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Three-Body N/D Equations. I. Integral Angular Momenta*

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Equations are derived for calculating three-body partial-wave amplitudes once all singularities other than the unitarity discontinuity are known. The equations can easily be reduced to Fredholm equations. They have a similar structure to the two-body equations, but are more complicated owing to the presence of disconnected diagrams. The D function is related to the form factor in the usual way, and it has the expected zeros and cuts on the unphysical sheet. A subsidiary problem which is treated is the determination of a function with kinematical branch points when the discontinuities across the dynamical cuts on all sheets are known. Also, a brief discussion is given of the many-channel analog of the Omnès equation. This discussion may be useful to those who do not wish to study the mathematical theory in detail.

I. INTRODUCTION

IT is the aim of this paper to obtain N/D equations for the relativistic three-body problem analogous to the familiar two-body N/D equations. For the two-body system in simple cases, the problem is to use unitarity to find a partial-wave scattering amplitude when the left-hand discontinuity is given. In all but the simplest cases a two-body partial-wave amplitude may have singularities in the complex plane, and a three-body amplitude will certainly have such complex singularities. The problem may then be stated as follows: Given the contributions of all singularities except the right-hand cut to a partial-wave amplitude, it is required to find the complete partial-wave amplitude.

One need hardly emphasize that the problem which has just been formulated is only a small part of the three-body problem. The singularities other than the right-hand discontinuity would have to be determined from the right-hand discontinuity by crossing, and a self-consistency procedure would have to be developed. Further, if fixed angular-momentum states are used at all, one would have to use complex angular momenta in order to avoid the difficulties associated with the divergence of the partial-wave expansion. Since such a program has not yet been carried out consistently for the two-body problem, it would obviously be impractical at the moment to apply it to the three-body problem. We are certainly not deriving the N/D equations with the aim of using them as they stand in numerical calculations. However, one may be able to carry out much simpler but cruder treatments with further approximations in order to make qualitative estimates of three-body effects. Moreover, it appears to be essential to treat resonances on a par with particles in two-body calculations if the results are to be at all accurate. Since a state consisting of a particle and a resonance is really a three-particle state, the three-body equations may be helpful in treating doubtful points in the equations for particle-resonance scattering.

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Another application of three-body N/D equations is to the study of the complex J plane in three-body systems. For this purpose it is unnecessary to know the left-hand and complex singularities, since the discontinuities across them can be proved to be holomorphic functions of J. In the present paper we shall concern ourselves with integral values of J only, and we shall leave the question of complex angular momenta to a subsequent paper.

The three-body problem may be looked upon as a two-body problem with a continuous infinity of channels. However, there are complications in the three-body problem which do not exist in the two-body problem. Such complications are associated with disconnected diagrams in which two particles scatter while the third is unaffected. The equations giving the discontinuity across the right-hand cut will now be more complicated than for the two-body system. Besides the discontinuity when the total energy is real and above threshold, there will be discontinuities when the center-of-mass energy of any pair of incoming or outgoing particles is real and above threshold. For reasons which we shall discuss in the following section, the unitarity equation will take a convenient form only if the variables are suitably chosen. Furthermore, the kinematics associated with the correct variables will be much simpler when the physical "Dalitz" region retains its shape with increasing energy as it does in the nonrelativistic case, than when it does not. Since a system with a Dalitz region of fixed shape possesses all the essential features of the problem without the kinematical complications, we shall begin by discussing such a system. The general system will be examined in the second half of the paper.

In the following section we shall deal with the choice of variables and, for the problem with a fixed Dalitz region, we shall choose a set in terms of which the unitarity condition takes a simple form. In Sec. 3 we shall express the scattering amplitudes of the two-body subchannels in terms of these variables and, in Secs. 4 and 5, we shall derive the N/D equations and shall separate off the disconnected parts. In Secs. 6–9 we shall investigate the problem with the more complicated kinematics necessary when the shape of the Dalitz region is not fixed. We shall show that the equations obtained do not differ in principle from those of the simpler system, except that the unitarity condition must now be applied in the physical region for reactions such as $A+B\rightarrow\bar{C}+D+E+F$ as well as in the physical region for the reaction $A+B+C\rightarrow D+E+F$.

In the course of our work we shall come across two mathematical problems which may possibly be of interest apart from their application to the three-body system. The first is the multichannel integral equation with Cauchy singularities in the kernel, analogous to the familiar single-channel Omnès equation. The Omnès technique is only applicable to the single-channel equation, but an extensive theory has been developed for the multichannel equation.[1] As many physicists are probably unfamiliar with this theory, we have collected some of the results in Appendix I. The other mathematical problem we shall encounter is the determination of an analytic function which has kinematical branch points of known character due to the choice of variables, and which also has dynamical cuts with known discontinuities on each of the kinematical sheets.[2] The solution to this problem is an extension of ordinary dispersion relations and is given in Sec. 7.

Though all the work of this paper is done on the three-body problem, we feel that the n-body problem should not differ in principle from it. The three-body problem may therefore be regarded as a model for deriving general results applicable to the n-body problem.

II. CHOICE OF VARIABLES

We first write down the many-channel two-body N/D equations in order to observe their structure which will be essentially the same as the structure of the three-body N/D equations which we shall derive:

$$N_{ij}(s)=F_{ij}(s)+\frac{1}{\pi}\int_R ds'\sum_l\frac{F_{il}(s')-F_{il}(s)}{s'-s}\times k_l(s')N_{lj}(s'),\quad(2.1a)$$

$$D_{ij}(s)=1+\frac{1}{\pi}\int_R ds'\frac{k_i(s')N_{ij}(s')}{s'-s}.\quad(2.1b)$$

In addition to the variable s, there are the variables i and j which specify the channels and which take a discrete number of values only.

For the three-body problem we shall begin by taking the Omnès[3] variables. Besides the total angular momentum J there will be seven variables, which Omnès takes to be (i) the total energy S; (ii) the partial energies s_1, s_2, s_1' s_2' where, for instance, s_1 is the square of the sum of the energies of particles 2 and 3 in their own center-of-mass system. Unprimed variables refer to the initial state, primed to the final state. (iii) The helicities M and M' of the initial and final states, measured with respect to some body-centered axis. The set of variables s_1, s_2 and M may be regarded as channel indices, analogous to the variable i in the two-body case. Now, however, instead of having one discrete variable, we have one discrete and two continuous variables.

The third partial energy will be given by the equation

$$s_3=S+3m^2-s_1-s_2,\quad(2.2)$$

where we have assumed equal masses for simplicity.

We shall denote the connected part of the scattering amplitude by the symbol A_C, and that disconnected part containing a factor $\delta(s_i-s_i')$ $(i=1,2,3)$ by the symbol a_i. According to the unitarity condition, the imaginary part of A_C will consist of seven terms, one of the form $A_C{}^*A_C$, three of the form $a_i{}^*A_C$ and three of the form $A_C{}^*a_i$.[4] It might be thought that these terms represent the discontinuities in the variables S, s_i and s_i'. However, Bonnevay and, independently, Bronzan and Kacser[5] have shown that the formulas for the individual discontinuities are rather more complicated. They may certainly be written in the form $A_C{}^*A_C$, $a_i{}^*A_C$ or $A_C{}^*a_i$, but the integration over the intermediate partial energies may have to be taken over a contour which is deformed out of the physical region. Such a complication can occur even when the initial and final variables are within the physical region for three-particle scattering. On the other hand, the sum of the discontinuities across the cuts in all the variables S, s_i, and s_i' is equal to the imaginary part of A_C. As long as we are in the physical region, therefore, the sum of the discontinuities will be given by the straightforward unitarity condition, with the intermediate variables integrated over the physical region.

Owing to the complications in the unitarity condition it is not convenient to choose S as our dispersion variable and s_i as the parametric variables corresponding to the channel indices in the two-body case. The discontinuity of the scattering amplitude A_C across the cut in the S plane does not assume a simple form when the s_i's are kept fixed. However, one can replace the s_i's by the variables

$$y_i=\tfrac{2}{3}-\frac{s_i-4m^2}{S-9m^2},\quad y_i'=\tfrac{2}{3}-\frac{s_i'-4m^2}{S-9m^2}.\quad(i=1,2,3.)\quad(2.3)$$

The constant $\frac{2}{3}$ and the minus sign have no deep significance and are only inserted for comparison with the

[1] N. I. Muskhelishvilli, *Singular Integral Equations*, translated by J. R. M. Radok (Stechert-Hafner Service Agency, Inc., New York, 1953).

[2] The usual threshold branch points are of course regarded as dynamical singularities not kinematical singularities.

[3] R. Omnès, Phys. Rev. 133, B1543 (1964).

[4] In addition, there will be a term $\Sigma_{i\neq j}\,a_i{}^*a_j$ which need not concern us at the moment.

[5] N. Bonnevay, Nuovo Cimento 30, 1325 (1963); J. B. Bronzan and C. Kacser, Phys. Rev. 132, 2703, 2712 (1963).

variables which we shall use in the second part of the paper. The variables y_i and y_i' satisfy the relations

$$y_1+y_2+y_3=1 \quad y_1'+y_2'+y_3'=1. \tag{2.4}$$

If the y's are kept constant and positive, and S crosses the real axis to the right of $9m^2$, then all seven variables S, s_i, and s_i' will move across the real axis above their threshold. The discontinuity in S across the right-hand cut when the y's and y_i's are kept constant is therefore given by the straightforward unitarity condition, and is equal to the sum of the seven terms $A_C{}^*A_C$, $a_i{}^*A_C$ and $A_C{}^*a_i$, the intermediate variables being integrated over the physical region.

We shall therefore provisionally take our variables to be S, y_1, y_2, y_1', y_2', M, M', and the total angular momentum J. The total energy S will be the dispersion variable, the y's and M are the variables analogous to the channel indices of the two-body problem, and J is a fixed parameter. The y's have been plotted in triangular ω ordinates in Fig. 1. The boundary of the physical region is given by the equation

$$(3y_1-1)^2+3(y_2-y_3)^2-3m^{-2}(S-9m^2) \times(y_1-\tfrac{2}{3})(y_2-\tfrac{2}{3})(y_3-\tfrac{2}{3})-1=0. \tag{2.5}$$

As S increases from threshold to infinity, the physical region spreads out from the circle to the triangle shown in Fig. 1.

The phase-shape factor in the unitarity condition is constant over the entire physical region at any particular value of S, and is given by the formula

$$G(S)=(S-9m^2)^2/(256\pi^3S). \tag{2.6}$$

Another variable we shall require is $\cos\theta_1$, the cosine of the angle between the directions of motion of the particles 1 and 2 in the center-of-mass system of 2 and 3. It is given by the formula

$$\cos\theta_1=s_1^{1/2}T_1^{-1}(\tfrac{2}{3}-y_1)^{-1/2}(y_2-y_3), \tag{2.7}$$

where

$$s_1=(\tfrac{2}{3}-y_1)(S-9m^2)+4m^2, \tag{2.8a}$$

$$T_1^2=(S-9m^2)(\tfrac{1}{3}+y_1)^2+12m^2y_1. \tag{2.8b}$$

Equation (2.5) is just the condition $\cos\theta_1=1$.

As we shall see in a moment, one runs into difficulties if the shape of the physical region changes as the energy S is increased. Our present variables are therefore still not suitable for the problem. It is not hard to find variables in terms of which the shape of the physical region does not change, but the kinematics becomes rather complicated. We shall avoid these "inessential complications" by omitting the terms proportional to $S-9m^2$ in (2.5) and (2.7). The physical region is then given by the circle of Fig. 1 for all values of S. In other words, we shall adopt a nonrelativistic approximation in defining the physical region and in expressing $\cos\theta_1$ in terms of the y's. After solving the simplified problem we shall treat a system with full relativistic kinematics and we shall

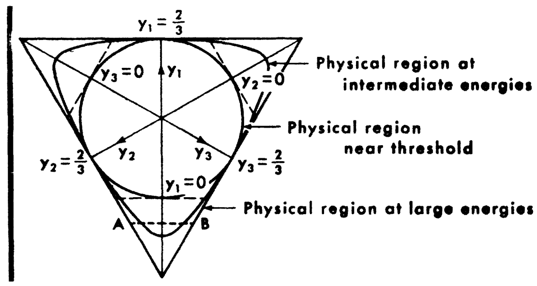

FIG. 1. Kinematical variables for the three-body problem.

find that the equations, though more complicated, are fairly similar in form.

In a nonrelativistic approximation we could replace the factor S in the denominator of (2.6) by $9m^2$. There is no need to do so and since this approximation, unlike the other nonrelativistic approximations, neglects a factor which increases without limit at large S and may affect the singular nature of the equations, we shall leave (2.6) as it stands.

We return to explain the difficulties which occur when the shape of the physical region changes with the energy S. The dispersion relations will be written at fixed values of the y's, and we shall therefore require the discontinuity in S for all values of y_1 and y_2 within the triangle of Fig. 1, even when we are outside the physical region. We have seen that the unitarity condition has a simple form within the physical region. We shall now show that the intermediate-state ranges of integration in the terms $a_i{}^*A_C$ can become complex if the initial state is outside the physical region.

The scattering amplitude a_1 has a delta function $\delta(s_1-s_1')$ and there is thus only one variable of integration in the term $a_1{}^*A$, which we may take to be $\cos\theta_1$. The integration in the unitarity condition must therefore be performed with y_1 fixed and with $\cos\theta_1$ ranging from -1 to 1. If $\cos\theta_1$ is replaced by y_2, Eq. (2.7) shows that the range of integration is

$$\frac{1-y_1}{2}-\frac{T_1(\tfrac{2}{3}-y_1)^{1/2}}{2s_1^{1/2}}<y_2<\frac{1-y_1}{2}+\frac{T_1(\tfrac{2}{3}-y_1)^{1/2}}{2s_1^{1/2}}. \tag{2.9}$$

Now, if

$$S-9m^2>-12m^2y_1/(\tfrac{1}{3}+y_1)^2 \tag{2.10}$$

the variable T_1 will be real, so that the range of integration (2.9) is real. In fact, the inequality (2.9) is precisely the condition for the point y_1, y_2 to be within the physical region. However, if $y_1<0$ and $S-9m^2$ decreases below the value given in (2.10), T_1 will become complex and the range of integration (2.8) becomes complex. Equation (2.9) definitely gives the range of integration in the unitarity term $a_1{}^*A_C$, so that this range will become complex if the inequality (2.9) is not satisfied.

In Fig. 1, the integral in the term $a_1{}^*A_C$ is to be taken along the line AB between the ends of the physical region. As S decreases the physical region decreases in size, and becomes tangential to the line AB when the equality corresponding to (2.9) is satisfied. If the value of S is decreased further, the range of integration becomes complex.

In the dispersion relations the variables y_1 and y_2 are to be kept real and constant, we have seen that they are analogous to the channel indices in the two-body problem. We cannot therefore work with these variables if the range of integration becomes complex, and we must choose our variables in such a way that the shape of the physical region does not change with energy.

