Pion-Pion Scattering. III*

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(Received 4 June 1965)

Continuing a program discussed in two earlier papers, we have calculated the left-hand partial-wave discontinuities in fourth-order perturbation theory. The s-wave and p-wave dispersion relations are solved by standard N/D techniques. It is found that the relative contributions arising from the double-spectral terms and the inelastic single-spectral functions are negligible. As a perturbation theory the procedure appears to be sensible for coupling constants $|\lambda|$ less than about 0.1. High-energy p-wave resonances can be made to occur for $|\lambda|$ of about 0.2, but with very large widths. It is suggested that the physical pion-pion system may not be describable in terms of a one-parameter theory, and this notion is given experimental meaning.

I. INTRODUCTION

THIS is the third and final paper of a series[1] describing an attempt to understand pion-pion scattering from an "almost Lagrangian" viewpoint. Under the circumstances it seems appropriate to begin with a brief review of the ideas underlying our work.

We take as our starting point the notion that the pion-pion interaction may not be very "strong." More precisely, if the value of the elastic π-π scattering amplitude at the "symmetry point" corresponding to isospin $\begin{pmatrix}0\\1\\2\end{pmatrix}$ is

$$A^I(s_0=t_0=u_0=\tfrac{4}{3})=\begin{pmatrix}5\\0\\2\end{pmatrix}\lambda\,, \qquad (\text{I.1})$$

then for λ smaller than about 0.2 it should be possible to utilize an expansion in powers of λ in order to obtain the scattering amplitude.[2] Since part of our interest will be the calculation of resonant amplitudes it is clear from the outset that conventional Feynman-Dyson perturbation theory is not appropriate. Thus, it is necessary to find the proper quantity to be expressed as a power series in λ.

In order to see how to proceed consider the energy-variable dispersion relations for the partial-wave scattering amplitudes.[3] As is well known, the dynamics (ignoring, for the moment, inelastic processes) is contained in the function that gives the discontinuity of a partial-wave amplitude across the real, negative, energy axis. If the exact discontinuity function is known then, of course, the (exact) partial-wave amplitude is obtainable as a solution of the dispersion relations. A more interesting observation, however, is that if the discontinuity function is well approximated over a low (negative)-energy region including the threshold, and if the behavior of the function for very large argument is not too singular, then the low-energy scattering amplitude will also be well approximated. From this point of view, then, we are instructed to seek an approximation technique that permits us to "work our way out" in energy for each partial wave.

As a matter of fact one knows that there is just such a perturbation theory available for potential scattering.[4] It is obtained by taking the negative energy discontinuity of the first n Born approximations as the nth approximation to the "left-hand" discontinuity. In the case of potential scattering this coupling constant expansion for the left-hand discontinuity gives the exact discontinuity over an increasing range of the left-hand cut.

From the potential-scattering analogy we are led to consider the applicability of the same technique in the relativistic problem. In this case the right-hand side of Eq. (1) is taken as the first Born approximation to the scattering amplitude. Repeated application of unitarity, analyticity arguments, and crossing symmetry then generates an expansion of the scattering amplitude from which the partial-wave left-hand discontinuities may be extracted. For low orders, at least, the iteration procedure is known to reproduce exactly by the Feynman-Dyson perturbation theory for a coupling proportional to $\lambda(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2$, thus providing a connection with Lagrangian field theory.[5]

The potential-scattering analogy is, of course, not an exact one. Primarily as a consequence of crossing sym-

* Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] K. Smith and J. L. Uretsky, Phys. Rev. **131**, 761 (1963); A. Saperstein and J. L. Uretsky, *ibid.* **133**, B1340 (1964).

[2] Estimates of λ have been made by Bipin R. Desai, Phys. Rev. Letters **6**, 497 (1961), and R. F. Sawyer and K. C. Wali, Phys. Rev. **119**, 1429 (1960), among many others. All such estimates known to us agree that $|\lambda|$ lies in the range 0.1–0.2, although there is no universal agreement as to the sign.

[3] For an introduction to the Mandelstam relation with applications and bibliography see G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin and Company, Inc., New York, 1961).

[4] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962).

[5] The idea of generating the discontinuity function from perturbation theory was first suggested by Stanley Mandelstam, Phys. Rev. **112**, 1344 (1958).

metry there is no order of the perturbation expansion that gives the exact discontinuity function anywhere on the left-hand cut. Nevertheless, inspection of the tractable Feynman diagrams suggests that the procedure we are discussing gives fairly rapidly convergent expressions for the discontinuity at fixed low (negative) energies for $|\lambda|$ of the order of 0.1 or less. Consequently, we have some confidence in our ability to calculate pion-pion scattering amplitudes provided that the interaction is not too strong and that it is governed by the $\lambda(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2$ coupling.

