **6**, 370 (1957). $\Gamma_\mu(p,\, p+l)$, the sum of all vertex graphs for the incident electron off the mass shell, $(p+l)^2=W^2$, is related to the Dyson irreducible vertex $V_\mu(p,\, p+l)$ by

$$\bar{u}(p)\Gamma_\mu(p,\, p+l)=\bar{u}(p)V_\mu(p,\, p+l)S_F'(p+l)S_F^{-1}(p+l),$$

i.e., $\Gamma_\mu(p,\, p+l)$ includes the self-mass insertions on the virtual electron leg. Therefore, from the identity of the above references,

$$\bar{u}(p)V_\mu(p,\, p+l)l^\mu=\bar{u}(p)S_F'(p+l)^{-1},$$

and we construct

$$\bar{u}(p)\Gamma_\mu(p,\, p+l)l^\mu=\bar{u}(p)S_F(p+l)^{-1}=\bar{u}(p)\boldsymbol{l}$$

using the vanishing of the inverse propagator at the pole:

$$\bar{u}(p)S_F'(p)^{-1}=\bar{u}(p)S_F^{-1}(p)=\bar{u}(p)(\boldsymbol{p}-m)=0.$$

[15] The term proportional to l_μ vanishes by time-reversal invariance; see F. J. Ernst, R. G. Sachs, and K. C. Wali, Phys. Rev. **119**, 1105 (1960).

[16] It has been proved that $F_2(\sigma^2)$, considered as a function of the photon mass σ^2 with the electron on the mass shell, must also vanish for $\sigma^2\to+\infty$ if the photon is to be described by an unsubtracted propagator and if, consequently, the vacuum polarization is to be a calculable quantity. See S. D. Drell, A. C. Finn, and A. C. Hearn, Phys. Rev. **136**, B1439 (1964).

discard all terms in Eq. (16) for $\nu_\mu{}^2$ that are proportional to l_μ when inserting into Eq. (18). Together with the identity, valid when $l^2=0$, $\boldsymbol{l}(-i\sigma_{\mu\tau}l^\tau)=l_\mu \boldsymbol{l}$ we can simplify the projection operator to

$$\nu_\mu{}^{(2)} \to (-m/2(W^2-m^2)^2)(\not p+m)(-i\sigma_{\mu\tau}l^\tau)u(p,s)=(-m^2/(W^2-m^2)^2)\sum_{\pm s'} u(p,s')\bar{u}(p,s')(-i\sigma_{\mu\tau}l^\tau)u(p,s)\,, \quad (20)$$

and cast the absorptive amplitude of Eq. (18) into a more transparent form:

$$\mathrm{Im}F_2{}^+(W^2)=\frac{-m^2}{8\pi}\left(\frac{-m^2}{W^2(W^2-m^2)}\right)\sum_{s,s',s''}\int_{-1}^{1} dx[\bar{u}(p,s)T_{\mu\nu}u(k,s'')][\bar{u}(k,s'')\gamma^\nu u(p,s')][\bar{u}(p,s')\{-i\sigma^{\mu\tau}l_\tau\}u(p,s)]\,. \quad (21)$$

Equation (21) is the product of a kinematic factor, the Compton amplitude $\bar{u}(p)T_{\mu\nu}u(k)$, the electron transition current from a state of momentum $\mathbf{p}$ to $\mathbf{k}$, $\bar{u}(k)\gamma^\nu u(p)$, and the spin projection operator. We can return to our previous nonrelativistic form by a systematic low-energy reduction from it or we can proceed directly to higher order corrections by including the full content of the low-energy Compton theorems in $T_{\mu\nu}$.

First, however, we may remark that the Klein-Nishina formula of lowest order perturbation theory,

$$\bar{u}(p)T^{\mu\nu}u(k)=-e^2\bar{u}(p)[\gamma^\mu(\not p+\not l-m)^{-1}\gamma^\nu +\gamma^\nu(\not p-\not q-m)^{-1}\gamma^\mu]u(k)\,, \quad (22)$$

when introduced in Eq. (21) gives

$$\mathrm{Im}F_2{}^+(W^2)=\tfrac{1}{2}\alpha((W^2-m^2)/W^2)(m^2/W^2)\,, \quad (23)$$

which reproduces the Schwinger correction $\frac{1}{2}(g-2)=\alpha/2\pi$. Equation (23) differs from our earlier two-component calculation of Eq. (10) only by a relativistic kinematic correction m^2/W^2 [recall that $\mathrm{Im}F_2{}^+(W^2)=\frac{1}{2}\,\mathrm{Im}g(W^2)$]. The ingredients of that earlier approach are apparent in the factors of Eq. (21); in particular, there appear the Thomson amplitude and the transition current from the initial electron with $\mathbf{p}$ to one with momentum $\mathbf{k}$. The detailed reduction of Eq. (21) to that result is carried out in the Appendix.