One consequence of our present choice of variables is that we shall not obtain the correct threshold behavior unless the left-hand cut satisfies certain conditions. The situation is analogous to that of higher partial waves in two-body systems, where the physical region shrinks to a point in the t plane at threshold. Since the scattering amplitude is analytic in t, we can conclude that the integral $A(s,t)P_l(1+t/2q^2)$ behaves like $(q^2)^l$. In the present case the physical region shrinks to a point in the space of the s_i's as well as in the t plane and the amplitude, besides behaving like $(q^2)^l$, must also become independent of s_1 and s_2. To obtain the precise threshold dependence we should expand the amplitude in a set of orthogonal polynomials in y_1 and y_2, and in y_1' and y_2'. A particular component will then behave like $(q^2)^{J+n_1+n_2}$ at threshold, where n_1 and n_2 are the degrees of the polynomials in y_i and y_i'.

If we make an arbitrary choice of the left-hand cut, the resulting amplitude will not have the correct threshold behavior. This feature must be regarded as an inaccuracy of the approximation scheme. The fact that the amplitude at threshold must not depend on the y's is a difference between the three-body problem and the many-channel two-body problem.

III. TWO-BODY AMPLITUDES IN TERMS OF THREE-BODY VARIABLES

The scattering amplitudes $a_i (i=1, 2, 3)$ which occur in the unitarity condition refer to the process where two particles scatter and the third remains fixed. In this section we shall obtain a formula relating the Jth partial wave of such an amplitude, regarded as a three-body amplitude and expressed in our present variables, to the ordinary two-body amplitudes. Most of the formulas will be able to be used unchanged when we treat the fully relativistic kinematics.

We shall derive our formulas for the amplitude a_1, analogous formulas will hold for a_2 and a_3. Before one makes the angular-momentum projection, the amplitude will be given by the expression

$$t_1(s_1, \cos\alpha_1)\delta(\mathbf{p}_1-\mathbf{p}_1'). \tag{3.1}$$

The symbol t_1 denotes the two-body amplitude, α_1 is the angle of scattering in the center-of-mass system of the particles 2 and 3, and $\mathbf{p}_1$ and $\mathbf{p}_1'$ are the initial and final momenta of the first particle. We can rewrite (3.1) as

$$t_1\{s_1, \cos\theta_1\cos\theta_1' +\sin\theta_1\sin\theta_1'\cos(\phi-\phi')\}\delta(\mathbf{p}_1-\mathbf{p}_1'), \tag{3.2}$$

where θ_1 and θ_1' are the initial and final angles between the directions of motion of the particles 1 and 2, in the center-of-mass system of the particles 2 and 3, while $\phi-\phi'$ is the azimuthal angle between the directions of initial and final motion of particle 2, with the direction of motion of particle 1 taken as the pole. The variables θ_1 and θ_1' are internal variables of the initial and final state, respectively. With our present kinematics, they are by the equation

$$\cos\theta_1=\{y_1(2-3y_1)\}^{-1/2}(y_2-y_3). \tag{3.3}$$

The azimuthal angles ϕ and ϕ' will be the same whether we measure them in the center-of-mass system of the particles 2 and 3 or of the particles 1, 2, and 3, since the relative velocity between these systems is directed along the pole. If, therefore, we take the body-centered axis of the Omnès variables to be along the direction of motion of the particle 1, i.e., along the direction of motion of the particle which is not scattered, the variables ϕ and ϕ' will be the Omnès azimuthal angles for the initial and final state.

The angular-momentum projection of the scattering amplitude will be given by multiplying (3.2) by $\mathfrak{D}_{MM'}{}^J$ and integrating over angles. Thus

$$a_1(S,s_1,s_2,s_1',s_2',M,M',J)\delta(S-S')=\int d\phi\, d\phi'\, d\Theta \times t_1\{s, \cos\theta_1\cos\theta_1'+\sin\theta_1\sin\theta_1'\cos(\phi-\phi')\}\delta(s_1-s_1') \times\delta(\mathbf{p}_1-\mathbf{p}_1')\mathfrak{D}^J{}_{MM'}(\phi,\Theta,\phi'). \tag{3.4}$$

The angle Θ in this equation is the angle of scattering of the particle 1, which is zero. As

$$d^J{}_{MM'}(0)=\delta_{MM'},$$

the integration over Θ will just give this Kronecker delta. The angular part of the function $\delta(\mathbf{p}_1-\mathbf{p}_1')$ has thereby been taken into account, and the expression $\delta(s_1-s_1')\delta(|p_1|-|p_1'|)/|p_1||p_1'|$ may be written $4ST_1^{-1}(S-9m^2)^{-3/2}\delta(y_1-y_1')\delta(S-S')$, the quantity T_1 being given by (2.7b). As the integrand depends only on $\phi-\phi'$ the integral over one of the ϕ's will be trivial, and (3.5) may be written

$$a_1(S,s_1,s_2,s_1',s_2',M,M',J) =8\pi ST_1^{-1}(S-9m^2)^{-3/2}\int d(\phi-\phi') \times t_1\{s_1, \cos\theta_1\cos\theta_1'+\sin\theta_1\sin\theta_1'\cos(\phi-\phi')\} \times e^{im(\phi-\phi')}\delta(y_1-y_1')\delta_{MM'}. \tag{3.5}$$

We can then make a change of variable

$$x=\cos\theta_1\cos\theta_1'+\sin\theta_1\sin\theta_1'\cos(\phi-\phi')$$

and we obtain the equation

$$a_1(S,s_1,s_2,s_1',s_2',M,M',J)=8\pi iST_1^{-1}(S-9m^2)^{-3/2}$$
$$\times\oint dx\{x^2+\cos^2\theta_1+\cos^2\theta_1'-1-2x\cos\theta_1\cos\theta_1'\}^{-1/2}$$
$$\times t_1(s_1,x)T_M\left\{\frac{x-\cos\theta_1\cos\theta_1'}{\sin\theta_1\sin\theta_1'}\right\}\delta(y_1-y_1')\delta_{MM'},\quad(3.6)$$

where the integral is to be taken around the branch points of the function in curly brackets. The function T_m is the mth Tchebycheff polynomial, i.e., the expression for $\cos m\theta$ in terms of $\cos\theta$.

If the partial-wave expansion converges, (3.6) can be integrated to give

$$a_1(S,s_1,s_2,s_1',s_2',M,M',J)$$
$$=8\pi ST_1^{-1}(S-9m^2)^{-3/2}\sum(2l+1)t_l(s_1)$$
$$\times P_l^M(\cos\theta_1)P_l^{-M}(\cos\theta_1')\delta(y_1-y_1')\delta_{MM'}.\quad(3.7)$$

With the present kinematics we are only interested in physical values of $\cos\theta_1$, so that (3.7) can always be used in place of (3.6) but, in the fully relativistic case, we shall also have to deal with unphysical angles. We could replace the summation in (3.7) by a Sommerfeld-Watson integral and we could then use it when the partial-wave expansion does not converge.

It is important to notice that the right-hand side of Eqs. (3.6) and (3.7) are independent of J.

The above equations are true for the particular case where the body-centered axis of the Omnès variables coincides with the unscattered particle. However, we can always transform from one axis to another using the formula

$$a'(M,M')=d^J{}_{MM''}(\cos\beta_a)$$
$$\times a(M'',M''')d^J{}_{M''',M}(\cos\beta_b),\quad(3.8)$$

where β_a is the angle between the old and the new axis for the initial state, β_b for the final state. With arbitrary body-centered axes (3.7) therefore becomes

$$a_1(S,s_1,s_2,s_1',s_2',M,M',J)=8\pi iST_1^{-1}(S-9m^2)^{-3/2}\oint dx\{x^2+\cos\theta_1+\cos^2\theta_1'-1-2x\cos\theta_1\cos\theta_1'\}$$
$$\times t_1(s_1,x)T_{M''}\left\{\frac{x-\cos\theta_1\cos\theta_1'}{\sin\theta_1\sin\theta_1'}\right\}\delta(y_1-y_1')\sum_{M''}d^J{}_{MM''}(\cos\beta_{B1})d^J{}_{M''M'}(\cos\beta_{B1}'),\quad(3.9)$$

where β_{B1} and β_{B1}' are the center-of-mass angles between the body-centered axis and particle 1 in the initial and final states. If the body-centered axis is taken along particle 2, β_{21} is given by the formula

$$\cos\beta_{21}=\{(S-9m^2)(y_1y_2-y_3)+10m^2+12m^2(y_1+y_2)\}/T_1T_2.\quad(3.10)$$

In the sequel, all amplitudes with a delta function (y_1-y_1') will be written with the subscript 1, as a_1. The coefficient of the delta function will be written $\tilde{a}_1$ so that

$$a_1=\tilde{a}_1\delta(y_1-y_1').\quad(3.11)$$

IV. THE N/D EQUATIONS WITH A FIXED DALITZ REGION

Construction of the Equations

Now that we have developed the kinematics, the actual writing down of the equations is a relatively simple matter and, in fact, the equations are a straightforward generalization of (2.1). We shall take as our unknown function the *total* scattering amplitude A, *including the disconnected parts*. If we do so, the unitarity equation will take the simple form

$$[A]=A^*GA.\quad(4.1)$$

If, on the other hand, we were to use the connected part of the scattering amplitude A_C as our unknown function, the unitarity equation would contain extra terms of the form $a_i^*A_C$ and $A_C^*a_i$. Needless to say, we shall have to separate the disconnected parts from our integral equation in order to make it nonsingular. We only do so after writing the total amplitude in the form

$$A=ND^{-1}\quad(4.2)$$

and writing the equations for N and D. The numerator and denominator functions will both contain disconnected parts. The connected parts of N and D will not be the same as the functions N_C and D_C, defined by the equation $A_C=N_CD_C^{-1}$.

We should emphasize that the process represented by Fig. 2 does not complicate the discontinuity equation, as it would if we had been keeping the s_i's rather than the y's constant. The singularity associated with Fig. 2 is a pole in a momentum transfer u. If the y's are kept

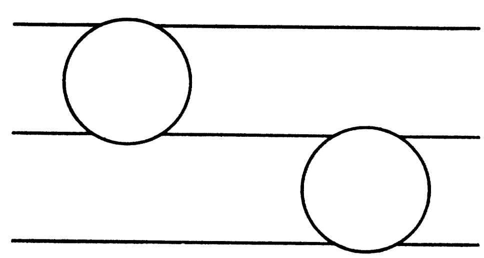

FIG. 2. Diagram corresponding to a pole in the physical region of the scattering amplitude.

constant, the variable S moves from $S-i\epsilon$ to $S+i\epsilon$ as u moves from $u-i\epsilon'$ to $u+i\epsilon'$, so that the singularity may be regarded as a normal right-hand cut in S. When the s_i's are kept constant, the variable S moves from $S+i\epsilon$ to $S-i\epsilon$ as u moves from $u-i\epsilon$ to $u+i\epsilon$, and the singularity structure in S is slightly more complicated.

In analogy with (2.1), we may now write the following equations for N and D:

$$N(S)=F(S)+\frac{1}{\pi}\int_{9m^2}^{\infty}dS'\frac{F(S')-F(S)}{S'-S}G(S')N(S'),\quad (4.3a)$$

$$D(S)=1-\frac{1}{\pi}\int_{9m^2}^{\infty}dS'\frac{G(S')N(S')}{S'-S},\quad (4.3b)$$

where

$$F(S)=\frac{1}{\pi}\int dS'\frac{L(S')}{S'-S},\quad (4.4)$$

L being $1/2i$ times the discontinuity across the left-hand cut. The y's and M's have been suppressed and will continue to be suppressed in subsequent equations. By a product such as $F(S)G(S')N(S')$ the integral

$$\int dy_1''\,dy_2''\sum_{M''}F(S,y_1,y_2,y_1'',y_2'',M,M'')\times G(S')N(S',y_1'',y_2'',y_1',y_2',M'',M')$$

is always implied.

Removal of the Delta-Function Singularities

The function F will contain disconnected parts and a connected part:

$$F=F_C+\sum f_i,\quad (4.5)$$

where the subscript i indicates the presence of a delta function $\delta(y_i-y_i')$. It follows from (4.3a) that the function N will also contain disconnected parts:

$$N=N_C+\sum n_i.\quad (4.6)$$

We can now substitute (4.6) and (4.5) into (4.3a). By equating coefficients of the three delta functions, as well as the terms without delta functions, we obtain the following equations for n_i and N_C:

$$\tilde{n}_i(S)=\tilde{f}(S)+\frac{1}{\pi}\int dS'\frac{\tilde{f}_i(S')-\tilde{f}_i(S)}{S'-S}G(S')\tilde{n}_i(S'),\quad (4.7)$$

$$N_C(S)=F_C(S)+\frac{1}{\pi}\sum_i\int dS'\frac{f_i(S')-f_i(S)}{S'-S}G(S')N_C(S')+\frac{1}{\pi}\sum_i\int dS'\frac{F_C(S')-F_C(S)}{S'-S}G(S')n_i(S')+\frac{1}{\pi}\sum_{i\neq j}\int dS'\frac{f_i(S')-f_i(S)}{S'-S}G(S')n_j(S')+\frac{1}{\pi}\int dS'\frac{F_C(S')-F_C(S)}{S'-S}G(S')N_C(S').\quad (4.8)$$

The functions $\tilde{n}_i$ and $\tilde{f}_i$ indicate the coefficients of the delta function in n_i and f_i. We shall examine (4.7) and (4.8) in turn.

It is not difficult to see that (4.7) is precisely the equation for the function n of the corresponding two-particle subchannel. The variable S, being equal to $(s_i-4m^2)y_i+9m^2$, is linearly proportional to s_i, and the dispersion relation in S may therefore be replaced by one in s_i. Analogously to (3.7), we may write

$$n_1=8\pi ST_1^{-1}(S-9m^2)^{-3/2}\sum(2l+1)n_{l,1}(s_1)\times P_l^M(\cos\theta_1)P_l^{-M}(\cos\theta_1')\delta(y_1-y_1')\delta_{MM'},\quad (4.9a)$$

$$f_1=8\pi ST_1^{-1}(S-9m^2)^{-3/2}\sum(2l+1)f_{l,1}(s_1)\times P_l^M(\cos\theta_1)P_l^{-M}(\cos\theta_1')\delta(y_1-y_1')\delta_{MM'},\quad (4.9b)$$

provided that the body-centered axis is taken along the direction of p_1. If we substitute (4.9) into (4.7), the angular integrals simply separate out the amplitudes of different l. The equation thus becomes

$$n_{l,1}(s_1)=f_{l,1}(s_1)+\frac{1}{\pi}\int ds_i'\frac{f_{l,1}(s_1')-f_{l,1}(s_1)}{s_1'-s_1}k(s_1')n_{l,1}(s'),\quad (4.10a)$$

where

$$k(s_1)=\frac{1}{16\pi}\left(\frac{s_1-4m^2}{s_1}\right)^{1/2}.\quad (4.10b)$$

Equation (4.9) is just the equation for the numerator function of the two-body channel, as was to be proved. We can obtain similar equations for a_2 and a_3 except that (4.9) must be replaced by an equation corresponding to (3.9) instead of (3.7).