In the preceding two papers[1] of this series we have reported upon calculations of the s- and p-wave amplitudes calculated from the second- and third-order (in λ) discontinuity functions. In performing those calculations it was presumed that the low-energy scattering (for small enough λ) was being predicted correctly. On the other hand it was recognized that the dominant observable feature of pion-pion scattering, namely, the ρ resonance, could hardly be considered a low-energy phenomenon, and an important test of the theory must be its usefulness for predicting the ρ. Consequently, it seemed imperative that we continue to the next order calculation in order to see whether the ρ resonance could be obtained for a small enough λ so that the low-energy predictions remained unchanged.

It is clear, of course, that a negative result from the fourth-order calculation would not constitute a proof that the ρ is not a consequence of a $\lambda(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2$ interaction among pions. One would always be left with the tantalizing possibility that one higher order in the perturbation expansion would be required in order to get "physics." From a practical viewpoint such a possibility is an empty one because the fourth-order calculation exhausts our present capabilities. In this sense the calculation reported here provides the crucial test of the present approach to pion-pion scattering. In a broader sense, however, one can look upon our work as outlining a methodology for treating medium-strong couplings so that the applicability to the pion-pion system need not be of primary interest.

The calculation of the fourth-order discontinuity functions from unitarity, crossing, and analyticity does have features that are intrinsically of interest. This is because the fourth order is the lowest order that admits of inelastic intermediate states, and one wants to discuss these as an iteration of the fundamental coupling in Eq. (1). Further, this is the lowest order that contributes double-spectral functions to a Mandelstam representation of the scattering amplitude, and it was thought that such terms would play a crucial role in the scattering at moderate and high energies. Section II, following, is devoted to a discussion of the derivation of the fourth-order amplitudes. The relevant formulas are compiled in the Appendix. Section III contains a discussion of the numerical results. Section IV contains the conclusions and some final remarks. Experimentalists reading this paper are advised to proceed directly to Sec. IV.

II. COMPUTATIONAL DETAILS

The elastic scattering amplitude, to third order in λ, may be written, for isospin $\begin{pmatrix}0\\1\\2\end{pmatrix}$,

$$A^I(\nu,z)=\begin{pmatrix}5\\0\\2\end{pmatrix}\lambda+\lambda^2\left\{\begin{pmatrix}25\\0\\4\end{pmatrix}F(s)+\begin{pmatrix}15\\5\\9\end{pmatrix}[F(t)+(-1)^IF(u)]\right\}$$
$$+\lambda^3\left\{\begin{pmatrix}125\\0\\8\end{pmatrix}H(s)+\begin{pmatrix}55\\35\\43\end{pmatrix}[H(t)+(-1)^IH(u)]+\begin{pmatrix}75\\0\\18\end{pmatrix}K(s)+\begin{pmatrix}55\\10\\28\end{pmatrix}[K(t)+(-1)^IK(u)]\right\},\quad\text{(II.1)}$$

where ν and z are, respectively, the square of the barycentric momentum of one pion and the barycentric scattering angle. The variables s, t, u are the usual Mandelstam ones defined by

$$s=4(\nu+1)\,,\quad\text{(II.2a)}$$

$$t=-2\nu(1-z)\,,\quad\text{(II.2b)}$$

$$u=-2\nu(1-z)\,,\quad\text{(II.2c)}$$

in units where h, c, and the pion mass are all unity. The functions F, H, and K are defined in the Appendix. The amplitude $A^I(\nu,z)$ is normalized so that the elastic unitarity relation reads (for $\nu>0$)

$$\mathrm{Im}A^I(\nu,z)=\frac{1}{4\pi}\left(\frac{\nu}{\nu+1}\right)^{1/2}\int d\Omega' A^I(\nu,z')A^{I*}(\nu,z'')\,,\quad\text{(II.3)}$$

where

$$z''=zz'+[(1-z^2)(1-z'^2)]^{1/2}\cos\phi'\,.\quad\text{(II.4)}$$

Finally, A^I is obtained by putting Eq. (II.3) into a dispersion relation, subject to the condition (I.1), and then applying the requirements of crossing symmetry.

When the expression on the right of Eq. (II.1) is inserted into the integral of Eq. (II.3) and the terms proportional to λ^4 isolated it will be found that these are of two kinds. Some, after the integration on angle, will be functions only of the energy variable s (or equivalently, ν). These will make contributions similar to those already encountered in the lower-order calculations and will be referred to as the elastic, single-variable contributions (see Appendix).