We also see in Eqs. (23) and (17) the dominant role of the low-mass region near threshold $W\sim m$ in the magnetic moment calculation. If we keep only the contribution between $m\leq W\leq \epsilon_{\max}\equiv\lambda m$ in Eq. (17) we find that

$$\tfrac{1}{2}(g-2)_\lambda=(\alpha/2\pi)(1-1/\lambda^2)\,, \quad (24)$$

which shows that more than 80% of the Schwinger correction comes from the mass region $<2.3m$ or within 0.65 MeV of threshold. Comparing Eq. (24) with Eq. (11) we see that the correct relativistic kinematics has converted the logarithmic cutoff dependence there into $(1-1/\lambda^2)$, a convergent factor as $\lambda\to\infty$.

Encouraged by this result we proceed to a calculation of higher order corrections to $\frac{1}{2}(g-2)$ by including the entire contents of the pole terms to the Compton scattering amplitude in Eq. (21). This means replacing the bare vertices γ_μ in Eq. (22), corresponding to a Dirac particle with $g=2$, by the vertex for a Dirac particle with arbitrary g:

$$\begin{aligned}\gamma_\mu &\to \Gamma_\mu(l)=\gamma_\mu+((g-2)/8m)[\gamma_\mu,\not l]\,,\\ \gamma_\nu &\to \Gamma_\nu(-q)=\gamma_\nu+((g-2)/8m)[\not q,\gamma_\nu]\,.\end{aligned} \quad (25)$$

The complete vertex of Eq. (14) reduces to Eq. (25) at the intermediate electron pole as seen in Eq. (15). As shown by Low, Gell-Mann, and Goldberger[5] the nonrelativistic reduction of Eq. (22) with the currents of Eq. (25) operating at the vertices as in Fig. 4, gives the exact low-energy Compton amplitude through first order terms in the energy ω, i.e.,

$$\begin{aligned}\epsilon_l{}^\mu\epsilon_q{}^\nu\bar{u}(p,s)T_{\mu\nu}u(k,s') \xrightarrow[(\omega=W-m\to 0)]{} \chi_s{}^*[&-(e^2/m)\boldsymbol{\varepsilon}_l\cdot\boldsymbol{\varepsilon}_q+(ie^2/m)(\omega/2m)\\ &\times\{(g-1)\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}_l\times\boldsymbol{\varepsilon}_q+(\tfrac{1}{4}g^2)\boldsymbol{\sigma}\cdot[(\boldsymbol{\varepsilon}_q\times\hat{q})\times(\boldsymbol{\varepsilon}_l\times\hat{l})]+(\tfrac{1}{2}g)[(\boldsymbol{\varepsilon}_q\cdot\hat{l})\boldsymbol{\varepsilon}_l\cdot(\boldsymbol{\sigma}\times\hat{l})-(\boldsymbol{\varepsilon}_l\cdot\hat{q})\boldsymbol{\varepsilon}_q\cdot(\boldsymbol{\sigma}\times\hat{q})]\}]\chi_{s'}\,,\end{aligned} \quad (26)$$

where $\omega=|\mathbf{l}|=|\mathbf{q}|$, $\hat{l}=\mathbf{l}/|\mathbf{l}|$, and $\hat{q}=\mathbf{q}/|\mathbf{q}|$. The corrections in Eq. (25) are proportional to α and to higher powers in the fine-structure constant and we turn to them, and through them to the full low-energy Compton scattering theorem of Eq. (26), as the major contributors to the higher order corrections to the electron $g-2$ value.

Including the α corrections by setting $\frac{1}{2}(g-2)=\alpha/2\pi$ in Eq. (25) we compute the α^2 contribution to the absorptive part. To the extent that we are able to reproduce in this way the known α^2 contribution to the $g-2$ value of the electron we are motivated to push on, including the value of $g-2$ through second order in α^2 in Eq. (25) to derive an approximation to the α^3 moment.

The absorptive part Eq. (21) to order α^2 is found using Eqs. (22) and (25) by direct calculation:

$$\begin{aligned}\{\mathrm{Im}F_2{}^+(W^2)\}_{(\alpha^2)}&=\frac{\alpha^2}{16\pi}\frac{(W^2-m^2)}{W^2}\left\{\frac{2m^2}{W^2-m^2}\left[\frac{W^2}{W^2-m^2}\ln\left(\frac{W^2}{m^2}\right)-1\right]-\frac{6W^2-5m^2}{W^2}\right\}\\ &\sim -(\alpha^2/3\pi)(W^2-m^2)^2/m^4\,,\quad W^2-m^2\to 0\\ &\sim -3\alpha^2/8\pi\,,\quad W^2\gg m^2\,.\end{aligned} \quad (27)$$