We now turn to Eq. (4.8). Since the function f_i in the kernel contains a delta function in s_i, it is not a Fredholm equation. We can convert it to a Fredholm equation by the method used by Weinberg[6] in the Schrödinger potential problem. To do this we consider the resolvent of (4.7), i.e., we define a function $\tilde{r}_i$ by the equation

$$\tilde{r}_i(S,S')=\frac{1}{\pi}\frac{\tilde{f}_i(S')-\tilde{f}_i(S)}{S'-S}G(S')+\frac{1}{\pi}\int dS''r_i(S,S'')\frac{\tilde{f}_i(S')-\tilde{f}_i(S'')}{S'-S''}G(S').\quad (4.11)$$

By writing r_i in the form (4.9), we can separate (4.11) into partial waves of the two-body channel, exactly as we did with (4.7). We now multiply both sides of (4.8) by the function

$$\delta(S-S')+\sum_i r_i(S,S').$$

On doing so, we find that certain terms cancel because of (4.11), and the remaining kernel contains no delta

[6] S. Weinberg, Phys. Rev. 133, B232 (1964).

functions. Thus

$$N_C(S)=I(S)+\int dS'K(S,S')N_C(S')dS', \quad (4.12a)$$

where

$$I(S)=F(S)+\frac{1}{\pi}\sum_i dS'\frac{F_C(S')-F_C(S)}{S'-S}G(S')n_i(S')+\frac{1}{\pi}\sum_i\int dS'\,r_i(S,S')F_C(S')$$
$$+\frac{1}{\pi}\sum_{i,j}\int dS'\,dS''\,r_i(S,S')\frac{F_C(S'')-F_C(S')}{S''-S'}G(S'')n_i(S''), \quad (4.12b)$$

$$K(S,S')=\frac{1}{\pi}\sum_{i\neq j}\int dS''r_i(S,S'')\frac{f_j(S')-f_j(S'')}{S'-S''}G(S')$$
$$+\frac{1}{\pi}\frac{F(S')-F(S)}{S'-S}G(S')+\frac{1}{\pi}\sum_i\int dS''r_i(S,S'')\frac{F_C(S')-F_C(S'')}{S'-S''}G(S''). \quad (4.12c)$$

The kernel (4.12c) contains no δ functions, since any delta function $\delta(s_i-s_i')$ in f_i or r_i becomes multiplied by a continuous function of s_i or s_i'. Equation (4.12) is thus a Fredholm equation which can be used to calculate the function N_C.

Having found the function N, we can use (4.3) to find D. It is easy to see that

$$D=1+\sum_i d_{iC}+D_C, \quad (4.13a)$$

where

$$d_{iC}=d_i-1, \quad (4.13b)$$

d_i being the d function for the two-body channel, related to the partial-wave functions $d_{l,i}$ by an equation analogous to (4.9). The connected part D_C is given by the equation

$$D_C(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}dS'\frac{G(S')N_C(S')}{(S'-S)}. \quad (4.14)$$

Finally, the equation for D^{-1} in terms of D will be an integral equation in the variables y_1 and y_2, with delta functions in the kernel. Thus, if we write

$$D^{-1}=1+\sum_i(d_i^{-1})_C+(D^{-1})_C \quad (4.15)$$

the equation will be

$$\{1+\sum_i d_{iC}+D_C\}\{1+\sum_i(d_i^{-1})_C+(D^{-1})_C\}=1. \quad (4.16)$$

We can separate (4.16) into the following equations for $(d_i^{-1})_C$ and $(D^{-1})_C$:

$$(d_i^{-1})_C=d_i^{-1}-1, \quad (4.17a)$$

$$(D^{-1})_C+\sum_i d_{iC}(D^{-1})_C+D_C(D^{-1})_C$$
$$+\sum_{i\neq j} d_{iC}(d_j^{-1})_C+D_C+D_C\sum_i(d_i^{-1})_C=0. \quad (4.17b)$$

On multiplying (4.17b) by $1+\sum_i(d_i^{-1})_C$, we find that

$$(D^{-1})_C+\sum_{i\neq j}(d_i^{-1})_C d_{jC}(D^{-1})_C+\{1+\sum_i(d_i^{-1})_C\}D_C(D^{-1})_C$$
$$+\{1+\sum_i(d_i^{-1})_C\}\{\sum_{i\neq j}d_{iC}(d_j^{-1})_C$$
$$+D_C+D_C\sum_i(d_i^{-1})_C\}=0, \quad (4.18)$$

which is a Fredholm equation for the function $(D^{-1})_C$. The whole problem has thus been reduced to the solution of Fredholm equations.

General Remarks

In the single-channel two-body problem, the function D^{-1} is the form factor for the process under consideration. In the multichannel two-body problem the function D^{-1} is not quite the form factor, since the D function has two-channel indices whereas the form factor has only one. The difference between the equation for the D function and for the form factor is that the inhomogeneous term in the former equation is δ_{ij}, whereas in the latter it is C_i, the numbers C depending on which form factor is being considered. The form factor is thus simply equal to $D_{ij}^{-1}C_j$. Similarly, for the three-body problem, the form factor will be $\int dy_1'dy_2'\times\sum_{M'}D^{-1}(S,y_1,y_2,y_1',y_2',M,M')C(y_1',y_2',M')$, where the function C depends on the particular form factor that is being considered.

If there is a resonance in the i subchannel, the function $(d_i^{-1})_C$ will have a pole on the unphysical sheet when $s_i=s_R$. The presence of both left and right factors $(d_i^{-1})_C$ in (4.18) indicates that $(D^{-1})_C$ will have poles on the unphysical sheet when $s_i=s_R$ and when $s_i'=s_R$. The function ND^{-1} will also have poles when $s_i=s_R$ and $s_i'=s_R$, since N contains disconnected terms of the form $\delta(s_i-s_i')$. The scattering amplitude will therefore have the expected poles when one of the subenergies is at a resonance.

In writing down our N/D equations we have taken all integrals along the positive real axis. Since, however, all functions appearing in them are analytic, we may deform the contour of integration into the complex plane. We thereby obtain a continuation of our amplitude onto the unphysical sheet. If one or more of the subchannels has a resonance, the associated amplitude a_i will have a pole in S (at fixed y_1 and y_2) on the unphysical sheet. If we wish to continue past this pole, we shall have to add extra terms to the equations. Such terms correspond to an extra two-particle channel, consisting of the resonance and the third particle, which communicates with the three-particle channel. One can thereby show that the scattering amplitude is meromorphic on the unphysical sheet except for the usual particle-resonance cuts.

V. THE N/D EQUATIONS WITH A FIXED DALITZ REGION, BUT WITHOUT ANALYTICITY OF THE DISCONNECTED PARTS

The method given in the previous section, which treated the whole scattering amplitude rather than the connected part, was the simplest for the particular problem under consideration. It made use of the fact that the disconnected diagrams satisfied the same analyticity requirements as the connected part, they were analytic in S when the y's were held fixed. With the fully relativistic kinematics we shall find that the disconnected diagrams are not analytic in S when the other variables (to be defined later) are held fixed, but that δ functions in S occur. In order to use a method that can be taken over to the case of interest, we shall therefore not make explicit use of the analyticity of the disconnected diagrams in S.

We denote the connected part of the amplitude by the symbol A_C. It will satisfy the unitarity condition

$$[A_C]=A_C^*GA_C+\sum_i a_i^*GA_C+\sum_i A_C^*Ga_i+R, \quad (5.1a)$$

where

$$R=\sum_{i\neq j} a_i^*Ga_j. \quad (5.1b)$$

The last term of (5.1) is the contribution to the discontinuity resulting from processes depicted in Fig. 2. We can also infer the existence of this term by writing $A=A_C+\sum a_i$ and using (4.1) together with the equation $[a_i]=a_i^*Ga_i$. From the remarks near the beginning of the last section we may conclude that this last term is an ordinary right-hand cut, and that the factors A_C^* and A_C in the unitarity condition refer to the values of A below and above this cut.

We now write A_C in the form

$$A_C=N_CD^{-1}. \quad (5.2)$$

It will turn out that the function D is the same as in the factorization (4.2). However, we shall not be able to obtain linear equations for N_C and D if we put all the right-hand discontinuities of A into D. Instead, we put the discontinuities associated with the first and third terms of (5.1) into D, and those associated with the second and fourth terms into N_C. More precisely, the discontinuities of N_C and D will be given by the formulas

$$[N_C]=LD+\sum a_i^*GN_{C+}+RD_+, \quad (5.3a)$$

$$[D]=-GN_{C+}-G\sum a_iD_+. \quad (5.3b)$$

The first term in (5.3a) is the discontinuity across the left-hand cut, the remaining terms are discontinuities across the right-hand cut. The subscript + indicates that the function is to be evaluated above the cut in the S plane. By using the formulas

$$\begin{aligned}[D^{-1}]&=-D_-^{-1}[D]D_+^{-1},\\ [N_CD^{-1}]&=N_{C-}[D^{-1}]+[N_C]D_+^{-1},\end{aligned} \quad (5.4)$$

we can easily show that (5.3) implies (5.1).

From (5.3), we can write down the following coupled integral equations for N_C and D:

$$N_C(S)=\frac{1}{\pi}\int_L dS'\frac{L(S')D(S')}{S'-S-i\epsilon}+\frac{1}{\pi}\int_{9m^2}^\infty dS'\frac{\sum a_i^*(S')G(S')N_C(S')}{S'-S-i\epsilon}+\frac{1}{\pi}\int_{9m^2}^\infty dS'\frac{R(S')D(S')}{S'-S-i\epsilon}, \quad (5.5a)$$

$$D(S)=1-\frac{1}{\pi}\int_{9m^2}^\infty dS'\frac{G(S')N_C(S')}{S'-S-i\epsilon}-\int_{9m^2}^\infty dS'\frac{G(S')\sum a_i(S')D(S')}{S'-S-i\epsilon}. \quad (5.5b)$$

Equations (5.5) involve the functions N_C and D on the right-hand cut, and also D on the left-hand cut. The first step in their simplification is to eliminate D on the left by substituting (5.5b) in the first term of (5.5a). Thus

$$N_C(S)=F_L(S)+\frac{1}{\pi}\int_{9m^2}^\infty dS'\frac{F_L(S')-F_L(S)}{S'-S}G(S')\{N_C(S')+\sum_i a_i(S')D(S')\}$$
$$+\frac{1}{\pi}\int_{9m^2}^\infty dS'\frac{\sum a_i^*(S')G(S')N_C(S')}{S'-S-i\epsilon}+\frac{1}{\pi}\int_{9m^2}^\infty dS'\frac{R(S')D(S')}{S'-S-i\epsilon}, \quad (5.6a)$$

$$D(S)=1-\frac{1}{\pi}\int_{9m^2}^\infty dS'\frac{G(S')N_C(S')}{S'-S-i\epsilon}-\int_{9m^2}^\infty dS'\frac{G(S')\sum a_i(S')D(S')}{S'-S-i\epsilon}, \quad (5.6b)$$

where

$$F_L(S)=\frac{1}{\pi}\int_L dS'\frac{L(S')}{S'-S}. \tag{5.6c}$$

The Eqs. (5.5) are Cauchy integral equations, with kernels containing dispersion denominators. A single-channel uncoupled equation of this form can be reduced to a Fredholm equation using Omnès method. His method is not applicable to coupled integral equations, even with a finite number of channels. However, there is a well-developed theory by means of which they can be reduced to Fredholm equations. That part of the theory which we require is treated very cursorily in the Appendix. To apply it to our example, we bring all terms of (5.6) to the left-hand side and regard it as a matrix equation in N and D. We then multiply on the left by the operator:

$$1+\frac{1}{\pi}\int\frac{dS''}{S''-S+i\epsilon}\begin{Bmatrix}\sum a_i^*(S'')G(S'') & R(S'')\\ -G(S'') & -G(S'')\sum a_i(S'')\end{Bmatrix}. \tag{5.7}$$

The resulting equation, which is free of Cauchy singularities, is

$$\begin{Bmatrix}1-2i\sum a^*(S)G(S) & -2iR(S)\\ 2iG(S) & 1+2iG(S)\sum a_i(S)\end{Bmatrix}\begin{matrix}N_C(S)\\ D(S)\end{matrix}=\begin{matrix}F_R(S)+J_N(S)\\ 1-\sum e_i(S)-J_D(S)\end{matrix}-\int dS'\{K(S,S')\}\begin{matrix}N_C(S')\\ D(S')\end{matrix}$$
$$+\frac{1}{\pi}\int\frac{dS'}{S'-S}\begin{Bmatrix}\sum\tilde{e}_i(S')-\sum\tilde{e}_i(S) & F_R(S')-F_R(S)\\ -C(S')+C(S) & -\sum e_i(S')+\sum e_i(S)\end{Bmatrix}\begin{Bmatrix}\sum a_i^*(S')G(S') & R(S')\\ -G(S') & -G(S')\sum a_i(S')\end{Bmatrix}\begin{matrix}N_C(S')\\ D(S')\end{matrix}, \tag{5.8a}$$

where

$$J_N(S)=F_L(S)+\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''}{S''-S+i\epsilon}\sum a_i^*(S'')G(S'')F_L(S''),$$

$$J_D(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''}{S''-S+i\epsilon}G(S'')F_L(S''),$$

$$e_i(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''}{S''-S+i\epsilon}G(S'')a_i(S''),\quad \tilde{e}_i(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''}{S''-S+i\epsilon}a_i^*(S'')G(S''),$$

$$F_R(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''}{S''-S+i\epsilon}R(S''),\qquad C(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''}{S''-S+i\epsilon}G(S''),$$

$$K(S,S')=\frac{1}{\pi}\frac{F_L(S')-F_L(S)}{S'-S}G(S')\begin{Bmatrix}1 & \sum a_i(S')\\ 0 & 0\end{Bmatrix}\frac{1}{\pi^2}\int_{9m^2}^{\infty}\frac{dS''}{(S''-S+i\epsilon)(S''-S')}$$
$$\begin{Bmatrix}\sum a_i(S'')G(S'')\{F_L(S')-F_L(S'')\}G(S') & \sum a_i(S'')G(S'')\{F_L(S')-F_L(S'')\}G(S')\sum a_i(S')\\ G(S'')\{F_L(S')-F_L(S'')\}G(S') & G(S'')\{F_L(S')-F_L(S'')\}G(S')\sum a_i(S')\end{Bmatrix}. \tag{5.8b}$$

It follows from (5.1b), and the unitarity equation for a_i, that

$$\begin{Bmatrix}1-2i\sum a^*(S)G(S) & -2iR(S)\\ 2iG(S) & 1+2iG(S)\sum a_i(S)\end{Bmatrix}^{-1}=\begin{Bmatrix}1+2iG(S)\sum a_i(S) & 2iR(S)\\ -2iG(S) & 1-2i\sum a^*(S)G(S)\end{Bmatrix} \tag{5.9}$$

and that the determinant of this matrix is unity. The factor on the left-hand side of (5.8a) can thus easily be brought to the right, and it follows from the results quoted in the Appendix that the original equation (5.5) is fully equivalent to the resulting equation. This equation has no Cauchy singularities, but it still has delta functions from the disconnected diagrams. They can be removed by the methods used in the previous section, and our equations have thus been reduced to Fredholm equations. The calculation of D^{-1} from D is identical to that given in the previous section.

