Origins of the Lee Triangle*

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The Lee triangle, $\sqrt{3}\langle\Sigma^+|p\pi^0\rangle-\langle\Lambda|p\pi^-\rangle=2\langle\Xi^-|\Lambda\pi^-\rangle$, is shown to be a consequence of any Hamiltonian which transforms as a member of an $SU(3)$ octet, and which satisfies one simple constraint. If the Hamiltonian is constructed in the form $\Sigma g_n[(\bar{B}\times B)_{(n)}\times\pi]_{(8)}$, where $n=8_D,8_F,10,10^*,27$, the required constraint is $g_{10}=g_{10^*}$. Thus the baryon-antibaryon decuplets must appear with equal weights, but the octets and (27)-plet are placed under no restrictions. This result is used to explain why some dynamical models and symmetry arguments predict the Lee triangle, while others do not. Within the framework of $SU(6)$, it also correlates the orbital angular momentum in $\Sigma^+\to n\pi^+$ with the Lee triangle.

1. INTRODUCTION

EVER since it was first proposed, the Lee triangle[1]

$$\sqrt{3}\langle\Sigma^+|p\pi^0\rangle-\langle\Lambda|p\pi^-\rangle=2\langle\Xi^-|\Lambda\pi^-\rangle \tag{1}$$

has been something of a puzzle. For one thing, it is the only new development arising from the use of unitary symmetry in nonleptonic hyperon decay, and for another, it is a consequence of several distinct arguments.[2] These arguments all start at the same point, namely, the assumption that weak interactions belong to an octet, but then they diverge, either in the direction of symmetry properties,[1,3] or toward dynamical models[4] which have no apparent symmetry. The question we wish to consider here is why they all lead to the Lee triangle.

Our answer is centered upon the effective decay Hamiltonian. If it transforms as a member of an octet, it will include five independent terms which engender observable decays. By virtue of the $\Delta T=\frac{1}{2}$ rule, there are only four observable amplitudes, and therefore the most general Hamiltonian yields no predictions beyond those of $\Delta T=\frac{1}{2}$. Furthermore, it is reasonable to suppose that one linear relation among these amplitudes will be predicted only when two constraints are imposed upon the five coupling constants. What is surprising about the Lee triangle is that this supposition is not correct: It can in fact, be derived from one constraint alone. The extra degree of freedom which this entails is then responsible for the variety of the triangle's derivations.

To prove these statements, we first use the $\Delta T=\frac{1}{2}$ rule to recast Eq. (1) into a form which involves only one π meson, namely, the π^-. Next, we construct the effective Hamiltonian by coupling baryons to antibaryons, and then to pseudoscalar mesons in order to form an over-all octet. The observable terms are

$$[(\bar{B}\times B)_{(n)}\times\pi]_{(8)},\quad n\equiv 8_D,8_F,10,10^*,27. \tag{2}$$

A comparison of the new version of Eq. (2) with the Hamiltonian reveals that the Lee triangle is automatically satisfied whenever the baryon-antibaryon system is an octet or (27)-plet; there is, however, only one combination of the decuplet and its conjugate which gives rise to Eq. (1). In other words, the Lee triangle is a natural consequence of three of the coupling schemes in Eq. (2), and it is for this reason that the Hamiltonian need be subject to one constraint instead of two.

As pointed out in an earlier paper,[5] the unitary symmetry scheme contains two types of weak-symmetry R conjugation and T-L invariance. In general, a Lee-constrained Hamiltonian [i.e., one which predicts Eq. (1)] satisfies neither of them, and so we may anticipate nonsymmetric dynamical models for the Lee triangle. We can also explain the existence of purely symmetric derivations by noting that certain symmetries include the Lee constraint as one of their necessary conditions.

Our basic result is proved in the next section, and it is examined from the standpoint of weak symmetries in the third. To illustrate its wider implications, we use the result to show that if weak interactions transform according to the 35-dimensional representation of $SU(6)$,[6] then the Lee triangle forces $\Sigma^+\to n\pi^+$ to be a pure S-wave decay. Some of the more detailed mathematics is confined to an Appendix.

2. THE LEE-CONSTRAINED HAMILTONIAN

By means of the relation

$$\langle\Sigma^+|p\pi^0\rangle=\langle\Sigma^0|p\pi^-\rangle \tag{3}$$

which follows from $\Delta T=\frac{1}{2}$, we can rewrite Eq. (1) in

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[1] B. W. Lee, Phys. Rev. Letters **12**, 83 (1964); H. Sugawara, Progr. Theoret. Phys. (Kyoto) **31**, 213 (1964).

[2] Summaries of the theoretical arguments leading to the Lee triangle, and of its experimental status, have been given by A. Pais and S. B. Treiman, in *Proceedings of the 12th Annual International Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965); and by R. H. Dalitz, from lectures given at the International School of Physics "Enrico Fermi" on Weak Interactions, Varenna, Italy, 1964 (to be published).