The other kind of contribution will come from cross terms of the form $F(t')F(t'')$ and will still have a t- (or equivalently, z-) dependence after the angular integration is carried out. Taking the imaginary part with respect to the t variable of the result one then obtains the lowest order elastic contribution to the double-spectral function which is

$$\rho_e{}^I(s,t) = -\frac{4\lambda^4}{\pi}\begin{bmatrix}225\\25\\81\end{bmatrix}[s(s-4)]^{-1/2}\int_4^\infty dt_1\left(1-\frac{4}{t_1}\right)^{1/2} \times\int_4^\infty dt_2\left(1-\frac{4}{t_2}\right)^{1/2}K(s;tt_1t_2)\,, \quad \text{(II.5)}$$

where

$$K^{-2}(s;tt_1t_2) = t^2+t_1^2+t_2^2-2(tt_1+tt_2+t_1t_2) -4tt_1t_2/(s-4) \quad \text{(II.6)}$$

is the same function that is discussed in Mandelstam's original work.[5] The integrations in Eq. (II.5) are confined to values of t_1, t_2 for which K^{-2} is positive, thus giving rise to the further restrictions

$$s\geqslant 4\,,\quad t\geqslant 16\,. \quad \text{(II.7)}$$

One sees, thereby, that $\rho_e{}^I(s,t)$ arises from the exchange of two pair of s-wave mesons in the t channel. The double-spectral function is so normalized that the contribution to the amplitude is (neglecting the required subtractions at s_0, t_0)

$$\pi^{-2}\int ds'(s'-s)^{-1}\int dt'(t'-t)^{-1}\rho_e{}^I(s',t')\,. \quad \text{(II.8)}$$

There is now another contribution to the (s,t) spectral function that arises from the exchange of two pairs of s-wave mesons in the s channel. This inelastic contribution is obtained from Eq. (II.5) as a consequence of crossing symmetry and is

$$\rho_i{}^I(s,t) = \sum_{I'=1}^{3} M_{II'}\rho_e{}^{I'}(t,s)\,, \quad \text{(II.9)}$$

where $M_{II'}$ is the usual crossing matrix[4]

$$\{M_{II'}\} = \tfrac{1}{6}\begin{bmatrix}2 & 6 & 10\\ 2 & 3 & -5\\ 2 & -3 & 1\end{bmatrix}. \quad \text{(II.10)}$$

The total (s,t) spectral function may now be denoted by

$$\rho_1{}^I(s,t) = \rho_e{}^I(s,t)+\rho_i{}^I(s,t) \quad \text{(II.11)}$$

and the (s,u) spectral function is readily seen to be

$$\rho_2{}^I(s,u) = (-1)^I\rho_1{}^I(s,u)\,. \quad \text{(II.12)}$$

Finally, one more application of crossing symmetry gives the third, or (t,u), contribution as

$$\rho_3{}^I(t,u) = \sum_{I'} M_{II'}\rho_2{}^{I'}(t,u)\,. \quad \text{(II.13)}$$

At this point all that is lacking are the inelastic single-variable terms and the contribution of the "X" diagram of Fig. 1. These are not obtainable from elastic-unitarity so that we must now take into consideration the lowest-order (λ^2) amplitudes for four-pion production. In order to obtain these as an iteration of the fundamental coupling of Eq. (I.1) it is necessary to consider first the three-meson-to-three-meson scattering which will then give the two-meson-to-four-meson amplitude by crossing one of the lines.

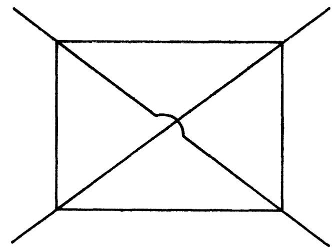

FIG. 1. The fourth-order X diagram which was omitted from the calculation. It is obvious by inspection that this diagram contributes symmetrically in t and u, and, in consequence, enters only into even angular momentum amplitudes.

The lowest-order three-particle scattering amplitude corresponds to the disconnected diagram where two of the particles scatter[6] and one goes "straight through," and where the scattering is given by the right-hand side of Eq. (I.1). Iteration of this amplitude by three-particle unitarity then gives a second-order "straight through" contribution plus nine δ-function terms that may be identified as the imaginary parts of Feynman propagators. One more such term is then added in order to maintain crossing symmetry. One of the incoming legs is then "crossed over" in order to convert the three particle scattering amplitude into a production amplitude. The result is simple in form but tedious to write and will not be reproduced here. It is, of course, also readily obtainable by application of the Feynman rules to a $(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2$ interaction.

When the contribution of the production amplitudes is added to the unitarity relation (II.3) there again result two classes of contributions. As expected, there are the single-variable inelastic terms which are also tabulated in the Appendix. In addition there are the double-variable terms which include the contributions already obtained from crossing arguments, Eq. (II.9), and the X diagram. This last has not been included in our work since it can only contribute to the high-energy, even angular-momentum states and is algebraically awkward to deal with.

III. NUMERICAL COMPUTATIONS AND RESULTS

The major computational problem in the fourth-order calculation was to evaluate the contributions of the double spectral-functions which are given in the Appendix as three-dimensional integrals. It seemed reasonable at this stage to adopt the most direct approach and perform the three dimensional integrals numerically on Argonne's CDC 3600 computer. The

[6] David I. Olive, Phys. Rev. **135**, B745 (1964).

numerics was slightly complicated by the occurrence of square-root singularities in some of the integrands; however, it was not difficult to obtain results that were accurate to within a few percent. The time required for the total double-spectral contribution to the three partial-wave discontinuity functions at a fixed energy was about one minute.