As we pointed out in this section, the present method differs from that of the previous section by writing only the connected part of the scattering amplitude in the form $N_C D^{-1}$. Once the equations have been solved we can easily write the whole amplitude in the form

$A=ND^{-1}$ by defining

$$N=N_C+\sum a_i D \quad (5.10)$$

and leaving D unchanged. It is obvious that the amplitude ND^{-1} is equal to the total amplitude, and it follows from (5.10) and (5.3) that N and D satisfy the familiar conditions $[N]_R=0$, $[D]_R=GN$.

VI. THE VARIABLES WITH FULL RELATIVISTIC KINEMATICS

Definition of our Variables

We have emphasized that it is necessary to define our variables in such a way that the shape of the physical region is independent of the variable S. This can be achieved by taking as our starting-point the center-of-mass momenta of the three particles rather than the partial energies. If the center-of-mass momenta of the incoming particles are denoted by p_1, p_2, and p_3, the variables x_i $(i=1, 2, 3)$ are defined by

$$x_i=p_i^2/(p_1^2+p_2^2+p_3^2). \quad (6.1)$$

The x's clearly satisfy the equation

$$x_1+x_2+x_3=1. \quad (6.2)$$

The boundary of the physical region is defined by the triangular inequalities in p and, expressed in terms of the x's, it is given by the equation

$$3(x_1-\tfrac{1}{3})^2+(x_2-x_3)^2=\tfrac{1}{3}. \quad (6.3)$$

This is the same as the equation for the non-relativistic physical region in terms of the y's and, if the x's are plotted in triangular coordinates, it will be a circle as before. In the nonrelativistic region $S-9m^2\approx 0$, the x's become equal to the y's.

We define variables x' for the final state in the same way. Our variables for the problem will then be S, x_1, x_2, x_1', x_2', M, M', and J, of which J is a constant parameter, S is a variable which can be real or complex, and the remaining variables are real parameters corresponding to the channel indices of the two-body problem.

It is now necessary to obtain equations for the partial energies in terms of the new variables. These kinematical relations are unfortunately more complicated than they were for the nonrelativistic case. They are most easily expressed implicitly, as a function of the parameter

$$w=p_1^2+p_2^2+p_3^2. \quad (6.4)$$

The variables S and s_1 will then be given by the formulas

$$S^{1/2}=(x_1w+m^2)^{1/2}+(x_2w+m^2)^{1/2}+(x_3w+m^2)^{1/2}, \quad (6.5)$$

$$s_1=S+m^2-2\{S(x_1w+m^2)\}^{1/2}. \quad (6.6)$$

Equation (6.6) is a restatement of the formula for the center-of-mass energy of one particle (or subsystem) in a two-particle system:

$$e_1=\{S-m_1^2+m_2^2\}/2S^{1/2},$$

where m_1 and m_2 are the masses of the particles (or subsystems), S is the square of the center-of-mass energy of the whole system, and e_1 is the center-of-mass energy of the first subsystem.

Equations (6.5) and (6.6) provide a parametric formula for s_1 in terms of x_1, x_2, x_3, and S. The parameter w can be eliminated, to give a quartic equation for s_1 in terms of S. The equation adopts a slightly simpler form if we replace s_1 by the variable

$$\sigma_1=S+m^2-s_1. \quad (6.7)$$

It then takes the form

$$D\sigma_1^4-4\sigma_1^3Sx_1(2x_1-1)+2\sigma_1^2S\{Sx_1(4x_1-1)-x_1-D\} -4\sigma_1S^2\{Sx_1^2+x_1(-3x_1+1)\}+S^2\{S^2x_1^2 +2Sx_1(-3x_1+1)-3x_1^2+2x_1+D\}=0, \quad (6.8a)$$

where

$$D=3(x_1-\tfrac{1}{3})^2+(x_2-x_3)^2-\tfrac{1}{3}. \quad (6.8b)$$

We shall also need the formula for x_1 in terms of s_1, s_2, s_3, and S. This is most easily obtained from the following explicit formula for w:

$$w=(1/4S)\{s_1^2+s_2^2+s_3^2+S^2-14m^2S-3m^4\}. \quad (6.9)$$

Substitution of (6.9) into (6.6) gives us the required formula immediately.

Remarks on the Solution of the Kinematical Equations

In order to study the character of the solutions of the equations we have just derived, it is easier to use the parametric form (6.5)–(6.6) than the explicit form (6.8). There will be a number of solutions, depending upon the signs chosen for the square roots. The square root in (6.6) must have the same sign as in (6.5). We notice that a change of sign of all the square roots will not affect the relation between S and s_1, so that there will be four possibilities, all the signs may be positive or any one may be negative. To study the solutions further we must know the relative magnitudes of the quantities x_1, x_2, and x_3, let us suppose that $x_2>x_1>x_3$. The quantity $S^{1/2}$ will then be real as long as $-m^2/x_2<w<\infty$. If we let the parameter w go from $+\infty$ to $-m^2/x_2$ and back to $+\infty$, the square root $(x_2w+m^2)^{1/2}$ will change sign when w touches the point $-m^2/x_2$. Thus, the solution with all square roots positive will pass over into that with $(x_2w+m^2)^{1/2}$ negative, while the solution with the square root $(x_1w+m^2)^{1/2}$ negative will pass over into that with $(x_3w+m^2)^{1/2}$ negative (since a change in the sign of all the square roots will not alter the solution).

As the parameter w goes from $+\infty$ to $-m^2/x_2$ and back to $+\infty$, there will thus be two distinct solutions. These have been plotted as the two upper curves of Fig. 3. It is not difficult to show that the variable s_1 goes from $+\infty$ to $-\infty$ without repetition in both solutions. Since $S^{1/2}$ is real, the variable S is positive or zero. From (6.6) we observe that, if it is zero, $s_1=m^2$, and one can

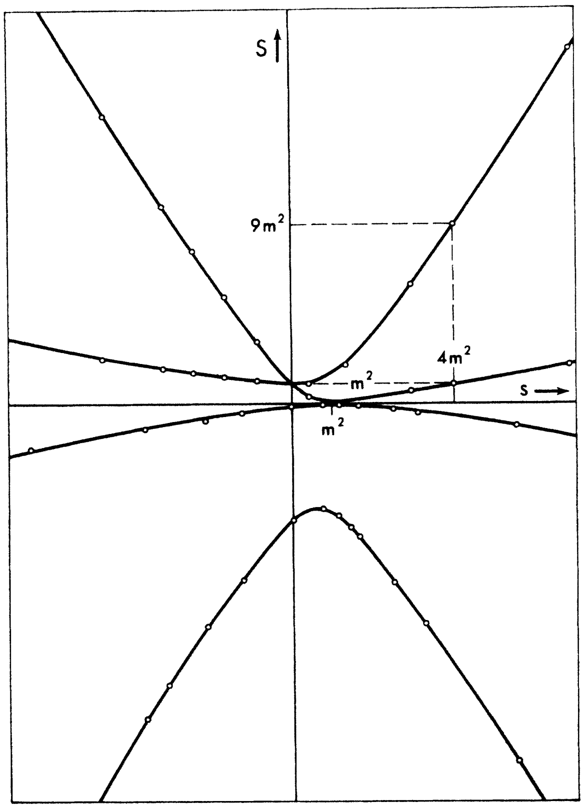

FIG. 3. A plot of S against s_1 for fixed values of x_1 and x_2.

easily show that the point $s_1=m^2$, $S=0$ occurs in one solution. As $w\to\infty$, $S\to+\infty$. The relation between S and s_1 thus has the character indicated by the upper curves of Fig. 3.

We can obtain further solutions by allowing w to go from $-m^2/x_3$ to $-\infty$. The square roots will then all be imaginary so that, from (6.5) and (6.6), $S^{1/2}$ will be imaginary, and the variables s_1 and S will both be real. We can then let the parameter w go from $-\infty$ to $-m^2/x_3$ and back to $-\infty$ and, when w touches the point $-m^2/x_3$, the square root $(x_3w+m^2)^{1/2}$ will change sign. Again, therefore, one of the four solutions will pass into another at this point, and there will be two distinct solutions. These have been plotted as the lower curves of Fig. 3. Since $S^{1/2}$ is imaginary, S will be negative or zero, and it will again be zero only if $s_1=m^2$.

Let us now investigate some particular points of Fig. 3. First, if we put $w=0$, we obtain the following four solutions:

$$s_1=4m^2,\quad S=9m^2;$$
$$s_1=4m^2,\quad S=m^2;$$
$$s_1=0,\quad S=m^2;$$
$$s_1=0,\quad S=m^2.$$

The first solution is the normal threshold, while the second is the "other normal threshold," familiar in the two-body unequal-mass case at the point $s=(m-\mu)^2$. Since we have two solutions at $s_1=0$, $S=m^2$, this will be a point of self-intersection. Further, since we have seen that two solutions touch at the point $s_1=m^2$, $S=0$, in the manner indicated in Fig. 3, this will be a point of osculation.

We have plotted Fig. 3 for the case $x_2>x_1>x_3$. If x_1 is not between x_2 and x_3, the solutions will join onto one another differently. As before, there will be two solutions for $S>0$ and two for $S<0$, but, for one or other of the cases $S>0$, $S<0$, the two right-hand solutions and the two left-hand solutions will join onto one another, instead of a right-hand solution joining onto a left-hand solution. There will then be an interval of s_1 within which there are only two real solutions for S instead of four. However, we can verify that there are always four real solutions for S if $s_1\geqq 4m^2$. Thus, the unitarity cuts for the s_1 reaction and the associated thresholds will always lie along the real (positive or negative) S axis.

We observe from the above reasoning that, if any of the four variables S, s_1, s_2, and s_3 is real and above its threshold ($9m^2$ or $4m^2$), the square roots in (6.5) will either all be real or all pure imaginary. We can therefore classify the solutions according to the signs of the square root. That solution with all square roots of the same sign will be denoted by the subscript 0, while the solutions with the root $(x_rw+m^2)^{1/2}$ of opposite sign to the others will be denoted by the subscript r ($r=1, 2, 3$). The variable s_1 is greater than $4m^2$ when S is greater than $9m^2$ for the zeroth solution, or when S is greater than m^2 for the first solution. For the second and third solutions s_1 will not be greater than $4m^2$ if S is positive, but there will be negative values of S for which $s_1>4m^2$. Corresponding results hold for s_2 and s_3.

One can also plot the kinematics by taking a fixed value of S, and drawing the lines of constant s_1 within the physical region of the (real) x_1-x_2-x_3 plane. We have carried this out for the physical solution in Fig. 4(a). If $S-4m^2$ is not too large, the lines will be practically horizontal and, in the nonrelativistic limit, they will be exactly horizontal. The other solution for $S>0$, $s>4\mu^2$ will have a similar character.

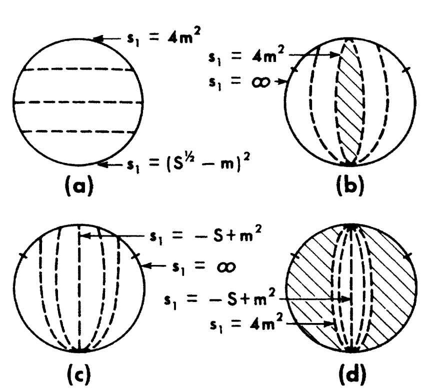

FIG. 4. Lines of constant s_1 in the x_1-x_2 plane.

The character of the negative $-S$ solutions will be different. If $-S>3m^2$, there will be at most one solution for s_1 which is greater than $4m^2$. This has been plotted in Fig. 4(b), there is no solution greater than $4m^2$ in the shaded region. As $-S$ approaches $3m^2$, the shaded region shrinks to zero and, for larger values of S, there may be two solutions for s_1 which are greater than $4m^2$. The two solutions for $-S<3m^2$ are shown in Figs. 4(c) and 4(d); for the shaded region of Fig. 4(d) there is only one solution. As the variable $-S$ approaches infinity, the unshaded region of Fig. 3(d) expands to fill the whole circle. If we move along one of the lines of constant s_1 in Fig. 4(c) or 4(d), the variable s_2 will change from ∞ to $-\infty$ (or from $-\infty$ to ∞) at the bottom of the curve, the variable s_3 will similarly pass from $-\infty$ to ∞ (or from ∞ to $-\infty$). We have already remarked that, for negative S, s_1 can be greater than $4m^2$ only in the second and third solutions, similarly s_2 can be greater than $4m^2$ only in the first and third solutions and s_3 greater than $4m^2$ in the second and third solutions. It follows that we pass from the second to the third solution, or vice versa, as we pass the bottom of a constant s_1 curve in Fig. 4(c) or 4(d).

Some Useful Formulas

We now quote the formula for the phase-space factor in the unitarity integral. First we note the formula

$$R_1/x_1=R_2/x_2=R_3/x_3=w\,, \tag{6.10}$$

where

$$R_1=\{S^2-2S(s_1+m^2)+(s_1-m^2)^2\}/4S\,. \tag{6.11}$$

The volume element of phase space is then

$$Gdx_1dx_2\,,$$

where

$$G=\frac{(S-9m^2)^2}{256\pi^3S}\times\frac{4S^2w}{(S-9m^2)^2(x_1\sigma_2\sigma_3+x_2\sigma_3\sigma_1+x_3\sigma_1\sigma_2)} \tag{6.12}$$

the quantity σ_i being defined by (6.7). In the nonrelativistic limit, the second factor of (6.12) approaches unity.

When evaluating the contribution to the unitarity integral from the disconnected diagrams, which contain factors such as $\delta(s_1-s_1')$, we shall require the derivatives of s_1, s_2, and s_3 with respect to our new variables. They can be obtained from the formula

$$ds_1=\frac{2Sw}{x_1\sigma_2\sigma_3+x_2\sigma_3\sigma_1+x_3\sigma_1\sigma_2}\times\{dx_1[x_2(s_2-s_3)+S+m^2-s_2]-x_1dx_2(s_2-s_3)\}\,. \tag{6.13}$$

There will be similar formulas for ds_2 and ds_3.

Also of interest is the formula for $\cos\theta_1$ in terms of our new variables where θ_1 as before is the angle between the directions of motion of particles 1 and 2, in the center-of-mass system of particles 2 and 3. The formula is

$$\cos\theta_1=-\frac{x_2-x_3}{x_1}\left(\frac{s_1}{s_1-4m^2}\right)^{1/2}\frac{(4R_1S)^{1/2}}{S+s_1-m^2}\,. \tag{6.14}$$

Relation between the Solution of the Kinematical Equations and the Physical Region for Various Processes

When $S>9m^2$ or $s_i>4m^2$ ($i=1,2,3$) we shall have to evaluate discontinuities using the unitarity condition, and it is therefore necessary to investigate whether or not we are in the physical region. As long as $S>0$ there is no difficulty, since the square roots in Eq. (6.5) are real. When all the square roots have the same sign (corresponding to the zeroth solution) we are in the physical region for the process $1+2+3\rightarrow X$. If the square root $(wx_1\approx m^2)^{1/2}$ is negative while the other two are positive (corresponding to the first solution), we are in the physical region for the process $2+3\rightarrow 1+X$. In this region the variable s_1 is positive while s_2 and s_3 are negative, and the discontinuity equation for s_1 is thus a straightforward unitarity equation for s_1 in the physical s_1 channel.