[3] M. Gell-Mann, Phys. Rev. Letters **12**, 155 (1964); B. Sakita, *ibid.* **12**, 379 (1964); S. P. Rosen, *ibid.* **12**, 408 (1964); S. Okubo, Phys. Letters **8**, 362 (1964); S. Coleman, S. L. Glashow, and B. W. Lee, Ann. Phys. (N.Y.) **30**, 348 (1964).

[4] H. Sugawara, Nuovo Cimento **31**, 635 (1964); S. Coleman and S. L. Glashow, Phys. Rev. **134**, B671 (1964); B. W. Lee and A. R. Swift, *ibid.* **136**, B228 (1964).

[5] S. P. Rosen, Phys. Rev. **137**, B431 (1965).

[6] F. Gursey and L. A. Radicati, Phys. Rev. Letters **13**, 173 (1964); A. Pais, *ibid.* **13**, 175 (1964); B. Sakita, Phys. Rev. **136**, B1756 (1964).

the form

$$\langle \tfrac{1}{2}(\sqrt{3}\Sigma^0-\Lambda)\,|\,p\pi^-\rangle = \langle \Xi^-|\Lambda\pi^-\rangle\,. \tag{4}$$

The particles appearing in (4) correspond to the following components of the baryon and meson octets[5]:

$$\begin{aligned}&\Lambda\sim-(\sqrt{3}/\sqrt{2})B_3{}^3,\quad \tfrac{1}{2}(\sqrt{3}\Sigma^0-\Lambda)\sim-(\sqrt{3}/\sqrt{2})B_1{}^1,\\ &p\sim B_1{}^3,\quad \Xi^-\sim B_3{}^1,\quad \pi^-\sim-\pi_2{}^1,\end{aligned} \tag{5}$$

and so the equation is equivalent to

$$\langle B_1{}^1|\,B_1{}^3\pi_2{}^1\rangle = \langle B_3{}^1|\,B_3{}^3\pi_2{}^1\rangle\,. \tag{6}$$

It now follows that, if a Hamiltonian is to predict the Lee triangle, it must include the terms

$$G_a\equiv\bar{B}_1{}^1B_1{}^3,\quad G_b\equiv\bar{B}_1{}^3B_3{}^3 \tag{7}$$

in the combination

$$(G_a+G_b)\pi_2{}^1\,. \tag{8}$$

The point to notice about Eqs. (7) and (8) is that G_a is orthogonal to the unitary multiplet $(\bar{B}\times B)_{(10^*)}$ and G_b to $(\bar{B}\times B)_{(10)}$. [The (10) is antisymmetric in the upper indices and symmetric in the lower ones; for the (10*) these permutation symmetries are reversed.[7]] Therefore, if the Hamiltonian is constructed as in Eq. (2), Eq. (8) will give rise to at least one constraint, namely between the coupling constants of the $n=10$ and $n=10^*$ terms.

To show that no other constraints are needed, we express G_a and G_b in terms of the components of the $(\bar{B}\times B)_{(n)}$:

$$G_a=\tfrac{1}{5}D_1{}^3+\tfrac{1}{3}F_1{}^3+\tfrac{1}{4}[10]_{11}{}^{13}+\tfrac{1}{4}[27]_{11}{}^{13}, \tag{9}$$

$$G_b=\tfrac{1}{5}D_1{}^3+\tfrac{1}{3}F_1{}^3+\tfrac{1}{4}[10^*]_{13}{}^{33}+\tfrac{1}{4}[27]_{13}{}^{33}. \tag{10}$$

D and F are the usual R-symmetric and R-antisymmetric octets, respectively, and the [10], [10*], and [27] terms are defined by Okubo[7]; the detailed derivation of (9) and (10) is given in the Appendix. Next, we write the observable part of the Hamiltonian as

$$\begin{aligned}H_{\text{eff}}=f(D\pi)_2{}^3+g(F\pi)_2{}^3+h[[10]_2{}^3\pi+[10^*]_2{}^3\pi]\\ +h'[[10]_2{}^3\pi-[10^*]_2{}^3\pi]+k[27]_2{}^3\pi+\text{H.c.}\,,\end{aligned} \tag{11}$$

where

$$\begin{aligned}(LM)_\nu{}^\mu&\equiv L_\lambda{}^\mu M_\nu{}^\lambda\,,\\ N_\nu{}^\mu\pi&\equiv N_{\nu\rho}{}^{\mu\lambda}\pi_\lambda{}^\rho\,.\end{aligned} \tag{12}$$

The space-time structure of H_{eff} is irrelevant to the present discussion and has therefore been suppressed.