Inserting Eq. (27) into Eq. (17) and cutting off the logarithmically divergent integral at $W=\lambda m$ gives for the α^2 correction

$$\tfrac{1}{2}(g-2)_{\alpha^2}=-(\alpha/2\pi)^2[(\ln\lambda^2)(\tfrac{3}{2}+(2(\lambda^2-1))^{-1})-7/4+5/4\lambda^2]\begin{array}{l}\approx -0.28\alpha^2/\pi^2 \quad \text{for} \quad \lambda^2=5,\\ \approx -0.33\alpha^2/\pi^2 \quad \text{for} \quad \lambda^2=6,\end{array} \tag{28}$$

showing as in Eq. (24) that more than 80% of the correct fourth-order anomalous moment contribution comes the mass region $<2.3m$. Evidently the sum of contributions from radiative corrections to the vertex and to the Compton scattering amplitude in addition to the pole terms in Eq. (18), as well as the high-energy contributions to the dispersion integral, play only a minor role.

We proceed then to the next higher order in α, including the full pole term in the Compton scattering, setting the anomaly to its "experimental value"

$$\tfrac{1}{2}(g-2)=(\alpha/2\pi)-0.328\alpha^2/\pi^2 \tag{29}$$

and choosing the same cutoff indicated in Eq. (28). In this way we deduce the approximate α^3 moment according to the idea of low-energy dominance which was successful to orders α and α^2.

The total contribution to the absorptive part Eq. (21) to all orders of α coming from the pole terms, Eqs. (22) and (25), is

$$\mathrm{Im}F_2^+(W^2)=\frac{\alpha}{2}\frac{(W^2-m^2)}{W^2}\left[\frac{m^2}{W^2}+\frac{g(g-2)}{16}\left\{\frac{2m^2}{(W^2-m^2)}\left(\frac{W^2}{W^2-m^2}\ln\frac{W^2}{m^2}-1\right)-\frac{6W^2-5m^2}{W^2}\right\}+\frac{(g-2)^2}{4}\left(\frac{W^2-m^2}{W^2}\right)\right], \tag{30}$$

and the resulting $g-2$ value is

$$\frac{g-2}{2}=\frac{\alpha}{2\pi}-\frac{\alpha}{2\pi}\frac{g(g-2)}{4}\left[(\ln\lambda^2)\left\{\frac{3}{2}+\frac{1}{2(\lambda^2-1)}\right\}-\frac{7}{4}+\frac{5}{4\lambda^2}\right]+\frac{\alpha}{2\pi}\left(\frac{g-2}{2}\right)^2\left[\ln\lambda^2-1+\frac{1}{\lambda^2}\right]$$

$$\begin{aligned}&=\frac{\alpha}{2\pi}-0.28\frac{\alpha^2}{\pi^2}+0.14\frac{\alpha^3}{\pi^3} \quad \text{for} \quad \lambda^2\approx 5,\\ &=\frac{\alpha}{2\pi}-0.33\frac{\alpha^2}{\pi^2}+0.17\frac{\alpha^3}{\pi^3} \quad \text{for} \quad \lambda^2\approx 6,\end{aligned} \tag{31}$$

or approximately

$$\frac{g-2}{2}\approx\frac{\alpha/2\pi}{1+0.6\alpha/\pi}.$$

Equation (31) represents our final result and is presented here as our answer to Feynman's challenge quoted in the first paragraph.

The analyticity properties of Feynman graphs together with the exact low-energy Compton scattering limits for photon-electron scattering have been used as a basis for an approximation scheme. The Schwinger correction of $\alpha/2\pi$ is reproduced approximately using only the exact Thomson limit at zero energy to the Compton amplitude and the two-component Pauli current for a nonrelativistic electron. These physical inputs also fix the sign of $g-2$ correction. Retaining relativistic kinematics as well as the full content of the low-energy Compton scattering theorem gives the exact Schwinger correction as well as a good approximation to the Petermann-Sommerfield term, $-0.328\alpha^2/\pi^2$. On the basis of this success we conjecture that our α^3 prediction of $\approx+0.15\alpha^3/\pi^3$ is correct in sign and approximately valid in magnitude.

In order to complete this program and to establish the validity our approximation of dominance of the low-energy amplitudes two calculations must be undertaken. One includes a more complete treatment of the current, represented by $\bar{u}(k)\gamma^\nu u(p)$ in Eq. (21) for the incident electron to produce the 1γ ,$1e$ state, by retaining the full vertex structure of Eq. (14). This analysis is currently in progress and necessitates a discussion of the compensating infrared divergences buried in $F_2^-(W^2)$ as well as in the 2γ, $1e$ intermediate states. The aim here is to compute as much as possible, if not all, of the contributions to $\mathrm{Im}F_2^+(W^2)$ that are proportional to $(W^2-m^2)^2$ near threshold $(W^2-m^2)\to 0$, in addition to the exact threshold term Eq. (10). It is readily established that the additional contributions to $\mathrm{Im}F_2^+(W^2)$ from the $F_2^\pm(W^2)$ terms in Eq. (14) are proportional to $(W^2-m^2)^2$ as $W^2-m^2\to 0$ and therefore do not disturb the exact low-energy behavior of the absorptive amplitude given by Eq. (10). The terms in Eq. (14) proportional to $F_3^\pm(W^2)$ vanish when contracted with the Compton amplitude according to current conservation, i.e., in Eq. (18), $\gamma_\nu\to q_\nu F_3^\pm(W^2)$ and by Eq. (19), $\{\bar{u}(p)T^{\mu\nu}u(k)\}q_\nu=0$.