For the cases $S<0$, $s_1>4m^2$ (corresponding to the solutions 2 and 3) the situation is slightly more complicated, and we shall now prove that these regions correspond to real energies but imaginary angles for the process $2+3\rightarrow 1+X$. The parameter w in (6.5) is then negative and greater than m^2/x_i ($i=1,2,3$), so that the momenta and energies of all three particles are imaginary. On transforming to the center-of-mass system of the particles 2 and 3, one can easily see that the momenta of all particles in the direction of motion of the particle 1 are real, but the momenta perpendicular to the direction of motion of the particle 1 are pure imaginary. We are therefore in the physical energy region for the reaction $2+3\rightarrow 1+X$, but the angle θ_1 between the directions of motion of the particles 1 and 2 is pure imaginary. As we move along a dotted line of Fig. 4(c) or (d) the variable $\cos\theta_1$ will increase from the value 1 which it has at the edge of the circle and, at the bottom of the curve, it will reach the value ∞ [Eq. (6.13)]. It then goes to $-\infty$, and increases to -1 when we reach the edge of the circle again.

Though the interior of the circle does not strictly represent the physical region when $S<0$, we shall show in Sec. 8 that the contour of integration in the unitarity equation can be deformed to values of $|\cos\theta_1|$ greater than 1. The unitarity integral is thus always performed over values of the x's within the circle.

With our relativistic kinematics, the scattering amplitude will not be uniquely determined by the variables S, x_1, x_2, x_1', x_2', M, M', and J, since we also have to specify which of the four possible values the quantities s_1 and s_1' take. In other words, we require indices α and α', running from zero to three, in addition to the above variables. We shall denote the amplitude by $A_{\alpha\alpha'}$. The disconnected parts a_i will similarly be denoted by the symbols $a_{i,\alpha\alpha'}$, where the first subscript denotes the presence of a delta function $\delta(s_i-s_i')$, and the last two indicate the solutions of the kinematics. Owing to the presence of the delta function, not all combinations of α and α' occur in the disconnected terms. Thus, when $i=1$, and $s_1>4m^2$, we cannot have a transition between the zeroth solution, where $s_1<S$ and the first, where $s_1>S$. We also cannot have a transition between either of these and the second or third solution, where $S<0$. We can however have a transition between the second and third solution, since we have seen that the constant $-s_1$ lines in Figs. 4(b) and (c) go from the second to the third solution, or vice versa, at the bottom of the diagram. The combinations which occur are thus $a_{1,00}$, $a_{1,11}$, $a_{1,22}$, $a_{1,33}$, $a_{1,23}$, and $a_{1,32}$. Similar combinations occur for $a_{2,\alpha\alpha'}$ and $a_{3,\alpha\alpha'}$.

VII. DISPERSION RELATIONS FOR MULTI-SHEETED FUNCTIONS

General Methods

Our partial-wave amplitude is an analytic function of the variables S, s_i, and s_i', except for the usual left- and right-hand cuts. However, the variables s_i are obtained in terms of S and x_i by solving quartic equations, so that these variables, expressed as functions of S with x_i kept fixed, will have branch points. The partial-wave amplitudes will therefore have branch points as functions of S when the x's are kept fixed. We shall call these branch points kinematical branch points, while the branch points associated with the left- and right-hand cuts will be called dynamical branch points.

In the usual analyticity problem of relativistic quantum mechanics one requires to find a function when the discontinuities across its cuts are known. Our present problem is rather different. We are given a Riemann surface divided into several sheets by the kinematical branch points. The function which we require to find is defined on this Riemann surface, and it has dynamical cuts in addition to the kinematical cuts. The discontinuity across the dynamical cuts on *each* of the kinematical sheets is known, but the discontinuity across the cuts separating the kinematical sheets is not known.

In order to illustrate the method, we shall first consider an example with a simple kinematical sheet structure, and shall treat a function of a variable ξ with two kinematical sheets connected by branch points at $\xi=\pm a$. In other words, the function has the same sheet structure as $(\xi^2-a^2)^{1/2}$. It will also have certain dynamical cuts with known discontinuities. A general function with these properties will satisfy the dispersion relation

$$\phi(\xi)=\frac{1}{\pi}\int d\xi'\frac{f_1(\xi')}{\xi'-\xi}+\frac{(\xi^2-a^2)^{1/2}}{\pi}\int d\xi'\frac{f_2(\xi')}{\xi'-\xi}, \quad (7.1)$$

where the integrals are to be taken over the dynamical cuts. If the discontinuities across the dynamical cuts on the two kinematical sheets are f_a and f_b, then

$$f_a(\xi)=f_1(\xi)+(\xi^2-a^2)^{1/2}f_2(\xi), \quad (7.2a)$$

$$f_b(\xi)=f_1(\xi)-(\xi^2-a^2)^{1/2}f_2(\xi). \quad (7.2b)$$

Since the functions f_a and f_b are given, Eqs. (7.1) and (7.2) provide the solution of our problem.

We may require our function f to approach zero as ξ approaches infinity. The solution (7.1) does not have this property, owing to the factor $(\xi^2-a^2)^{1/2}$ in front of the second integral. However, we can easily construct a solution with the required property, namely,

$$\phi(\xi)=\frac{1}{\pi}\int d\xi'\frac{f_3(\xi')}{\xi'-\xi}+\frac{(\xi^2-a^2)^{1/2}-\xi}{\pi}\int d\xi'\frac{f_4(\xi')}{\xi'-\xi}. \quad (7.3)$$

The functions f_3 and f_4 will be given by replacing $(\xi^2-a^2)^{1/2}$ by $(\xi^2-a^2)^{1/2}-\xi$ in (7.2). Since the factor outside the second integral in (7.3) does not increase with ξ on the kinematical sheet where the square root $(\xi^2-a^2)^{1/2}$ is positive, the function $\phi(\xi)$ defined by (7.3) will tend to zero in this limit. The function $\phi(\xi)$ will not tend to zero when ξ approaches infinity on the other kinematical sheet and, in fact, it is impossible to construct a solution which approaches zero at infinite ξ on both sheets unless f_a and f_b have special properties.

In the example considered, the functions $\eta=(\xi^2-a^2)^{1/2}$ and ξ are related by an equation of the second degree. Now let us consider a problem where the kinematical sheet structure is the same as that of a function η which is related to ξ by an equation of the nth degree. There will be n functions $\zeta_{\alpha 1}\cdots\zeta_{\alpha n}$ (one of which may be unity) which have the same sheet structure as η but no dynamical cuts, and which are not related by an equation of the form

$$\sum_{\lambda} p_\lambda(z)\zeta_{\alpha\lambda}=0,$$

where the p_λ are polynomials. The subscript α denotes the sheet. [It would fix the sign of the square root in Eqs. (7.1)–(7.3).] As an example of the functions ζ, one could take $\zeta_{\alpha 1}=1$, $\zeta_{\alpha n}=(\eta_\alpha)^{n-1}$. A function with the given sheet structure, together with the dynamical cuts, will then be given by the equation

$$\phi_\alpha(\xi)=\sum_{\lambda=1}^{n}\frac{\zeta_{\alpha\lambda}(\xi)}{\pi}\int d\xi'\frac{f_\lambda(\xi')}{\xi'-\xi}. \quad (7.4)$$

If $f_\alpha(\xi)$ $(\alpha=0,\cdots,n-1)$ are the given discontinuities on the n sheets, the functions f_λ in (7.4) can be obtained from the equations

$$f_\alpha(\xi)=\sum_{\lambda=1}^{n}\zeta_{\alpha\lambda}f_\lambda(\xi) \tag{7.5a}$$

or

$$f_\lambda(\xi)=\{\zeta\}^{-1}{}_{\lambda\alpha}(\xi)f_\alpha(\xi)\,, \tag{7.5b}$$

where $\{\zeta\}^{-1}{}_{\lambda\alpha}$ is the matrix reciprocal of $\zeta_{\alpha\lambda}$. We discuss below the possibility of the matrix $\zeta_{\alpha\lambda}$ being singular.

One can define the functions ζ in such a way that they remain finite as ξ approaches infinity on one of the sheets. For, if $\eta\to C\xi$ on that sheet, one can take $\zeta_{\alpha1}=1$, $\zeta_{\alpha n}=(\eta_\alpha')^{n-1}$, where $\eta_\alpha'=\eta_\alpha-C\xi$. With that choice the function $\phi(\xi)$, defined by (7.4), will approach zero at infinite ξ provided discontinuity across the cut approaches zero. On the other sheets the ζ's may approach infinity like ξ^{n-1} at infinite ξ, so that the function ϕ may behave like ξ^{n-2}.

The question now arises whether it is possible to make a choice of the ζ's different from the original choice 1, $\eta,\cdots,\eta^{n-1}$, which has the property that all the ζ's approach infinity less rapidly than ξ^{n-1} on all the sheets. One can show that such a choice is not possible unless the equation connecting ξ with η has exceptional points, such as points of self-intersection or points of osculation. If there are exceptional points it is possible to make a choice of functions with the desired property. Suppose, for instance, that there is a point of self-intersection at $\xi=\xi_0$. Two of the η_α's will then be equal at this point, so that it is possible to find a set of n constants a_r, not all of which are zero, satisfying the equations

$$\sum_{r=0}^{n-1}a_r\{\eta_\alpha(\xi_0)\}^r=0\,. \tag{7.6}$$

The function

$$\{\sum_{r=0}^{n-1}a_r(\eta_\alpha)^r\}/(\xi-\xi_0) \tag{7.7}$$

will be regular at $\xi=\xi_0$, will have the same sheet structure as η, and it will behave like ξ^{n-2} at infinity. This function, together with the functions $1,\eta,\cdots,\eta^{n-2}$, provide a choice with the desired properties.

One can prove in a similar way that each point of self-intersection enables us to reduce the degree of one of our functions ζ_λ by 1. Thus, with no points of self-intersection, the degrees of our functions ζ are 0, $1,\cdots,n-1$. With one point of self-intersection, one can find functions ζ of degree $0,1,\cdots,n-2,n-2$. If there are two further points of self-intersection, making three in all, one can make a choice of the ζ's of degree $0,1,\cdots,n-3,n-3,n-3$. In counting points of self-intersection, a point of osculation with $d\xi/d\eta=0$ will count as two points of intersection. At such a point there are two coincident double roots $\eta_1(\xi),\eta_2(\xi)$ and $\eta_3(\xi),\eta_4(\xi)$, and one can consider it as a point of self-intersection between η_1 and η_3 and between η_2 and η_4.

Once the ζ's have been redefined in the manner just described, the quantities $\zeta_{\alpha\lambda}(\xi)$ will be linearly independent everywhere except at the kinematic branch points. This follows from the fact that the row matrices $1,\eta_\alpha\cdots\eta_\alpha^{n-1}$ ($\alpha=1,\cdots,n$) can only be linearly dependent if two of the η_α's are equal. The case of equality corresponds either to kinematic branch points or to points of self-intersection which we have just treated. The matrix reciprocal $\{\zeta\}^{-1}{}_{\lambda\alpha}$ therefore exists everywhere except at the kinematic branch points. These branch points will not occur within the ranges of the integral equations to be derived.

We shall require the relation between the behavior of ζ and $\{\zeta\}^{-1}$ at high ξ. Let us suppose that

$$\zeta_{\alpha\lambda}(\xi)\to C_{\alpha\lambda}\xi^{n(\alpha)}\,,\quad \xi\to\infty\,, \tag{7.8a}$$

where $n(\alpha)$ is an integer. If the asymptotic behavior of $\zeta_{\alpha\lambda}$ depends on λ, we take $n(\alpha)$ equal to the largest asymptotic power for that value of α and put $C_{\alpha\lambda}=0$ for those components with slower increase. Further, we can assume that the ζ's have been defined so that the c's are linearly independent; if not, we can take new linear combinations of the ζ's with slower asymptotic behavior than is given by (7.7a). It then follows that

$$\{\zeta\}^{-1}{}_{\lambda\alpha}(\xi)\to C'_{\lambda\alpha}\xi^{-n(\alpha)}\,,\quad \xi\to\infty\,. \tag{7.8b}$$

We have chosen the ζ's so that they do not increase asymptotically on one kinematical sheet. If we denote this sheet by $\alpha=0$, then

$$n(0)=0\,. \tag{7.8c}$$

Applications to the Problem of Interest

We can now apply these results to the problem under consideration. The partial-wave amplitude is a function of S, s_1, s_2, s_1', s_2'. The functions s_1 and s_2 have the same sheet structure, since they are related by the equation

$$2s_2=S+3m^2-s_1+\frac{x_1-x_3}{x_1}\frac{s_1^2-2s_1(S+m^2)-(S-m^2)^2}{S+s_1-m^2} \tag{7.9}$$

as may easily be shown from (6.10) and (6.11). Thus, if we neglect the primed variables for the moment, the partial-wave amplitude must have the same kinematical structure as the function $s_1(S)$. Now s_1 satisfies a quartic equation in S at given values of x_1 and x_2, but we saw in the previous section that the equation has a point of self-intersection as well as a point of osculation with $ds_1/dS=0$. We must therefore be able to find four functions ζ of degree 0,1,1,1 with the required sheet structure and which are not related by an equation of the form (7.4).

One could obtain the functions by the systematic method which we have just given, but this is not necessary. It follows from (7.9) and (6.10) that the functions

s_2 and w have the same sheet structure in S as s_1, so that a possible choice of the functions ζ would be 1, s_1, s_2, w. It may be verified that these functions are not related by an equation of the form (7.4). However, we wish to define our functions in such a way that they do not become infinite as S approaches infinity on the physical sheet, i.e., the sheet obtained by taking positive signs for all square roots in (6.5). Now, according to (6.5) and (6.6)

$$s_1 \to \frac{-x_1^{1/2}+x_2^{1/2}+x_3^{1/2}}{x_1^{1/2}+x_2^{1/2}+x_3^{1/2}} S, \quad S\to\infty\,,$$

$$s_2 \to \frac{x_1^{1/2}-x_2^{1/2}+x_3^{1/2}}{x_1^{1/2}+x_2^{1/2}+x_3^{1/2}} S, \quad S\to\infty\,,$$

$$w \to \frac{1}{\{x_1^{1/2}+x_2^{1/2}+x_3^{1/2}\}^2} S, \quad S\to\infty\,.$$

We may therefore define the four functions

$$\zeta_{\alpha 1}(S)=1(\text{independent of } \alpha)\,, \tag{7.10a}$$

$$\zeta_{\alpha 2}(S)=s_{1,\alpha}-\frac{-x_1^{1/2}+x_2^{1/2}+x_3^{1/2}}{x_1^{1/2}+x_2^{1/2}+x_3^{1/2}} S, \tag{7.10b}$$

$$\zeta_{\alpha 3}(S)=s_{2,\alpha}-\frac{x_1^{1/2}-x_2^{1/2}+x_3^{1/2}}{x_1^{1/2}+x_2^{1/2}+x_3^{1/2}} S, \tag{7.10c}$$

$$\zeta_{\alpha 4}(S)=w_\alpha-\frac{1}{\{x_1+x_2+x_3\}^2} S. \tag{7.10d}$$

The subscript α denotes the kinematical sheet, and it takes the values 0, 1, 2, and 3 according to the prescription given in the previous section.