The results were surprising and disappointing. It turned out that the total double-spectral contribution to the discontinuity functions was never more than a few percent of the contributions from the fourth-order, elastic, single-variable terms. Further, the single-variable inelastic terms contributed only to the very distant left-hand cut and were also estimated to be a small correction. Thus, for our purposes it was adequate to use only the elastic, single-variable discontinuity functions in calculating the partial-wave amplitudes.

It had originally been our idea to introduce the perturbation theory inelasticities into the right-hand singularities of the partial-wave amplitudes. However, as will be seen, the results obtained using just elastic unitarity suggested that such an added complication would probably not be worth the trouble.

The calculations that were finally carried out were performed on the CDC 3600 using the basic program constructed by K. Smith.[1] To this had been added subprograms for calculating the third- and fourth-order partial-wave discontinuities. As before, the partial-wave dispersion relations were solved by means of the N/D technique.[3]

The main features of our results are depicted in Figs. 2 through 5. Fig. 2 compares the three scattering lengths as calculated in second, third, and fourth orders, thus

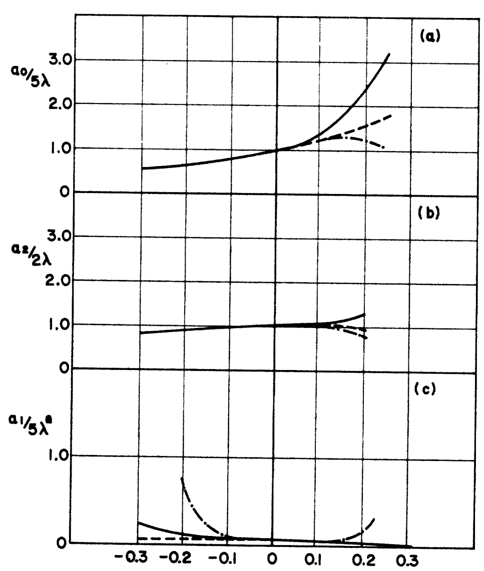

FIG. 2. The scattering lengths a_l in second (dashes), third (solid), and fourth (dash-dot) order.

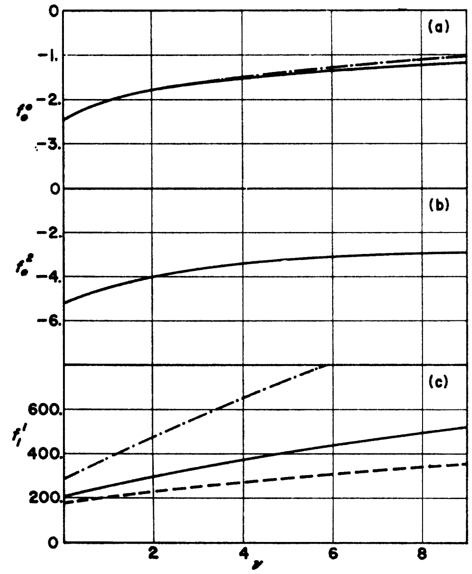

FIG. 3. The real part of the inverse amplitude, $f_l^I=[\nu^l/(\nu+1)]^{1/2}\times\cot\delta_l^I$ in second (dash-dot), third (solid), and fourth (dashes) order. Where fewer than three curves are shown the results are indistinguishable. The calculations are for $\lambda=-0.1$.

indicating the limits of applicability of our perturbation approach. Another measure of the convergence of the coupling constant expansion (for the discontinuity functions) is given by the relative magnitude of the fourth-order contribution near the threshold. For λ of 0.1 it turns out to be about 30% which suggests that for this magnitude of coupling the convergence will already be slow. It also turns out that the discontinuity functions for very large argument become more and more singular (higher powers of $\ln|\nu|$) in higher orders, but such singularities are weak enough to have little effect upon the low-energy scattering.

In Fig. 3 we show the energy dependence of the three solutions, each in second, third, and fourth order for a coupling constant $-\lambda$ equal to 0.1. These graphs show explicitly how the higher orders of approximation have their important influence upon the higher-energy behavior of the scattering amplitude.

In our earlier papers we pointed out that the notion of crossing-symmetry is not well-defined when one deals with only a finite number of partial-wave amplitudes. One may, however, speak of approximate crossing symmetry over a region of the left-hand cut of the partial-wave amplitudes. Figure 4 shows the extent to which our calculation satisfies such approximate crossing symmetry, again for λ of 0.1. We consider that the agreement between the two discontinuity functions is reasonable in view of the fact that the partial-wave sum must diverge anyhow at $\nu=-9$.