The second calculation is of course the complete α^3 perturbation calculation including the $(3\gamma,1e)$ intermediate states which must be faced up to eventually.

Our prediction of $\approx +0.15\alpha^3/\pi^3$ cannot be tested against experiment until more accurate determinations of the muonium (μ^+e^-) hyperfine structure in the ground state, or of the deuterium or helium fine structure, yield a more accurate determination of the fine structure constant.[17] The present limit of accuracy

$$\alpha^{-1}=137.0388\pm 0.0006\,,$$

when included in writing the $\alpha/2\pi$ term in the formula Eq. (12), leads to an uncertainty

$$\delta(\tfrac{1}{2}(g-2))=\delta(\alpha/2\pi)=\pm 0.4\alpha^3/\pi^3\,,$$

due to the experimental uncertainty in α alone. Thus more than a more accurate $g-2$ measurement itself is needed before a test of the sixth-order anomalous moment is achieved.

IV. MUON g-2 VALUE

An alternative dispersion approach to the calculation of g-2 is to study the electromagnetic vertex as a function of the momentum transfer introduced by the electromagnetic current, with both fermion lines on their mass shells. This is the form of dispersion relation familiar in the electromagnetic form factor studies,[18] viz.,

$$F_2(q^2)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{d\sigma^2\,\mathrm{Im}F_2(\sigma^2)}{\sigma^2-q^2-i\epsilon}\,. \qquad (32)$$

In this approach the absorptive amplitude is obtained by multiplying the current to produce a lepton pair by the Bhabha scattering amplitude in the $3S_1$ and $3D_1$ states. Once again a perturbation approach reproduces the Schwinger correction and the positive sign of the moment correction corresponds to the fact that the electron and positron *attract* one another via their Coulomb interaction in the dominant S-state interaction. We have preferred the sidewise dispersion relations of the previous section as being of more direct intuitive appeal, although similar results to order α^2 and α^3 can be obtained from Eq. (32) by including anomalous moment contributions to the e^-e^+ rescattering[19]; no exact low-energy theorem can be cited, however.

When we turn to a study of the muon $g-2$ value,

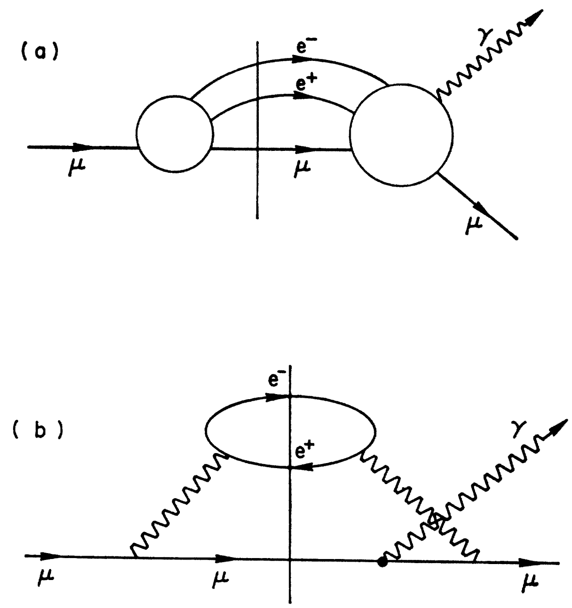


FIG. 6. Vacuum polarization contribution to the muon current.

however, our previous considerations fail. This is because we have relied heavily on the assumption that the low-energy behavior near the Compton scattering threshold plays the dominant role. For muons, however, there exist states such as the one-muon plus e^-e^+ pair states shown in Fig. 6(a) which has a very low-lying threshold at

$$(m_\mu+2m_e)^2\simeq m_\mu^2[1+4m_e/m_\mu]\simeq 1.02m_\mu^2\,,$$

and must not be ignored. This absorptive amplitude arises from the indicated cut through the vacuum polarization bubble in Fig. 6(b). Whereas this vacuum polarization contribution is totally negligible for the electron $g-2$ value[20] and has a threshold of $3m_e$ above and beyond our low-energy region, it plays an important role in the muon problem with its low-lying threshold.