Because $D_1{}^3$ appears with the same weight in (9) as it does in (10), the first term of H_{eff} includes G_a and G_b in the combination of Eq. (8). Similarly, the second term also includes them in the required combination. The last term engenders decays involving a π^- meson via $[27]_{21}{}^{32}\pi_2{}^1$; from the properties of the (27)-plet,[7] it follows that

$$[27]_{21}{}^{32}\pi_2{}^1=-\{[27]_{11}{}^{13}+[27]_{13}{}^{33}\}\pi_2{}^1. \tag{13}$$

Comparing the right-hand side of this equation with (9) and (10), we see that G_a and G_b again appear as in Eq. (8). Therefore, any Hamiltonian of the form

$$\lambda[(\bar{B}\times B)_{(8)}\times\pi]_{(8)}+\mu[(\bar{B}\times B)_{(27)}\times\pi]_{(8)} \tag{14}$$

will always give rise to the Lee triangle.

From the permutation symmetries and traceless conditions[7] of $[10]_{\nu\rho}{}^{\mu\lambda}$ and $[10^*]_{\nu\rho}{}^{\mu\lambda}$, it follows that

$$\begin{aligned}[10]_{21}{}^{32}\pi_2{}^1&=[10]_{11}{}^{13}\pi_2{}^1,\\ [10^*]_{21}{}^{32}\pi_2{}^1&=[10^*]_{13}{}^{33}\pi_2{}^1.\end{aligned} \tag{15}$$

Consequently, the required combination of G_a and G_b appears in the third term of H_{eff} but not in the fourth [see Eqs. (9)–(12)]. It is now evident that the only constraint needed to predict the Lee triangle is

$$h'=0. \tag{16}$$

Henceforth we shall refer to Eq. (16) as the Lee constraint, and to a Hamiltonian which satisfies it as a Lee-constrained Hamiltonian.

3. WEAK SYMMETRIES

There are two weak symmetries available in $SU(3)$. One, R conjugation, is defined by

$$R:\quad X_\nu{}^\mu\rightarrow X_\mu{}^\nu\quad(X\equiv\bar{B},B,\pi) \tag{17}$$

and the other, the T-L transformation, interchanges the indices 2 and 3:

$$2\leftrightarrow3\,. \tag{18}$$

A given Hamiltonian may be even under (18), or odd, or neither even nor odd. The even case corresponds to T-$L(1)$ invariance and the odd case to T-$L(2)$ invariance.[5]

In general the Lee-constrained Hamiltonian has no definite symmetry. The D and F terms of Eq. (11) transform in opposite ways under (17), and they are related by (18) to unobservable interactions which are not included in H_{eff}, e.g.,

$$(D\pi)_2{}^3\rightarrow(D\pi)_3{}^2\equiv D_\lambda{}^2\pi_3{}^\lambda\,. \tag{19}$$

It is, therefore, not surprising that there exist non-symmetric cynamical models[2,4] for the Lee triangle. An example of such a model is one based upon meson poles,[8] e.g.:

$$\Lambda\xrightarrow[\text{strong}]{}p+K^-\underset{\text{weak}}{\searrow}\pi^-. \tag{20}$$

By virtue of the strong vertex, the effective interaction takes the form $[(\bar{B}\times B)_{(8)}\times\pi]_{(8)}$ and is therefore Lee-constrained. It is not T-L-invariant, and unless there are accidental cancellations at the two vertices, it is not R-invariant.

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[7] S. Okubo, Progr. Theoret. Phys. (Kyoto) **28**, 24 (1962); see also the Appendix below.

[8] B. W. Lee and A. R. Swift, Ref. 4, discuss this type of model in some detail.

Despite this lack of symmetry in the general case, the Lee constraint [Eq. 16)] does happen to be a necessary condition for certain symmetries. To see which they are, we assume that time-reversal invariance is valid, and that the pion field is coupled to baryon fields through its derivative. The terms of interest in $H_{\rm eff}$, namely those involving baryon-antibaryon decuplets [see Eq. (11)], then take the form

$$(h_\lambda+h_\lambda')([10]_2{}^3\pi-[10^*]_3{}^2\pi)\,,\\ +(h_\lambda-h_\lambda')([10^*]_2{}^3\pi-[10]_3{}^2\pi) \quad (21)$$

for both parity-conserving ($\lambda\equiv$p.c.) and parity-violating ($\lambda\equiv$p.v.) decays. The coupling constants h_λ and h_λ' are all real, and the second term in each $SU(3)$ expression is the Hermitian conjugate of the first (see the Appendix for details).