Finally, in Fig. 5 we show the effect upon the p-wave amplitude of varying λ. It is at this point that we are really hunting for the ρ resonance which we would take seriously if it were obtainable at the correct energy

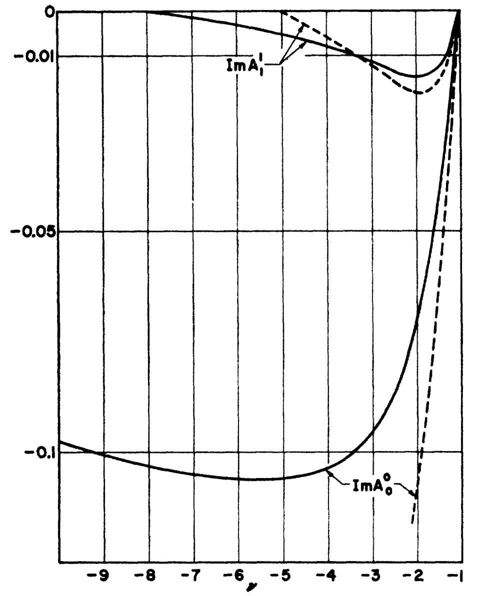

FIG. 4. The left-hand spectral functions as calculated from fourth-order perturbation theory (broken) and, *a posteriori*, from crossing (solid). The calculations are for $\lambda=0.1$.

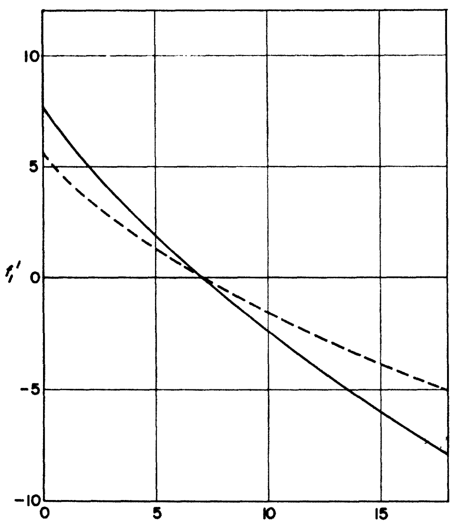

FIG. 5. The real part of the inverse p-wave amplitude for λ chosen to produce a resonance at the position of the ρ. The solid curve is for $\lambda=+0.213$, the dashed curve for $\lambda=-0.2005$. See Fig. 3 caption for definition of f_l^I.

($\nu\approx 6.25$), with a reasonable width ($\Gamma\approx 1$) and for a reasonably small value of $\lambda(|\lambda|\approx 0.1)$. It may be recalled from our previous work that in the second and third order it was just not possible to make the p wave resonate at such a high energy for any value of λ. In the present calculation the situation is quite different and very-high-energy resonances may be obtained as a result of the more singular high-energy behavior of the discontinuity functions.

Nevertheless, the fourth-order calculation does not appear to be capable of providing a ρ resonance. First, in order to obtain a p-wave resonance near the correct energy it is necessary to have λ as large as about 0.2 (resonances occur for both signs of λ), and this is too large for our perturbation approach to make sense. Secondly, the widths come out to be more than five times too large so that it is very hard to believe that these resonances bear any relationship to the ρ.

IV. CONCLUSIONS AND COMMENTS

The first question to be faced is whether the techniques described here constitute a reasonable approach to the physics of particles whose interactions are of moderate strength. With some qualifications it would seem that such is indeed the case. That is to say, it appears that a perturbation expansion of the "left-hand cut" discontinuity functions permits one to explore increasing areas of the energy angular-momentum plane in a convergent manner. In doing so one is able to impose unitarity, analyticity, and crossing symmetry to the extent that the latter notion is meaningful.

The major qualification has to do with the convergence of our procedure which obviously cannot be studied by calculating a finite number of contributions. All we really have to go on, in fact, is the weak analogy with potential theory and the trends that are indicated by the calculations that have been performed. It is clear, from this point of view, that there are some difficult mathematical questions to be answered before we can lay claim to a rigorous theory.

It has already been seen that (up to fourth order, at any rate) the ρ resonance does not seem to be a consequence of our model. It now becomes relevant to ask where the model has deviated from "physics" and whether it is capable of making any predictions whatsoever concerning physical pion-pion scattering. As will be seen, it will be necessary to call upon the experimental physicist for additional information in order to answer this question.