In order to study the muon $g-2$ value we turn then to Eq. (32) and attempt to calculate the *difference* between the muon and electron moments due to the vacuum polarization contributions. Our aim here is to give a very simple physical basis for understanding this difference in sign and approximate magnitude. As computed with perturbation theory[21] through order α^2 it equals

$$\tfrac{1}{2}(g-2)_\mu-\tfrac{1}{2}(g-2)_e=0.75\alpha^2/\pi^2 -(-0.328\alpha^2/\pi^2)\simeq 1.08\alpha^2/\pi^2\,. \qquad (33)$$

We achieve this by observing simply that a vacuum polarization contribution necessarily enhances the pho-

[17] W. E. Cleland, J. M. Bailey, M. Eckhause, V. W. Hughes, R. M. Mobley, R. Prepost, and J. E. Rothberg, Phys. Rev. Letters **13**, 202 (1964).

[18] G. F. Chew, R. Karplus, S. Gasiorowicz, and F. Zachariasen, Phys. Rev. **110**, 265 (1958); P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. **112**, 642 (1958). Here $4m^2$ is the threshold for a state containing a fermion-antifermion pair. There are states containing photons only with a threshold at $\sigma^2=0$, but these first contribute to $F_2(q^2)$ to order α^3.

[19] H. R. Pagels, Ph.D. thesis, Stanford University Physics Department, 1965 (unpublished). The idea of using the approach with Eq. (32) to answer the Feynman challenge occurred independently to A. Petermann and to one of us (SDD) in 1962 and was discussed then as a way of relating the positive sign of the Schwinger correction with the attractive nature of the electron-positron interaction. See S. D. Drell and F. Zachariasen, Phys. Rev. **111**, 1727 (1959).

[20] It contributes

$$\alpha^2/\pi^2(119/36-\tfrac{1}{3}\pi^2)=0.016\alpha^2/\pi^2.$$

[21] H. Suura and E. Wichmann, Phys. Rev. **105**, 1930 (1957); A. Petermann, Phys. Rev. **105**, 1931 (1957).

ton propagator and therefore increases the e^-e^+ attraction since the spectral function $\pi(\sigma^2)$ is always greater or equal to zero in[22]

$$D_F'(q^2)_{\mu\nu} = -g_{\mu\nu}D_F' = g_{\mu\nu}\left[\frac{1}{q^2}+\int_0^\infty \frac{\pi(\sigma^2)d\sigma^2}{q^2-\sigma^2}\right].$$

For scattering $q^2=-|q|^2$ and, since $\pi(\sigma^2)\geq 0$,

$$D_F'(-|q|^2)=\left[\frac{1}{|q|^2}+\int_0^\infty \frac{\pi(\sigma^2)d\sigma^2}{\sigma^2+|q|^2}\right]\geq\frac{1}{|q|^2}.$$

Physically this inequality records the fact that the vacuum polarization cloud shields the bare electron or muon charge and thus a probe of the electron or muon charge to within the range $\hbar/2mc\sim 2\times 10^{-11}$ cm of the vacuum polarization cloud "sees" a larger charge and hence a stronger attraction. In the Lamb shift, for example, vacuum polarization contributes[23] an added binding of ≈ -27 Mc/sec lowering the $2S_{1/2}$ relative to the $2P_{1/2}$ level since in the latter state the electron and proton remain outside of their vacuum polarization clouds due to the centrifugal barrier.

The difference [Eq. (33)] in the vacuum polarization contributions to the muon and electron $g-2$ values arises solely from the difference in their scales of energy and momentum transfers, and hence of impact parameters of the rescattering of the $\mu^+\mu^-$ or e^+e^- pairs in Fig. 7. The energy scale is determined by the rest masses of the particles, i.e., by the threshold $4m^2$ in Eq. (32). The momentum transfer in the scattering, $t=q^2$, is related to the total energy $s=\sigma^2$ according to

$$t=-\tfrac{1}{2}(s-4m^2)(1-x)\,, \tag{34}$$

where x is the cosine of the center-of-mass scattering angle which is integrated over in the scattering. The angular momentum and parity selection rules assure us that only the $3S_1$ and $3D_1$ scattering channels contribute (i.e., $J=1$ and $C=-1$ corresponding to a photon) and

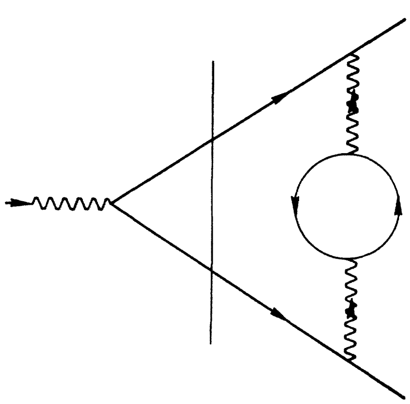


FIG. 7. Cut vacuum-polarization Feynman graph.

[22] G. Källén, Helv. Phys. Acta. **25**, 417 (1952).

[23] Page 66 of Ref. 1 and A. Petermann, Fortshcr. Physik **6**, 505 (1958).