From (7.10), we find the following asymptotic behavior for the ζ's:

$$\zeta_{0\lambda}(S)\to C_{0\lambda}\,,\quad \zeta_{\alpha\lambda}(S)\to C_{\alpha\lambda}\,,\quad S\to\infty\,. \tag{7.11a}$$

The constant $C_{\alpha 1}$ is actually zero, since $\zeta_{\alpha 1}=1$. It follows from (7.8) that

$$\{\zeta\}^{-1}{}_{\lambda 0}\to C_{\lambda 0}'\,,\quad \{\zeta\}^{-1}{}_{\lambda\alpha}\to C_{\lambda\alpha}'S^{-1}\,,\quad S\to\infty \tag{7.11b}$$

since it can easily be verified that the c's are linearly independent.

We can now use these functions ζ to construct a solution to our analyticity problem, using Eqs. (7.4) and (7.5). Our scattering amplitude will have separate kinematical branch points associated with the initial and final states, since the branch points arise from the branch points of s_i or s_i' as a function of S with the x's and x''s fixed. We have already pointed out in the last section that a scattering amplitude has two kinematical subscripts α and α', and Eqs. (7.4) and (7.5) must be modified by including ζ factors corresponding to both subscripts. It is in fact easier to work in λ space than in α space. We thus define amplitudes $A_{\lambda\lambda'}$ by the formulas

$$A_{\lambda\lambda'}(S)=\{\zeta\}^{-1}{}_{\lambda\alpha}(S)A_{\alpha\alpha'}(S)\{\zeta\}^{-1}{}_{\lambda'\alpha'}(S) \tag{7.12a}$$

or

$$A_{\alpha\alpha'}(S)=\zeta_{\alpha\lambda}(S)A_{\lambda\lambda'}(S)\zeta_{\alpha'\lambda'}(S)\,. \tag{7.12b}$$

The function $A_{\lambda\lambda'}$ will satisfy an ordinary dispersion relation

$$A_{\lambda\lambda'}(S')=\frac{1}{\pi}\int dS'\frac{[A_{\lambda\lambda'}(S')]}{S'-S}\,. \tag{7.13}$$

Equations (7.12) and (7.13) are together equivalent to (7.4), with two factors ζ instead of one.

In our problem, we are given equations for the discontinuity of the function $A_{\alpha\alpha'}$, we use (7.12) to convert them into equations for the discontinuity of $A_{\lambda\lambda'}$. We then solve the problem in terms of the amplitudes $A_{\lambda\lambda'}$, using the simple dispersion relation (7.13) in conjunction with the discontinuity formulas. Finally, we convert to the amplitudes $A_{\alpha\alpha'}$ using (7.12). All products of scattering amplitudes become matrix products in the λ's, as well as in our other variables, during the course of our calculations. Notice that we have independent functions associated with each subscript in λ space, whereas the functions pass into one another at the branch points in α space.

The functions ζ have been chosen to remain constant at high energy on the zero sheet, but they increase with energy on the other sheets. The scattering amplitudes which we calculate will therefore tend to zero at high energy on the physical sheet for three-particle scattering, but not on the other kinematical sheets. In the exact problem the amplitudes might be expected to approach zero on all sheets, and this would be the case provided the input left-hand discontinuities had certain properties. The failure of the scattering amplitude to approach zero on the other sheets may be regarded as a failure of the approximation scheme and, since the approximation scheme is not expected to be accurate at high energies, it is not a serious failure.

One might suspect that the increasing asymptotic behavior of our scattering amplitudes would give us integral equations which are singular at infinity. Fortunately it turns out, as a consequence of the asymptotic behavior (7.11), that the integral equations for $A_{\lambda\lambda'}$ are no more singular than they would have been if the complications associated with the kinematical branch points had been absent.

VIII. THE UNITARITY CONDITION

Let us begin by recalling the relation between the four solutions of the kinematics and the physical regions for the various processes. In the zeroth solution we shall be in the physical region for three-particle scattering, provided $S>9m^2$. All seven variables S, s_i, s_i' will then be above their thresholds, and there will be terms in the unitarity condition of the form A^*GA, a_i^*GA, and

A^*Ga_i. In the other three solutions, we shall not be in the physical region for three-particle scattering, but we may be in the physical region for an s_i or an s_i' channel, i.e., for a reaction in which two particles go into four. The term A^*GA will then not contribute to the discontinuity, but there will be contributions from the terms a_i^*GA and A^*Ga_i.

We now divide the unitarity discontinuity into seven parts as follows:

$$[A]=[A]_S+\sum_i [A]_{s_i}+\sum_i [A]_{s_i'}\,. \qquad (8.1)$$

The term $[A]_S$ is the contribution from the physical region for three-particle scattering, including the terms a_i^*GA and A^*Ga_i. The term $[A]_{s_i}$ is the contribution from the case where $s_i>4m^2$, but where we are not in the physical region for three-particle scattering. According to this separation, the term $[A]_S$ comes from an intermediate state with the zeroth solution of the kinematics, while the terms $[A]_{s_i}$ and $[A]_{s_i'}$ come from intermediate states with the other three solutions. We could have included the terms $a_i^*GA_i$ in the contribution $[A]_{s_i}$, even when they come from the physical region for three-particle scattering. However, it turns out to be more convenient to include all discontinuities from the physical three-particle region in one term. To avoid confusion, we should emphasize that all terms of (8.1) refer to the discontinuity in S when the x's are kept fixed. The s_i's are not regarded as the variables in our problem, and discontinuities corresponding to the s_i channels become reflected as discontinuities in S.

With this decomposition, the formula for $[A]_S$ will be

$$[A_{\alpha\alpha'}]_S=\theta(S-9m^2)\{A^*{}_{\alpha0}GA_{0\alpha'} +\sum_{i=1}^{3}\delta_{\alpha0}a^*{}_{i,00}GA_{0\alpha'}+\sum_{i=1}^{3}\delta_{\alpha'0}A^*{}_{\alpha'0}Ga_{i,00}\}\,. \qquad (8.2)$$

The subscripts indicate the solution to the kinematics, as has been explained in Sec. 6.

We next treat the discontinuity $[A]_{s_1}$. We have seen in Sec. 6 that s_1 will be real and greater than $4m^2$ in solution 1 of the kinematics when S is real and greater than m^2, and also in solutions 2 and 3 when S has certain real negative values. The discontinuity $[A_{\alpha\alpha'}]_{s_1}$ will therefore be equal to $a^*{}_{1,\alpha1}GA_{1\alpha'}$ when $S>0$ and to $a^*{}_{1,\alpha2}GA_{2\alpha'}+a^*{}_{1,\alpha3}GA_{3\alpha'}$ when $S<0$. At the end of Sec. 6 we pointed out that certain components of $a_{\alpha\alpha'}$ vanished, so that we may write

$$[A_{\alpha\alpha'}]_{s_1}=\delta_{\alpha1}\theta(S-m^2)a^*{}_{1,11}Ga_{1\alpha'}-(\delta_{\alpha2}+\delta_{\alpha3}) \times\theta(s_1-4m^2)\{a^*{}_{1,\alpha2}GA_{2\alpha'}+a^*{}_{1,\alpha3}GA_{3\alpha'}\}\,. \qquad (8.3)$$

The minus sign in front of the second term occurs because the imaginary parts of the variables s_1 and S have opposite signs when $S<0$, $s_i>4m^2$ and the x's are real.

As the amplitude a_i^* has a delta function $\delta(s_1-s_1')$, the unitarity integrals in (8.3) will be over a single variable which may conveniently be taken as $\cos\theta_1$. We have in fact to integrate over the dotted lines in Fig. 4. However, we must bear in mind that the interior of the circles in Fig. 4(b), (c), and (d) correspond to values of $\cos\theta_1$ greater in magnitude than unity when $S<0$, whereas the unitarity integral should be taken over the range $-1<\cos\theta_1<1$. In order to perform the integrals within the circle one has to deform the contour to cover the range $|\cos\theta_1|>1$ and, in the deformation, there are several points to be considered.

First, we may encounter singularities of the amplitude A, because a three-body amplitude will in general have singularities in the complex $\cos\theta$ plane. Since we are assuming that the discontinuities across such cuts are known, they will not cause any difficulty. We simply have to include an extra term $I_{\alpha\alpha',s_1}$ on the right-hand side of (8.3), and this term will in general be known. Thus

$$[A_{\alpha\alpha'}]_{s_1} =\delta_{\alpha1}\theta(S-m^2)a^*{}_{1,11}Ga_{1\alpha'}-(\delta_{\alpha2}+\delta_{\alpha3})\theta(s_1-4m^2) \times\{a^*{}_{1,\alpha2}GA_{2\alpha'}+a^*{}_{1,\alpha3}GA_{3\alpha'}\}+I_{\alpha\alpha',s_2}\,. \qquad (8.4a)$$

Next we enquire whether we encounter any singularities of the amplitude a^*. This amplitude is given by (3.6) or (3.9), and we assume that the two-body amplitude t is analytic in x with cuts along the real axis. We now insert (3.6) in (8.4a), and attempt to continue the initial-state variable $\cos\theta_1$ from a value between -1 and 1 to a value greater in magnitude than 1 and, at the same time, to deform the unitarity integral implied by (8.4a) to cover the range $|\cos\theta_1'|>1$. We find that we can continue the variable $\cos\theta_1$ to a point just above the real axis and deform the contour of integration to cover the range $|\cos\theta_1'|>1$, $\mathrm{Im}(\cos\theta_1')=\epsilon$, or vice versa, but that we must keep the signs of the small imaginary parts of $\cos\theta_1$ and $\cos\theta_1'$ the same.

To see what this implies, we recall that the amplitude $A_{2\alpha'}$ will have s_3 as well as s_1 positive when $S<0$, and $A_{3\alpha'}$ will have s_2 as well as s_1 positive. Each of these amplitudes therefore has a discontinuity $[A]_{s_2}$ or $[A]_{s_3}$ in addition to $[A]_{s_1}$, and we must investigate whether the amplitudes A in (8.4a) are to be evaluated above or below this discontinuity. Now, if $\cos\theta_1$ has a positive imaginary part, the variable s_2 will have a positive imaginary part in the region for which it is positive, while s_3 will have a negative imaginary part. If, therefore, we do the unitarity integral in (8.4a) with $\cos\theta_1$ and $\cos\theta_1'$ above the real axis, we conclude that the amplitudes A on both sides of (8.4a) are to be taken below the cut $[A]_{s_2}$, but above the cut $[A]_{s_3}$ (we again recall the reversal of sign between $\mathrm{Im}s_i$ and $\mathrm{Im}\,S$ when $\mathrm{Re}S<0$). We could have taken all amplitudes above the cut $[A]_{s_2}$ and below the cut $[A]_{s_3}$, but further possibilities are not allowed.

The equations for the discontinuities $[A_{\alpha\alpha'}]_{s_2}$ and $[A_{\alpha\alpha'}]_{s_3}$ are given by formulas similar to (8.4a):

$$[A_{\alpha\alpha'}]_{s_2}=\delta_{\alpha 2}\theta(S-m^2)\times a^*_{2,22}GA_{2\alpha'}-(\delta_{\alpha 1}+\delta_{\alpha 3})\theta(s_2-4m^2)\times\{a^*_{2,\alpha 3}GA_{3\alpha'}+a^*_{2,\alpha 1}GA_{1\alpha'}\}+I_{\alpha\alpha',s_2}, \quad (8.4b)$$

$$[A_{\alpha\alpha'}]_{s_3}=\delta_{\alpha 3}\theta(S-m^2)\times a^*_{3,33}GA_{3\alpha'}-(\delta_{\alpha 1}+\delta_{\alpha 2})\theta(s_3-4m^2)\times\{a^*_{3,\alpha 1}GA_{1\alpha'}+a^*_{3,\alpha 2}GA_{2\alpha'}\}+I_{\alpha\alpha',s_3}. \quad (8.4c)$$

When $S<0$, the amplitudes in (8.4a) are to be taken below the cut $[A]_{s_2}$, but above the cut $[A]_{s_3}$, the amplitudes in (8.4b) are to be taken below the cut $[A]_{s_3}$ but above the cut $[A]_{s_1}$, and the amplitudes in (8.4c) are to be taken below the cut $[A]_{s}$, but above the cut $[A]_{s_2}$.

In (8.4) we have used the symbol a^* for the amplitude at complex as well as real values of $\cos\theta_1$. What is implied is the analytic continuation of a^* from the physical region, even though it is not equal to the complex conjugate of a.

Since the range of integration in the second term of (8.4) includes the point $\cos\theta_i=\infty$, the equations will only be meaningful as they stand if the amplitudes a_i decrease sufficiently rapidly at infinite momentum transfer. Nevertheless, we shall show that our integral equations can be constructed for the general case if the two-body amplitudes have a Regge asymptotic behavior.

The formulas for the discontinuities $[A_{\alpha\alpha'}]_{s_i'}$ will be precisely the same as those for $[A_{\alpha\alpha'}]_{s_i}$ and they need not be written down separately.

The terms I in (8.4) are known discontinuities of the scattering amplitude, and they are therefore ordinary left-hand cuts. We must include contributions from them in the function F_L but, since we are taking this function as given, we may henceforth neglect the terms I in (8.4).

IX. THE N/D EQUATIONS WITH FULL RELATIVISTIC KINEMATICS

We now combine the unitarity condition (8.2) and (8.4) with the dispersion relations in order to calculate the scattering amplitude. The discontinuity equations (8.5) for $[A]_{s_i}$ are linear equations of the Omnès type, whereas the equations (8.2) resemble the equations which we encountered with the nonrelativistic kinematics. Our method will be first to eliminate the discontinuities $[A]_{s_i}$ and $[A]_{s_i'}$ by solving the Omnès equations (with a continuous infinity of channels). We will then be left with equations similar to those of Sec. 5.

Elimination of the Discontinuities $[A]_{s_i}$ and $[A]_{s_i'}$

According to the procedure developed in Sec. 7, we write all integral equations for the amplitude $A_{\lambda\lambda'}$, where $A_{\lambda\lambda'}$ is related to the actual amplitude of interest $A_{\alpha\alpha'}$ by (7.11). The discontinuity formulas of the previous section, which have been written for the amplitude $A_{\alpha\alpha'}$, will be converted into formulas for $A_{\lambda\lambda'}$ as we go along.