First consider one of the basic assumptions of the calculation, namely, that the effective π-π coupling constant (λ) is a small number. If this assumption is true (within the context of the computational scheme) then there is a definite prediction made concerning the ratio of the s-wave scattering lengths. That is, the ratio must be positive and approximately

$$a_0/a_2\approx\tfrac{5}{2}. \qquad \text{(IV.1)}$$

Deviations from this ratio then provide a measure of the coupling strength. It is worth noting that the only way that a_0/a_2 can be negative in our scheme is for there to be an s-wave bound state in the $I=0$ channel. In this case we could no longer consider λ to be small and our particular perturbation scheme would not be applicable anyhow. If a_0/a_2 is negative and there is no s-wave bound state then the concept of a one-parameter calculation would have to be discarded. If a_0/a_2 is positive but greatly different from the ratio (IV.1) then we would guess that λ is not small and again the perturbation scheme would be inapplicable. In this event, however, one might still be permitted to retain (*in principle*) the notion of a unique $(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2$ coupling. Finally, if Eq. (IV.1) is approximately satisfied, thereby indicating that the coupling is not very strong, then we would suspect very strongly that additional parameters (perhaps in the form of unstable "elementary" particles) are needed in π-π calculations.

In this connection it is certainly not unthinkable that the ρ itself is in fact an "elementary" particle, at least in the context of π-π scattering. This is, in fact, suggested by the convenient way that the ρ fits into the SU_3 symmetry scheme.[7] The conventional expectation is that such an elementary particle should manifest itself as a CDD zero[8] in the p-wave amplitude, and the presence of such a zero may be experimentally observable if there is a substantial s-wave background with which it can interfere.

That the ρ is an elementary particle is, of course, only one possibility. There have also been suggestions of s-wave resonances near 400 MeV[9] and (or) 720 MeV[10] (barycentric total energy), neither of which come out of the present calculation. It may be that at least one of these needs to be "put in by hand;" and if this is the case then the entire complexion of the calculation would be greatly altered.

APPENDIX

1. The three functions appearing in Eq. (II.1) are

$$F(x)=(1/\pi)\{\alpha-(1-4/x)^{1/2}\ln Y(x)\}\,, \tag{A1}$$

$$H(x)=[F(x)]^2\,, \tag{A2}$$

$$K(x)=-(4/3\pi^2)\{[x(x-4)]^{1/2}\times(\pi^2+\ln^2Y)\ln Y-(\tfrac{3}{2})\ln^2Y-3\pi(1+\alpha)F(x)+(\tfrac{3}{16})\alpha(\alpha^2-4\alpha-2\pi^2)\} \tag{A3}$$

where

$$Y(x)=[(1-4/x)^{1/2}+1]/[(1-4/x)^{1/2}-1]\,, \tag{A4}$$

$$\ln^n u\equiv(\ln u)^n\,, \tag{A5}$$

and the logarithm is real for negative values of x. Also,

$$\alpha\equiv\sqrt{2}\tan^{-1}(2\sqrt{2})\,. \tag{A6}$$

2. The single-variable contribution to the s-wave left-hand discontinuity functions is

$$\begin{aligned}\operatorname{Im}A_s(\nu)=&[4\lambda^4/(4-s)\pi^2]\{a_1[\mathrm{Li}_2(-1/Y)+(\pi^2/12)\\&-\tfrac{1}{2}\ln Y\ln(4-s)]+a_2D_3^+(Y)\\&+a_3[\ln YD_2^-(Y)-D_3^-(Y)]+a_4\ln^5Y+a_5\ln^4Y\\&+a_6\ln^3Y+\{a_Y+(a_8+a_9/\nu)[\nu(\nu+1)]^{1/2}\}\ln^2Y\\&+(a_{10}+a_{11}\nu)\ln Y+(a_{13}/\nu)[\nu(\nu+1)]^{1/2}\}\,.\end{aligned} \tag{A7}$$

Here ν is $(s-4)/4$ and $Y\equiv Y(s)$ has already been defined. $\mathrm{Li}_2(x)$ is the function[11] Dilog, and the $D_n{}^I(x)$, which are related to the Debye functions,[12] are given by

$$D_n{}^\pm(x)=\int_1^x du(u\pm1)^{-1}\ln^n u\,. \tag{A8}$$

The a_j's, for isospin $\binom{0}{2}$, are given by

$$\begin{aligned}
a_1&=8\left[3\alpha\binom{605}{473}-60\binom{5}{8}-\frac{2\pi^2}{3}\binom{185}{131}\right],\\
a_2&=-\frac{20}{3}\binom{185}{131},\\
a_3&=16\binom{235}{106},\\
a_4&=-\frac{2}{15}\binom{185}{101},\\
a_5&=\frac{1}{6}\binom{1865}{1079},\\
a_6&=\frac{1}{3}\left[66\alpha\binom{55}{28}+5\binom{1015}{751}-\frac{4\pi^2}{3}\binom{185}{101}\right],\\
a_7&=66\alpha\binom{55}{43}-120\binom{5}{8}-\frac{4\pi^2}{3}\binom{185}{131},\\
a_8&=-6\binom{1535}{931},\\
a_9&=-12\binom{235}{211},\\
a_{10}&=3\left[-\frac{37}{2}(\alpha^2-2\pi^2)\alpha+585\alpha^2\right]\binom{5}{3}\\
&\quad+6\alpha\binom{135}{-29}-18\binom{635}{411}-\pi^2\binom{1445}{1157},\\
a_{11}&=-12\left[363\alpha\binom{5}{3}+\binom{3205}{1913}\right],\\
a_{12}&=-2\left[a_{10}-a_{11}-2\pi^2\binom{235}{211}\right],\\
a_{13}&=-4\pi^2\binom{235}{211}.
\end{aligned} \tag{A9}$$

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[7] M. Gell-Mann and Y. Ne'eman, *The Eightfold Way* (W. A. Benjamin and Company, Inc., New York 1964), especially Chap. 12.