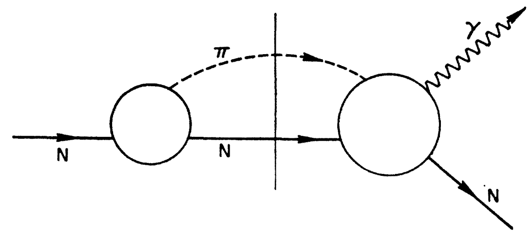


FIG. 8. Pion-nucleon intermediate-state contribution to the absorptive part of the nucleon current.

so the effective t values are just the energies $s\propto m^2$ for $x\simeq 0$ in Eq. (34). The $\mu^+\mu^-$ scattering is thus at small impact parameters, $\approx 1/m_\mu$ and the corresponding vacuum polarization contribution is large. In fact, we may simply extrapolate the Serber-Uehling[24] term for the vacuum polarization contribution for large $|t|\approx\bar{t}=m_\mu^2\gg m_e^2$ and enhance the muon $g-2$ calculation by just that amount

$$\left[\frac{1}{t}\left(1+\frac{\alpha}{3\pi}\ln\frac{\bar{t}}{m_e^2}\right)-\frac{1}{t}\right]\approx\frac{1}{t}\left[\frac{\alpha}{3\pi}\ln\frac{m_\mu^2}{m_e^2}\right],$$

i.e.,

$$\left(\frac{g-2}{2}\right)_\mu-\left(\frac{g-2}{2}\right)_e\approx\frac{\alpha}{2\pi}\left[\frac{\alpha}{3\pi}\ln\frac{m_\mu^2}{m_e^2}\right]. \tag{35}$$

Up to terms of order unity relative to $\ln(m_\mu^2/m_e^2)$, Eq. (35) just accounts for the more accurately computed difference, Eq. (33). The sign of the difference is now clearly seen to arise from the increased attraction between the $\mu^+\mu^-$ pair as they scatter with $J=1$ at small impact parameters, $\sim 1/m_\mu$, within the vacuum polarization cloud.

V. NUCLEON g-2 VALUE

As a final application of the ideas in this paper and as independent evidence supporting the point of view exploited here—namely that of dominance of the threshold contributons to the absorptive amplitude—we turn to the nucleon anomalous moment calculation as first studied by Bincer[7] as a function of the incident fermion mass W^2. As illustrated in Fig. 8, the absorptive amplitude at threshold $W^2=(M+\mu)^2$ corresponds now to the pion emission amplitude multiplied by photopion production. The exact threshold behavior of the photopion amplitude—i.e., S-wave production of charged pions—is given, for $\mu/M\to 0$ for the external pion line, by the renormalized perturbation term according to the Kroll-Ruderman theorem[25] which replaces the Thomson limit for this application. Its relativistic dispersion theoretic form, i.e., the complete pole term, gives rise to both P- and S-wave amplitudes with $J=\frac{1}{2}$ that are relevant[7] in our applications.

[24] R. Serber, Phys. Rev. **48**, 49 (1935); E. Uehling, Phys. Rev. **48**, 55 (1935).

[25] N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954).

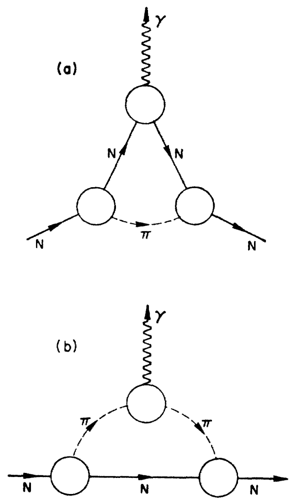


FIG. 9. Perturbation contributions to the nucleon current.

Keeping just the leading term at threshold, $W^2-M^2\to 0$, letting $\mu/M\to 0$, repeating the calculations of Sec. III with the renormalized Born amplitude for photopion production replacing Eq. (22), and with

$$e\bar{u}(k)\gamma^\nu u(p)\to g\bar{u}(k)i\gamma_5\boldsymbol{\tau}u(p) \quad (36)$$

in Eq. (21), we find in analogy with Eq. (23) [the $1/W^2$ comes from phase space as found in Eq. (2)] that

$$\begin{aligned}\mathrm{Im}F_2{}^P(W^2)&=-\mathrm{Im}F_2{}^N(W^2)=\mathrm{Im}F_2{}^V(W^2)\\&\simeq\tfrac{1}{2}(g^2/4\pi)(W^2-M^2)/W^2,\quad\text{as}\quad W^2\to M^2.\end{aligned} \quad (37)$$

The superscripts P, N, and V denote proton, neutron, and isovector, respectively, and the $\boldsymbol{\tau}$ in Eq. (36) are the isotopic Pauli matrices; $g^2/4\pi\approx 15$ is the pion-nucleon coupling strength. The resulting anomalous moments are

$$\Delta\mu^P=-\Delta\mu^N=\frac{1}{2\pi}(g^2/4\pi)\int_{M^2}^{\lambda^2M^2}\frac{dW^2}{W^2}=\frac{1}{2\pi}(g^2/4\pi)\ln\lambda^2$$
$$\approx 2\text{ for }\lambda^2\sim 2.3. \quad (38)$$