At the moment we assume that the two-body amplitudes decrease sufficiently rapidly at infinite momentum transfer for equations (8.4) to be meaningful as they stand.

To eliminate the discontinuity (8.4a), we write

$$A_{\lambda\lambda'}=M^{(1)}{}_{\lambda\mu}A'_{\mu\lambda'}, \quad (9.1)$$

where the function $M^{(1)}$ satisfies the equation

$$M^{(1)}{}_{\lambda\lambda'}(S)=\delta_{\lambda\lambda'}+\frac{1}{\pi}\int_{m^2}^{\infty}\frac{dS'}{S'-S-i\epsilon}\{\zeta\}^{-1}{}_{\lambda 1}(S')a^*_{1,11}(S')G(S')\zeta_{1\mu}(S')M^{(1)}{}_{\mu\lambda'}(S')-\frac{1}{\pi}\int\frac{dS'}{S'-S+i\epsilon}\{\zeta\}^{-1}{}_{\lambda\alpha}(S')$$
$$\times(\delta_{\alpha 2}+\delta_{\alpha 3})\theta(s_1'-4m^2)\{a^*_{1,\alpha 2}(S')G(S')\zeta_{2\mu}(S')+a^*_{1,\beta 3}(S')G(S')\zeta_{3\mu}(S')\}M^{(1)}{}_{\mu\lambda'}(S'). \quad (9.2)$$

The reason for the sign of the $i\epsilon$ in the second denominator is again that points above the real axis in the s_1 plane correspond to points below the real axis in the S plane, so that an amplitude a_1^*, must be multiplied by an amplitude A below the cut in the S plane.

On using (8.5) and (9.2) to calculate the discontinuities of the function A', we find that $[A']_{s_\alpha}=0$, while

$$[A'_{\lambda\lambda'}]_{s_2}=\theta(S-m^2)M_{\lambda\mu-}{}^{(1)-1}\{\zeta\}^{-1}{}_{\mu 2}a^*_{2,22}G\zeta_{2\nu}M_{\nu\rho}{}^{(1)}A'_{\rho\lambda'}-M_{\lambda\mu+}{}^{(1)-1}\{\zeta\}^{-1}{}_{\mu\alpha}(\delta_{\alpha 1}+\delta_{\alpha 3})\theta(s_2-4m^2)$$
$$\times\{a^*_{2,\alpha 3}G\zeta_{3\nu}M^{(1)}{}_{\nu\rho+}+a^*_{2,\alpha 1}G\zeta_{1\nu}M_{\nu\rho}{}^{(1)}\}A'_{\rho\lambda'}. \quad (9.3)$$

We have used the subscripts + and − to indicate whether M is to be taken above or below the cut in the S plane. The subscript − on the first factor M^{-1} in (9.3), and the subscript + on the factor M^{-1} at the beginning of the second term, arise from the formulas $[XY]=X_-[Y]+[X]Y_+$ and $[XY]=X_+[Y]+[Y]X_+$, which we have used to evaluate the discontinuities across the positive and negative real S axes, respectively. The subscript + on the first factor $M^{(1)}$ within the curly bracket results from the prescription we gave after writing down Eqs. (8.4), that the amplitudes in (8.4b) are to be evaluated above the cut $[A]_{s_1}$. The factor $\zeta_{1\nu}M^{(1)}{}_{\rho\nu}$ in the second term within the curly bracket of (9.3) will have no discontinuity in S along the negative real axis, as the variable s_2 is negative in the first solution of the kinematics (for $S>0$), and $\zeta M^{(1)}$ only has a discontinuity if $s_1>4m^2$. It is therefore unnecessary to indicate a + or − subscript on this factor. It is similarly unnecessary to indicate a + or − subscript on the factor $\zeta_{2\nu}M_{\rho\nu}{}^{(1)}$ in the first term.

We can now eliminate the discontinuity $[A'_{\alpha\alpha'}]_{s_2}$ by defining a new function

$$A'_{\lambda\lambda'}=M_{\lambda\mu}{}^{(2)}A''_{\mu\lambda'}\,, \tag{9.4}$$

where the function $M^{(2)}$ satisfies the equation

$$M^{(2)}{}_{\lambda\lambda'}(S)=\delta_{\lambda\lambda'}+\frac{1}{\pi}\int_{m^2}^{\infty}\frac{dS'}{S'-S-i\epsilon}M_{\lambda\mu-}{}^{(1)-1}(S')\{\zeta\}^{-1}{}_{\mu2}(S')a^*{}_{2,22}(S')G(S')$$
$$\times\zeta_{2\nu}(S')M^{(1)}{}_{\nu\rho}(S')M^{(2)}{}_{\rho\lambda'}(S')-\frac{1}{\pi}\int\frac{dS'}{S'-S+i\epsilon}M_{\lambda\mu+}{}^{(1)-1}(S')\{\zeta\}^{-1}{}_{\mu\alpha}(S')(\delta_{\alpha1}+\delta_{\alpha3})\theta(s_2'-4m^2)$$
$$\times\{a^*{}_{2,\alpha3}(S')G(S')\zeta_{3\nu}(S')M^{(1)}{}_{\nu\rho+}(S')+a^*{}_{2,\alpha1}(S')G(S')\zeta_{1\nu}(S')M^{(1)}{}_{\nu\rho}(S')\}M^{(2)}{}_{\rho\lambda'}(S')\,. \tag{9.5}$$

In the same way we can eliminate the discontinuity $[A]_{s_3}$ by defining the function

$$A''=M^{(3)}A'. \tag{9.6}$$

In writing down the equation for $M^{(3)}$, we shall use the notation

$$M^{(12)}{}_{\lambda\lambda'\pm\pm}=M^{(1)}{}_{\lambda\mu\pm}M^{(2)}{}_{\mu\lambda'\pm}\,.$$

If it is unnecessary to specify one of the $+$ or $-$ subscripts, we shall replace it by a dot. The equation then adopts the form

$$M^{(3)}{}_{\lambda\lambda'}(S)=\delta_{\lambda\lambda'}+\frac{1}{\pi}\int_{m^2}^{\infty}\frac{dS'}{S'-S-i\epsilon}M_{\lambda\mu--}{}^{(12)-1}(S')\{\zeta\}^{-1}{}_{\mu3}(S')a^*{}_{3,33}(S')G(S')$$
$$\times\zeta_{3\nu}(S')M^{(12)}{}_{\nu\rho++}(S')M^{(3)}{}_{\rho\lambda'}(S')-\frac{1}{\pi}\int\frac{dS'}{S'-S+i\epsilon}M_{\lambda\mu++}{}^{(12)-1}(S')\{\zeta\}^{-1}{}_{\mu\alpha}(S')(\delta_{\alpha2}+\delta_{\alpha3})\theta(s_3'-4m^2)$$
$$\times\{a^*{}_{3,\alpha1}(S')G(S')\zeta_{1\nu}(S')M^{(12)}{}_{\nu\rho\cdot+}(S')+a^*{}_{3,\alpha2}(S')G(S')\zeta_{2\nu}(S')M^{(12)}{}_{\nu\rho+-}(S')M^{(3)}{}_{\rho\lambda'}(S')\,. \tag{9.7}$$

Before proceeding further, we may note that the Cauchy singularities may easily be removed from the equations for the M's. The equations all have the form

$$M(S)=1+\frac{1}{\pi}\int\frac{dS'}{S'-S\mp i\epsilon}\{\pm h^*(S')\}M(S'), \tag{9.8}$$

where the upper and lower of the $\mp$ and $\pm$ signs are to be taken together, and the two signs refer to nonoverlapping parts of the range of integration. If we multiply (9.8) by the operator

$$1+\frac{1}{\pi}\int\frac{dS'}{S'-S\pm i\epsilon},$$

we find that

$$\{1-2ih^*(S)\}M(S)=1+r(S)+\frac{1}{\pi}\int dS'\frac{r(S')-r(S)}{S'-S}h(S')M(S'), \tag{9.9a}$$

where

$$r(S)=\frac{1}{\pi}\int dS'\frac{h(S')}{S'-S+i\epsilon}. \tag{9.9b}$$

Now the function corresponding to $h(S)$ in Eqs. (9.2), (9.5), and (9.7) satisfies the unitarity equation

$$\{1-2ih^*(S)\}\{1+2ih(S)\}=1$$

so that (9.9a) may be written

$$M(S)=\{1+2ih(S)\}\{1+r(S)+\frac{1}{\pi}\int dS'\frac{r(S')-r(S)}{S'-S}h(S')M(S')\}. \tag{9.9c}$$

From the theorems quoted in the Appendix, it follows that (9.9c) is fully equivalent to (9.8), provided that the amplitudes a have no bound states or Castillejo-Dalitz-Dyson (C.D.D.) poles. We shall discuss the question of bound states or C.D.D. poles below. We shall also deal with other possible singularities of our kernel arising from the δ functions in a or an increasing asymptotic behavior of a in the momentum transfer. Before doing so let us write down the N/D equations for the scattering amplitude.

Construction of the N/D Equations

Equations (9.1), (9.4), and (9.5) may be rewritten

$$A=MA^{(1)}, \tag{9.10a}$$

where

$$M=M^{(1)}M^{(2)}M^{(3)}. \tag{9.10b}$$

We can now eliminate the discontinuities $[A]_{s_i'}$ by defining amplitudes B and b as follows:

$$A_{\lambda\lambda'}=M_{\lambda\mu}B_{\mu\mu'}M^T{}_{\mu'\lambda'}\,, \tag{9.11a}$$

$$\delta_{\alpha0}a^*{}_{i,0\alpha'}=\zeta_{\alpha\lambda}M^*{}_{\lambda\mu}b_{i,\mu\mu'}M^\dagger{}_{\mu'\lambda'}\zeta_{\alpha'\lambda'}\,. \tag{9.11b}$$

From (8.2), we obtain the following equation for $[B]$

$$[B]=\theta(S-9m^2)\{B^*HB+\sum_{i=1}^{3} b_i^*HB+\sum_{i=1}^{3} B^*Hb_i+R_B\}\,, \quad (9.12a)$$

where

$$H_{\lambda\lambda'}=M^\dagger{}_{\lambda\mu}\zeta_{0\mu}G\zeta_{0\mu'}M_{\mu'\lambda'}\,, \quad (9.12b)$$

$$R_B=\sum_{i\neq j} b_i^*Hb_j\,. \quad (9.12c)$$

The subscripts λ have been suppressed in (9.12a) and (c) and will henceforth be suppressed. Matrix multiplication in these subscripts, as well as in the other variables, is always implied.

Equations (9.12) have the same form as (5.1), except that the quantity H is now a matrix. They can be solved in exactly the same way, and we can repeat line for line the reasoning leading to (5.1) and (5.8)–(5.9). Thus, if we write

$$B=N_CD^{-1}\,, \quad (9.13)$$

the functions N_C and D will satisfy the following equations:

$$\begin{pmatrix}N_C(S)\\ D(S)\end{pmatrix}=\begin{Bmatrix}1+2iH(S)\sum b_i(S) & 2iR_B(S)\\ -2iH(S) & 1-2i\sum b_i^*(S)H(S)\end{Bmatrix}$$
$$\times\Bigg\{\begin{pmatrix}F_{RB}(S)+J_{NB}(S)\\ 1-\sum e_{iB}(S)-J_{DB}(S)\end{pmatrix}-\int dS'(K_B(S,S'))\begin{pmatrix}N_C(S')\\ D(S')\end{pmatrix}$$
$$+\frac{1}{\pi}\int\frac{dS'}{S'-S}\Bigg[\begin{pmatrix}\sum \tilde{e}_{iB}(S')-\sum \tilde{e}_{iB}(S) & F_{RB}(S')-F_{RB}(S)\\ -C_B(S')+C_B(S) & -\sum e_{iB}(S')+\sum e_{iB}(S)\end{pmatrix}$$
$$\times\begin{pmatrix}\sum b_i^*(S')H(S') & R_B(S')\\ -H(S') & -H(S')\sum b_i(S')\end{pmatrix}\Bigg]\begin{pmatrix}N_C(S')\\ D(S')\end{pmatrix}\Bigg\}\,, \quad (9.14a)$$

where

$$J_{NB}(S)=F_{LB}(S)+\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''}{S''-S+i\epsilon}\sum b_i^*(S'')H(S'')F_{LB}(S'')\,,$$

$$J_{DB}(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''}{S''-S+i\epsilon}H(S'')F_{LB}(S'')\,,$$

$$e_{iB}(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''}{S''-S+i\epsilon}H(S'')b_i(S'')\,,\quad \tilde{e}_{iB}(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''}{S''-S+i\epsilon}b_i^*(S'')H(S'')\,,$$

$$F_{RB}(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''R_B(S'')}{S''-S+i\epsilon}\,,\qquad C_B(S)=\frac{1}{\pi}\int_{9m^2}^{\infty}\frac{dS''H(S'')}{S''-S+i\epsilon}\,,$$

$$F_{LB,\lambda\lambda'}(S)=\frac{1}{\pi}\int_L\frac{dS''}{S''-S+i\epsilon}M^{-1}{}_{\lambda\mu}(S'')\{\zeta\}^{-1}{}_{\mu\alpha}(S'')L_{\alpha\alpha'}(S'')\{\zeta\}^{-1}{}_{\mu'\alpha'}(S'')M^{T-1}{}_{\mu'\lambda'}(S'')\,,$$

$$K_B(S,S')=\frac{1}{\pi}\frac{F_{LB}(S')-F_{LB}(S)}{S'-S}H(S')\begin{Bmatrix}1 & \sum b_i(S')\\ 0 & 0\end{Bmatrix}+\frac{1}{\pi^2}\int_{9m^2}^{\infty}\frac{dS''}{(S''-S+i\epsilon)(S''-S')}$$
$$\times\begin{Bmatrix}\sum b_i(S'')H(S'')\{F_{LB}(S')-F_{LB}(S'')\}H(S') & \sum b_i(S'')H(S'')\{F_{LB}(S')-F_{LB}(S'')\}H(S')\sum b_i(S')\\ H(S'')\{F_{LB}(S')-F_{LB}(S'')\}H(S') & H(S'')\{F_{LB}(S')-F_{LB}(S'')\}H(S')\sum b_i(S')\end{Bmatrix}\,. \quad (9.14b)$$

Investigation of Possible Singularities in the Equations

Equations (9.9c) and (9.14) contain no Cauchy singularities. Before we can assert that they are Fredholm equations, we have to investigate three other possible singularities: (i) possible singularities from delta functions in the kernel, (ii) possible singularities at high S, and (iii) possible singularities due to the increase of the two-body amplitudes a_i at high-momentum transfer.