[8] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 456 (1956); Freeman J. Dyson, *ibid.* **106**, 157 (1957).

[9] R. Del Fabbro, M. De Pretis, R. Jones, G. Marini, A. Odian, G. Stoppini, and L. Tau, Phys. Rev. **139**, B701 (1965). We are indebted to Dr. Odian for a discussion of this experiment.

[10] M. Feldman, W. Frati, J. Halpern, A. Kanofsky, M. Nussbaum, S. Richert, P. Yamin, A. Choudry, S. Devons, and J. Grunhaus, Phys. Rev. Letters **14**, 869 (1965).

[11] *Handbook of Mathematical Functions* (*AMS 55*), compiled by U. S. Department of Commerce, National Bureau of Standards (U. S. Government Printing Office, Washington, D. C., 1964), p. 1004.

[12] Reference 11, p. 998.

For the p wave we have

$$\begin{aligned}\mathrm{Im}A_1{}^1(\nu)=(5\lambda^4/2\nu 2)\{&64[(\nu+1)\ln Y+2(\nu(\nu+1))^{1/2}]D_2(Y)-64(\nu+1)D_3{}^-(Y)\\&+(380/3)\nu D_3{}^+(Y)+(6/5)(\nu+1)\ln^5 Y+[12(\nu(\nu+1))^{1/2}-(37/2)(\nu+1)+(95/6)]\ln^4 Y\\&-[(64/3)(\nu(\nu+1))^{1/2}+b_1+(1565/6)]\ln^3 Y+[b_2(\nu(\nu+1))^{1/2}+b_3\nu]\ln^2 Y\\&+6[119(\nu+1)^2+b_4(\nu+1)+b_5]\ln Y+[b_6(\nu+1)-12b_5](\nu(\nu+1))^{1/2}\\&\qquad+12[156\nu+b_7][\mathrm{Li}_2(-1/Y)+\pi^2/12-\tfrac{1}{2}\ln Y\ln(4-s)]\}\,,\end{aligned}\tag{A10}$$

where

$$\begin{aligned}b_1&=4(11\alpha+811/12-\pi^2)\,,\\b_2&=-24(11\alpha+1561/24-\pi^2)\,,\\b_3&=-6(77\alpha-40-38/9\pi^2)\,,\\b_4&=(1/12)[37\alpha^3-1170\alpha^2-(4020+74\pi^2)\alpha-8(857-(1027/12)\pi^2)]\,,\\b_5&=-(1/24)[37\alpha^3-1170\alpha^2-(5146+74\pi^2)\alpha-(7283-(3458/3)\pi^2)]\,,\\b_6&=726\alpha+1543\,,\\b_7&=77\alpha+38-(74/9)\pi^2.\end{aligned}\tag{A11}$$

3. The contributions of the double-spectral functions may be written in the form

$$\begin{aligned}\mathrm{Im}A_{Dl}{}^I(s)=\frac{8\lambda^4}{(4-s)^2}\bigg\{&\int_4^{4-s}dt\,P_l[1+2t(4-s)]J^I(s,t)\\&+\int_{16}^{4-s}dt\,P_l[1+2t(4-s)]L^I(s,t)+\int_4^{-s}dt\,\mathrm{Re}Q_l[1+2t/(4-s)]M^I(s,t)\bigg\}\,,\end{aligned}\tag{A12}$$

where

$$\begin{aligned}J^I(s,t)\equiv\int_4^{\bar t_1}dt_1\left(1-\frac{4}{t_1}\right)^{1/2}\int_4^{\bar t_2}dt_2\left(1-\frac{4}{t_2}\right)^{1/2}\\\times\left\{\begin{bmatrix}225\\25\\81\end{bmatrix}[j(s;tt_1t_2)-j(s_0;tt_1t_2)]+\begin{bmatrix}185\\5\\101\end{bmatrix}[j(4-s-t;tt_1t_2)-j(s_0;tt_1t_2)]\right\}\,,\end{aligned}\tag{A13}$$

where $\bar t_2=(t^{1/2}-t_1{}^{1/2})^2$; $\bar t_1=(t^{1/2}-2)^2$.