The contribution to the isoscalar absorptive amplitude vanishes $\propto (W^2-M^2)^2$ at threshold. If we retain the entire Born amplitude for arbitrary W^2 but again approximating $\mu/M\to 0$ for simplicity we find

$$\begin{aligned}\mathrm{Im}F_2{}^S(W^2)&=\frac{3}{4}\left(\frac{g^2}{4\pi}\right)\left[\frac{(3W^2-M^2)M^2}{2M^4}-\frac{M^2\ln(W^2/M^2)}{W^2-M^2}\right]\\&\simeq-\frac{(W^2-M^2)^2}{4M^4}\left(\frac{g^2}{4\pi}\right)\text{ for }W^2\to M^2,\end{aligned} \quad (39)$$

$$\mathrm{Im}F_2{}^V(W^2)=\frac{1}{4}\left(\frac{g^2}{4\pi}\right)\left[\frac{(W^2-3M^2)M^2}{2W^4}+\frac{M^2\ln(W^2/M^2)}{W^2-M^2}\right].$$

Although it has an extra factor of (W^2-M^2) at threshold, the absorptive amplitude for the isoscalar moment grows rapidly until for $W^2/M^2\gg 1$,

$$\mathrm{Im}F_2{}^S(W^2)\approx -3\,\mathrm{Im}F_2{}^V(W^2).$$

It is here that we see explicitly the failure of the perturbation calculations of the anomalous moments which have long been known to predict much too large an isoscalar part.[18] Evidently if we insert Eq. (39) into the dispersion integral

$$\Delta\mu^{V,S}=F_2{}^{V,S}(M^2)=\frac{1}{\pi}\int_{M^2}^{\infty}\frac{dW^2}{W^2-M^2}\mathrm{Im}F_2{}^{V,S}(W^2), \quad (40)$$

and perform the integral over the full range of energies $M^2\le W^2\le\infty$, we are just reproducing relativistic perturbation theory with the familiar unsuccessful result

$$\Delta\mu^P=(1/4\pi)(g^2/4\pi)=-\tfrac{1}{4}\Delta\mu^N;\quad \mu/M\to 0.$$

The same failure has been noted in the dispersion calculations first performed with the photon mass as the variable and with the nucleons on the mass shell. There the large unwelcome isoscalar contribution originated from the nucleon current of Fig. 9(a) and was discarded due to its high threshold, $q_t{}^2=4M^2$, compared with the purely isovector contribution with threshold $q_t{}^2=4\mu^2$ arising from the pion current in Fig. 9(b).

If we insert Eq. (39) into Eq. (40), but cut off the absorptive integral at the low threshold $\lambda^2=2.3$, i.e. $M\le W\le 1.5M$, we find that the role of the unwanted isoscalar term is greatly suppressed and that

$$\Delta\mu^P=\frac{1}{2\pi}\frac{g^2}{4\pi}\left[\frac{\ln\lambda^2}{\lambda^2-1}+\frac{1}{2}-\frac{3}{2\lambda^2}\right]\approx 1.2\approx 0.7(\Delta\mu_{\text{expt}}{}^P),$$

$$\Delta\mu^N=-\frac{1}{2\pi}\frac{g^2}{4\pi}\left[2-\frac{2\ln\lambda^2}{\lambda^2-1}\right]\approx -1.7\approx 0.9(\Delta\mu_{\text{expt}}{}^N). \quad (41)$$

As in the electron $g-2$ calculation we see that the low-energy region plays a major role. Inserting just the pole terms, we reproduce the measured moments with some success. We conclude with this independent evidence in support of the point of view exploited in the calculations throughout this paper.

APPENDIX

Here we shall demonstrate the nonrelativistic reduction of the calculation of the Schwinger correction $\alpha/2\pi$ and show that all that enters the final result is the Thomson scattering amplitude multiplying the Pauli current at the vertex. We begin with Eq. (21) which we write in the form

$$\mathrm{Im}F_2^+(W^2)=-\frac{m^2}{4\pi}\left[\frac{W^2-m^2}{W^2}\right]\sum_{s,s',s''}\int_{-1}^{+1}dx[\bar{u}(p,s)T_{\mu\nu}u(k,s'')][\bar{u}(k,s'')\gamma^\nu u(p,s')]\times\left[\bar{u}(p,s')\left(\frac{m^2}{2(W^2-m^2)^2}i\sigma^{\mu\tau}l_\tau\right)u(p,s)\right], \quad \text{(A1)}$$

where the factor $(W^2-m^2)/W^2$ arises from considerations of relativistic kinematics. To obtain the reduction of Eq. (A1) as $\omega=W-m\rightarrow 0$, we may consider each term in the brackets separately.