The kernels in our integral equations involve the two-body scattering amplitudes a_i, which contain factors $\delta(s_i-s_i')$. However, the delta functions can be removed by iteration of the kernel. In fact, we have already iterated the kernel to remove the Cauchy singularities, and Eqs. (9.9) and (9.14) contain products of at least two factors a_i in the kernel. The reason that such products do not contain delta functions with the present kinematics, whereas they did with the nonrelativistic kinematics, is that the argument of the delta function is not now simply the difference between two x's. The functions a_i all occur in (9.9) and (9.14) in the form $\int dS'' a_1(S'')a_1(S')\chi(S'',S')$, where the function χ does not contain delta functions. When written out, this integral is

$$\int dS''dx_1''dx_2''\,\tilde{a}_1(S'',x_1,x_2,x_1'',x_2'')\tilde{a}_1(S',x_1'',x_2'',x_1,x_2) \times\chi(S'',S')\delta\{s_1(S'',x_1,x_2)-s_1(S'',x_1'',x_2'')\} \times\delta\{s_1(S',x_1'',x_2'')-s_1(S',x_1,x_2)\}\,. \quad (9.15)$$

The functions $s_1(S'',x_1,x_2)$ and $s_1(S',x_1,x_2)$ are different functions of x_1 and x_2 (if $S''\neq S'$). The restrictions imposed by the delta functions will thus fix the values of the variables of integration x_1'' and x_2'', but they will not give delta functions in the x's or x''s. The integration over x_1'' and x_2'' thus becomes a simple product of the two a's, together with the Jacobian

$$\begin{vmatrix} \dfrac{\partial s_1(S'',x_1'',x_2'')}{\partial x_1''} & \dfrac{\partial s_1(S'',x_1'',x_2'')}{\partial x_2''} \\ \dfrac{\partial s_1(S',x_1'',x_2'')}{\partial x_1''} & \dfrac{\partial s_1(S',x_1'',x_2'')}{\partial x_2''} \end{vmatrix}^{-1} \quad (9.16)$$

which may be calculated from (6.13). It is of course necessary to check that the iterated kernel is nonsingular, for instance at the points where the Jacobian vanishes, but this can be done. In the nonrelativistic case, the Jacobian is identically zero, and the delta function cannot be removed by simple iteration.

Products of the form a_ia_j $(i\neq j)$ can be treated in the same way as the product a_ia_j.

The investigation of singularities in the integral equation at high energy proceeds analogously to the corresponding investigation of the two-body problem. The result in that case is that the S-wave equation is not singular provided that the input left-hand cuts decrease at infinity. The same is true of the P-wave equation with a threshold behavior of $(s-4m^2)$, provided one writes the equation with a subtraction in D at some finite value s_0 of s. The phase shift will only tend to zero logarithmically at high energy. The D-wave equation is singular if one attempts to enforce a threshold behavior of $(s-4m^2)^2$. If we do not attempt to enforce the threshold behavior, all partial waves are equivalent. With the three-body problem, the phase-space factor increases proportionally to S instead of remaining constant, so that the equations with a threshold behavior of $(S-9m^2)^J$ are equivalent to the two-body equations with a threshold behavior of $(s-4m^2)^{J+1}$. If we allowed a threshold behavior of $(S-9m^2)^{-1}$, and constructed equations for the function $(S-9m^2)A(S)$, the equations would be true as written and the phase shifts would tend to zero like S^{-1} at high energy. If we did not allow a singularity at threshold, the equations would have to be modified by making a subtraction in the D function, and the phase shifts would tend to zero logarithmically at high energies. One cannot enforce a threshold behavior of $(S-9m^2)^n$ where $n\geqq 1$.

In general, therefore, one would not be able to enforce the threshold conditions, though they would be satisfied automatically if the correct left-hand discontinuity were inserted. If one knew that the problem had a solution where the phase shift decreased more quickly than $1/\ln S$, one could allow a threshold behavior of $(S-9m^2)^{-1}$ and write the dispersion relations for N and D without subtractions. It would turn out that the singularity at threshold was in fact absent in the solution. We remarked in Sec. 2 that there are further threshold conditions relating to the dependence of the amplitude on the x's, which can only be satisfied if the left-hand cut has certain properties.

We should point out that the asymptotic increase of the functions ζ does not give rise to singularities at high S. The ζ's occur in the discontinuity formulas in the combinations $\zeta_{0\lambda}\zeta_{0\lambda'}$ and $\zeta_{\alpha\lambda}\{\zeta\}^{-1}{}_{\lambda'\alpha'}$ $(\alpha=1,2,3)$ and, from the asymptotic formulas (7.11), we observe that these products do not increase asymptotically.

If the two-particle amplitudes a_i increase as the momentum transfer becomes infinite, the equations have to be modified, since the region over which we integrate the unitarity equation for $[A]_{s_i}(s_i>4m^2, S<0)$ includes the point $\cos\theta_i=\infty$. We have seen that this point corresponds to the bottom of Fig. 4(b), (c), and (d) (when $i=1$), where $x_1=0$, $x_2=x_3=\frac{1}{2}$. The terms a_i in the kernel would become infinite at this point, so that the integral equations for M would become singular.

The difficulty can be overcome by inserting the Regge asymptotic behavior explicitly. We therefore substitute into the formula (3.6) for a

$$t(s_1,x)=t_0(s_1,x)+\frac{\beta(s_i)}{\sin\pi\alpha(s_i)}P_{\alpha(s_i)}(x)\,. \quad (9.17)$$

We now divide the function $M^{(i)}$ $(i=1,2,3)$ into two parts:

$$M^{(i)}(S,x_1,x_2,x_1',x_2') =M_0(S,x_1,x_2,x_1',x_2')+\{\sin\pi\alpha(s_1)\}^{-1} \times\phi\left\{S,\frac{(x_2-x_3)^2}{x_1},x_1',x_2'\right\}(\cos\theta_1)^{\alpha(s_1)}, \quad (9.18)$$

where the function M_0 does not increase asymptotically with $\cos\theta_1$. As usual the variables s_1 and $\cos\theta_1$ are to be regarded as functions of S, x_1, and x_2. If $\alpha>0$, it will be necessary to add further terms to (9.18) with factors

$(\cos\theta_1)^{\alpha-1}$, etc. The reason why we have chosen the variable $(x_2-x_3)^2/x_1$ is that s_1 is a function of this variable near the point of divergence $x_1=0$, $x_2=x_3$. From the analyticity of M it is easy to see that the function ϕ is analytic in S.

The unitarity equation for M now gives us linear equations for the discontinuities of both M_0 and ϕ, so that we obtain coupled equations similar to (9.2) for these two functions. We shall not enter into the details of these equations. If there are several Regge poles in one two-body subchannel, or if more than one subchannel contains Regge poles, we shall have to add further terms to (9.18). After calculating the functions M, but before solving the N/D equations, we may recombine the terms of (9.18) into one, since the N/D equations do not involve the region for which $\cos\theta_1$ is large.

A word may be said about the case where one or more of the two-particle subchannels has bound states. In solving the N/D equations (9.14) we would take these states into account explicitly, just as with the nonrelativistic kinematics. On the other hand, in the Eqs. (9.2), (9.5), and (9.7) for the functions M, the bound states would automatically be taken into account by using the representation (9.18). As in the usual Regge formula, they arise from the factor $\sin\pi\alpha(s_i)$ in the denominator of (9.18). Furthermore, this method of treating the bound states enables us to overcome the problem of the unique solubility of (9.8). Our treatment was based on the theorem quoted in the Appendix, to the effect that the usual Fredholm results are true as long as the kernel of (9.9) does not develop a pole when the function h is increased continuously from zero to its final value. If the two-body channel does have a bound state, this theorem cannot be applied directly to (9.8), as the function h becomes discontinuous at threshold when the state just becomes bound. By using the representation (9.18), we can eliminate this difficulty, since the kernel of the integral equation will not then develop a discontinuity when the state becomes bound. There will be a singularity in the factor $\{\sin\pi\alpha(s_i)\}^{-1}$ of (9.18), but this factor does not enter explicitly into the integral equation.

If the two-body scattering amplitudes have C.D.D. poles the methods developed in this paragraph require modification, since we assumed that there were no C.D.D. poles in proving the unique solubility of the Eqs. (9.2), (9.5) and (9.7). We shall not deal with C.D.D. poles here. It turns out that the equations for M are then soluble, but not uniquely. Nevertheless, if we demand that the functions MN_C and $M^{T-1}D$ (which are the numerator and denominator functions of the amplitude A) have no poles, the final result is unique.

General Remarks

As in the nonrelativistic case, we can show that the functions $M(N_C+\sum a_iD)$ and $M^{T-1}D$ satisfy the usual discontinuity equations associated with the numerator and denominator functions, so that they may be regarded as the numerator and denominator functions of the complete amplitude. The function $D^{-1}M^T$ will satisfy the form factor equations, and the form factor will be equal to

$$\int dx_1'dx_2' \sum_{M',\alpha'} (D^{-1}M^T)_{\alpha\alpha'}(x_1,x_2,x_1',x_2',M,M') \times C_{\alpha'}(x_1',x_2',M').$$

The situation is again the same as in the nonrelativistic case, except that there is now an additional summation over the kinematic index α.

The remarks which we made in the nonrelativistic case with regard to deformation of the contour of the S integration in our main term (9.14) apply here too. The presence of the terms a_i^*GA and AGa_i in the unitarity condition imply that the function A has a pole on the unphysical sheet when one of its initial or final subenergies is resonant. Since the numerator function is holomorphic, the denominator function must have a zero eigenvalue at this point. If we deform the contour past this point, we must include the resonance explicitly in our calculation as a bound state, and we can then prove the existence of cuts on the unphysical sheet.

Our final conclusion to the rather complicated treatment of the fully relativistic case is thus that the kinematics complicates all the equations but that the essential results are unchanged. All the familiar results of the N/D approach to the two-body problem can be taken over into the three-body problem.

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APPENDIX. SOME RESULTS OF THE THEORY OF CAUCHY INTEGRAL EQUATIONS

We are interested in the equation

$$f_\mu(x)=1+\frac{1}{\pi}\int dx'\frac{a_{\mu\nu}(x')f_\nu(x')}{x'-x-i\epsilon}, \qquad \text{(A1)}$$

where the indices μ and ν may be discrete or continuous, and will hereafter be suppressed. Let us write it in the form

$$f(x)-1-\frac{1}{\pi}\int dx'\frac{a(x')f(x')}{x'-x-i\epsilon}=0. \qquad \text{(A2)}$$

We now multiply it by the operator

$$\delta(x-x')+\frac{1}{\pi}\frac{a(x')}{x'-x+i\epsilon}. \qquad \text{(A3)}$$

In other words, we consider the equation

$$\int dx''\left\{\delta(x-x'')+\frac{1}{\pi}\frac{a(x'')}{x''-x+i\epsilon}\right\}\times\left\{f(x'')-1-\frac{1}{\pi}\int dx'\frac{a(x')f(x')}{x'-x''-i\epsilon}\right\}=0. \quad \text{(A4)}$$

On multiplying out and using the equation

$$\frac{1}{x'-x-i\epsilon}-\frac{1}{x'-x+i\epsilon}=2\pi i\delta(x'-x),$$

we find that

$$\{1-2ia(x)\}f(x)=1+b(x)-\frac{1}{\pi}\int dx'\frac{b(x')-b(x)}{x'-x}a(x')f(x'), \quad \text{(A5a)}$$

where

$$b(x)=\frac{1}{\pi}\int dx''\frac{a(x'')}{x''-x+i\epsilon}. \quad \text{(A5b)}$$

Thus

$$f(x)=\{1-2ia(x)\}^{-1}\times\left\{1+b(x)+\frac{1}{\pi}\int dx'\frac{b(x')-b(x)}{x'-x}a(x')f(x')\right\}. \quad \text{(A6)}$$

As long as the factor $1-2ia(x)$ has no zero, the equation (A6) will be nonsingular.

It is important that the sign of the $i\epsilon$ in the denominator of (A3) be opposite to that in the denominator of (A2). If we had used an $i\epsilon$ of the same sign, the kernel in (A5a) would have been

$$\frac{b_+(x')-b_-(x)}{x'-x}a(x'),$$

where the function b_- is defined by (A5b), and the function b_+ by (A5b) with a $-i\epsilon$ in the denominator. This kernel would still contain a Cauchy singularity and would not be a Fredholm kernel.

Equation (A4), and therefore (A6) is a necessary consequence of (A2). However, to go in the reverse direction, one must know that the homogeneous integral equation corresponding to the operator (A3) has no solution. It turns out that the case where this homogeneous equation does have a solution cannot be regarded as exceptional. One can find functions $a(x)$ such that the homogeneous equation corresponding to (A3) is soluble and continues to be soluble for all sufficiently small changes of a. Another complication which may occur is that the homogeneous equation corresponding to (A1) may be soluble, and that the inhomogeneous equation may also be soluble even if an extra arbitrary inhomogeneous term is added to the right-hand side. The solution of the inhomogeneous equation would not be unique. Again this is not an exceptional situation, and it may be contrasted with the Fredholm theory, where the inhomogeneous term must be restricted by as many linear conditions as there are solutions of the homogeneous equation.

In the one-channel case where a has the form $e^{i\delta}\sin\delta$ (δ real), these two complications correspond to the conditions $\delta(\infty)-\delta(0)=-n\pi$ and $\delta(\infty)-\delta(0)=n\pi$, respectively, n being a positive integer.

We shall not go into the criteria which can be used to distinguish between the various cases, but shall quote a theorem which can be used in a large class of problems to show that the complications do not occur: *If the function $a(x)$ can be varied continuously from zero to its final value in such a way that the factor $\{1-2ia(x)\}^{-1}$ in (A6) never has a pole within the range of integration and is zero at the ends of the range, then the homogeneous equations corresponding to (A1) and (A3) are not soluble. In such cases equation (A6) is uniquely soluble and its solution is the unique solution of (A1).* This theorem thus reduces (A1) to the Fredholm equation (A6) when it is applicable.

We are often interested in equations where the function $a(x)$ satisfies a unitarity condition

$$\{1-2ia(x)\}^{-1}=1+2ia^*(x). \quad \text{(A7)}$$

If we increase all eigenphase shifts of a from zero to their final values, Eq. (A7) will hold throughout the process, and the function $\{1-2ia(x)\}^{-1}$ will not have a pole. The conditions of the theorem are therefore applicable, and Eq. (A1) is equivalent to (A6). It will only be possible to increase the phase shifts continuously from zero to their final values if $\delta(\infty)-\delta(0)=0$. One cannot therefore use this theorem if $\delta(\infty)-\delta(0)=n\pi$.

Equation (A1) can be generalized by adding a term $\int dx'K(x,x')f(x')$ to the right-hand side, where K is a Fredholm kernel. The equation can also be reduced to the Fredholm form by applying the operator (A3). If the conditions of the above theorem hold, we can again conclude that the original equation is fully equivalent to the reduced equation and that all the results of Fredholm theory apply to the original equation. It may now happen that the homogeneous equation instead of the inhomogeneous equation is soluble, just as in the case of a normal Fredholm equation [with the unmodified equation (A1) this could not happen if the conditions of the theorem held]. We can also quote another theorem that is useful: *If the determinant of the quantity $1-2ia(x)$ is equal to one for all values of x, the original Cauchy equation is fully equivalent to the reduced Fredholm equation, and the results of Fredholm theory apply to the original equation.*