$$\begin{aligned}L^I(s,t)=\int_4^{\infty}dt_1\left(1-\frac{4}{t_1}\right)^{1/2}\int_4^{\infty}dt_2\left(1-\frac{4}{t_2}\right)\\\times\left\{\begin{bmatrix}235\\30\\76\end{bmatrix}[l(t;st_1t_2)-l(t;s_0t_1t_2)]+\begin{bmatrix}185\\-5\\101\end{bmatrix}[l(t;4-2-tt_1t_2)-l(t_0;s_0t_1t_2)]\right\}\,,\end{aligned}\tag{A14}$$

$$M^I(s,t)=\begin{bmatrix}185\\-5\\101\end{bmatrix}\int_4^{\infty}dt_1\left(1-\frac{4}{t_1}\right)^{1/2}\int_4^{\infty}dt_2\left(1-\frac{4}{t_2}\right)^{1/2}\left[m(t;4-s-tt_1t_2)+\begin{bmatrix}1\\-1\\1\end{bmatrix}m(4-s-t;tt_1t_2)\right].\tag{A15}$$

We have defined

$$\begin{aligned}j(s;tt_1t_2)&=[s(s-4)]^{-1/2}K(s;tt_1t_2)\ln Z_1(s;tt_1t_2)\,,\\l(t;st_1t_2)&=[t(t-4)]^{-1/2}K(t;st_1t_2)\ln Z_2(t;st_1t_2)\,,\\m(t;xt_1t_2)&=[t(t-4)]^{-1/2}K(t;xt_1t_2)\,,\end{aligned}\tag{A16}$$

with K defined in the text. Finally,

$$z_1(s;tt_1t_2)=\left[\left(1-\frac{4}{s}\right)^{1/2}K(\infty;tt_1t_2)+K(s;tt_1t_2)\right]\bigg/\left[\left(1-\frac{4}{s}\right)^{1/2}K(\infty;tt_1t_2)-K(s;tt_1t_2)\right], \tag{A17}$$

$$z_2(s;tt_1t_2)=\{[s_+(tt_1t_2)-s]^{1/2}+[s_-(tt_1t_2)-s]^{1/2}\}/\{[s_+(tt_1t_2)-s]^{1/2}-[s_+(tt_1t_2)-s]^{1/2}\}, \tag{A18}$$

where s_+ is the greater and s_- the lesser root of $K^{-2}(t;st_1t_2)$.

4. Lastly, we record the three inelastic single-spectral functions. These are

$$I(s)=+\left(\frac{16\lambda^4}{\pi^2}\right)\left(\frac{s}{s-4}\right)^{1/2}\int_4^{\hat{t}_1}dt_1\left(1-\frac{4}{t_1}\right)\int_4^{\hat{t}_2}dt_2K(\infty;st_1t_2)\ln A(s;t_1t_2)\ln A(t_2;st_1) \tag{A19}$$

with

$$\hat{t}_1=(s^{1/2}-2)^2,\quad \hat{t}_2=(s^{1/2}-t_1^{1/2}),$$

and

$$A(s;t_1t_2)=\left[s-t_1-t_2+\left(1-\frac{4}{s}\right)^{1/2}K^{-1/2}(\infty;st_1t_2)\right]\bigg/\left[s-t_1-t_2-\left(1-\frac{4}{s}\right)^{1/2}K^{-1/2}(\infty;st_1t_2)\right],$$

$$I_2(s)=\frac{10\lambda^4}{\pi^2 s}\int_4^{\hat{t}_1}dt_1\left(1-\frac{4}{t_1}\right)^{1/2}\int_{t_2'}^{t_2''}dt_2t_2^{-1}(t_2-1)^{-2}[(t_1-t_2-1)^2-4t_2]^{1/2}[(t_2-s-1)^2-4s]^{1/2}, \tag{A20}$$

where

$$t_2'=(t_1^{1/2}+1)^2;\quad t_2''=(s^{1/2}-1)^2,$$

$$I_3(s)=\frac{3\lambda^4}{2\pi^2 s}\int_4^{\hat{t}_1}dt_1\left(1-\frac{4}{t_1}\right)^{1,2}\int_{t_2'}^{t_2''}dt_2t_2^{-1}\ln[g_+(s,t_1,t_2)/g_-(s,t_1,t_2)], \tag{A21}$$

where

$$g_I(s,t_1,t_2)=s(t_1+t_2-1)-t_2(t_2-1)\pm[(t_2-t_1-1)^2-4t_1]^{1/2}[(t_2-s-1)^2-4s]^{1/2}.$$

I_2 corresponds to a "bubble" diagram where one of the exchanged pions itself has a three-pion bubble, and I_3 to a "bubble" inside of which a bubble is exchanged. The remaining possible graph gives rise to I_1. The three I's occur, respectively, with the isotopic spin factors

$$\begin{bmatrix}825\\0\\224\end{bmatrix},\quad\begin{bmatrix}25\\0\\8\end{bmatrix},\quad\begin{bmatrix}125\\0\\24\end{bmatrix}. \tag{A22}$$