The third term in Eq. (A1) is the projection operator. Since $\bar{u}(p)\sigma^{0\tau}l_\tau u(p)=0$, we need consider only the space component as $\omega\rightarrow 0$,

$$\bar{u}(p,s')\left(\frac{m^2}{2(W^2-m^2)^2}i\sigma^{k\lambda}l_\lambda\right)u(p,s)\rightarrow\frac{1}{8\omega^2}\chi_{s'}{}^*i(\boldsymbol{\sigma}\times\mathbf{p})^k\chi_s\,, \quad \text{(A2)}$$

where $\mathbf{p}=-\mathbf{l}$. The second term in Eq. (A1) is the familiar Dirac current and has the nonrelativistic reduction valid to order v/c, as in Eqs. (8) and (9)

$$\bar{u}(k,s'')\gamma^0 u(p,s')\rightarrow\chi_{s''}{}^*\chi_{s'}\,,$$
$$\bar{u}(k,s'')\gamma^i u(p,s')\rightarrow\chi_{s''}{}^*((\mathbf{p}+\mathbf{k})/2m+i[\boldsymbol{\sigma}\times(\mathbf{k}-\mathbf{p})](2m)^{-1})_i\chi_{s'}\,, \quad \text{(A3)}$$

in terms of the Schrödinger and Pauli current. The first bracket in Eq. (A1) is given by the low-energy theorem Eq. (26) as $\omega\rightarrow 0$. The time components, $\bar{u}(p)T_{k0}u(k)$ may be obtained from $\bar{u}(p)T_{ki}u(k)$ by invoking current conservation as in Eq. (19)

$$\sum_{j=1}^{3}q_iT_{ki}=\omega T_{k0}\quad \omega=|q|=q_0\quad \mathbf{q}=-\mathbf{k}.$$

Putting all this together we obtain from Eq. (A1) the reduction as $\omega\rightarrow 0$,

$$\mathrm{Im}F_2^+(W^2)=-\frac{m^2}{4\pi}\frac{W^2-m^2}{W^2}\sum_{s,s',s''}\int_{-1}^{+1}dx\left[\chi_s{}^*T_{ki}\chi_{s''}\chi_{s''}{}^*\left(\frac{\mathbf{p}+\mathbf{k}}{2m}+i\frac{(\mathbf{p}-\mathbf{k})}{2m}\times\boldsymbol{\sigma}\right)_i\chi_{s'}+(q_i/\omega)\chi_s{}^*T_{ki}\chi_{s''}\chi_{s''}{}^*\chi_{s'}\right][\chi_{s'}{}^*(i(\boldsymbol{\sigma}\times\mathbf{p})_k/8\omega^2)\chi_s]. \quad \text{(A4)}$$

Using the full low-energy theorem Eq. (26) and the fact that

$$\int_{-1}^{1}dx\,q_j=0\,,$$

it is seen that the term

$$\int_{-1}^{1}dx\frac{q_i}{\omega}\chi^*T_{ki}\chi\,,$$

corresponding to the contribution from intermediate longitudinal photons, gives no contribution. The Schrödinger current also gives no contribution in this limit which can be seen by substituting the Thomson limit

$$T_{ki}=-(e^2/m)\delta_{ki}$$

in Eq. (A4) and taking the trace. What remains is just the Thomson limit of Compton scattering in conjunction with the Pauli current,

$$\mathrm{Im}F_2^+(W^2)=-\frac{m^2}{4\pi}\frac{W^2-m^2}{W^2}\sum_{s,s',s''}\int_{-1}^{1}dx\,\chi_s{}^*\left(-\frac{e^2}{m}\delta_{ki}\right)\chi_{s''}\chi_{s''}{}^*\left(\frac{i\boldsymbol{\sigma}\times(\mathbf{k}-\mathbf{p})}{2m}\right)_i\chi_{s'}\times\chi_{s'}{}^*(i(\boldsymbol{\sigma}\times\mathbf{p})_k/8\omega^2)\chi_s=\tfrac{1}{2}\alpha(W^2-m^2)/W^2, \quad \text{(A5)}$$

in agreement with our previous result, Eq. (10).

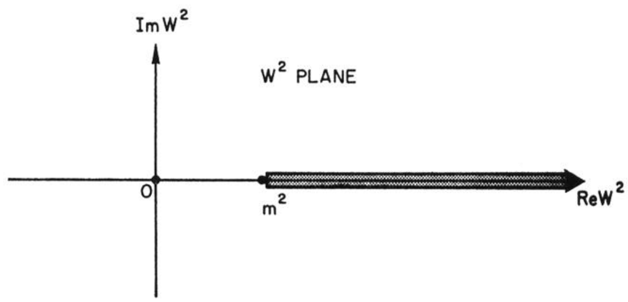


FIG. 5. Analytic properties of the invariant functions $F_i(W^2)$.