Under (17), the terms in (21) transform according to

$$[10]_2{}^3\pi\leftrightarrow[10^*]_3{}^2\pi \quad (22)$$

and under (18),

$$[10]_2{}^3\pi\leftrightarrow[10]_3{}^2\pi\,. \quad (23)$$

From (21)–(23), we see that the constraint

$$h_{\rm p.c.}'=h_{\rm p.v.}'=0 \quad (24)$$

is necessary for R invariance and for T-$L(2)$ invariance, but not for T-$L(1)$ invariance. In addition, RP invariance (P denotes parity) requires

$$h_{\rm p.c.}'=0 \quad (25a)$$

and T-$L(1)\times P$ requires

$$h_{\rm p.v.}'=0\,. \quad (25b)$$

Thus there are three symmetry arguments which predict the Lee triangle[9]: (i) R invariance; (ii) T-$L(2)$ invariance; and (iii) RP and T-$L(1)\times P$ invariance. Notice that if derivative coupling is replaced by nonderivative coupling in $H_{\rm eff}$, then each invariance I must be replaced by $I\times P^5$: for example the third argument becomes (iii′) R and T-$L(1)$ invariance.

The first of these arguments is marred by the fact that R invariance and derivative coupling predict the wrong relative sign for the asymmetry parameters in Λ and Ξ decay.[2,5] The second runs counter to Cabibbo's theory,[10] in which nonleptonic decays turn out to be T-$L(1)$ invariant.[5] We may therefore regard the last argument, especially in its nonderivative form (iii′), as the most attractive of the three. Whether it is also the "true" derivation of the Lee triangle is a question which we do not wish to discuss here.

[9] Symmetry arguments have been summarized by A. Pais and S. B. Treiman, Ref. 2, and by S. Coleman, S. L. Glashow, and B. W. Lee, Ref. 3.

[10] N. Cabibbo, Phys. Rev. Letters 10, 531 (1963).

4. SUMMARY AND DISCUSSION

We have shown that the Lee triangle follows from any octet interaction in which the baryon-antibaryon coupling has the form

$$f(\bar{B}\times B)_{(8_D)}+g(\bar{B}\times B)_{(8_F)}+h[(\bar{B}\times B)_{(10)}\\ +(\bar{B}\times B)_{(10^*)}]+k(\bar{B}\times B)\,. \quad (26)$$

The only constraint needed is that between the (10) and (10*) terms. Using this result, we have also been able to explain why certain dynamical models and symmetry arguments predict the triangle, while others do not. It is worth noting that time-reversal invariance is not necessary for the proof of Eq. (26) (Sec. 2), but it has been used in the subsequent discussion of weak symmetries (Sec. 3).

To illustrate some other uses of this result, we shall show how, within the framework of nonrelativistic $SU(6)$,[6] it correlates the Lee triangle with the properties of $\Sigma^+\to n\pi^+$. We assume that baryons belong to a (56)-plet and mesons to a (35)-plet. If the effective interaction also belongs to a (35)-plet, it will be a linear combination of two terms:

$$H_1\sim[(\bar{B}\times B)_{(35)}\times\pi]_{(35)}\,,\\ H_2\sim[(\bar{B}\times B)_{(405)}\times\pi]_{(35)}\,. \quad (27)$$

In parity-violating decays, the intrinsic spins of baryon and antibaryon are coupled to zero (S-wave decays), and in parity-conserving decays they are coupled to a resultant spin of one (P-wave decays).

Since the spin–unitary-spin content of the (35) is

$$(1,8)+(3,8)+(3,1)\,, \quad (28)$$

it follows that the baryon-antibaryon system forms an octet in both the S- and P-wave parts of H_1. Consequently, the observable amplitudes arising from H_1 always satisfy the Lee triangle. The spin-zero content of the (405) is[11]

$$(1,1)+(1,8)+(1,27) \quad (29)$$

and so the S-wave amplitude from H_2 also satisfies the triangle. The P-wave amplitude does not: It includes $(\bar{B}\times B)_{(10)}$ and $(\bar{B}\times B)_{(10^*)}$ with a relative phase opposite from that given in Eq. (26). Therefore, if the Lee triangle is to be satisfied, we must set

$$H_2(P\text{ wave})=0\,. \quad (30)$$

Now, because the system $\bar{\Sigma}^+n$ has isotopic spin $\frac{3}{2}$, it does not belong to an octet and hence it cannot appear in a (35)-plet. The decay $\Sigma^+\to n\pi^+$ must therefore be engendered by H_2 alone, and if Eq. (30) holds, its P-wave amplitude will vanish. In other words the Lee triangle forces $\Sigma^+\to n\pi^+$ to be a pure S-wave decay. If experiment should show that this decay is pure P-wave, we would have to conclude that the assignment of non-

[11] M. A. B. Bég and V. Singh, Phys. Rev. Letters 13, 418 (1964).

leptonic decays to the 35-fold representation of $SU(6)$ is incorrect.[12]

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APPENDIX

In this Appendix we derive the basic result leading to Eqs. (9) and (10). We also fill in some of the details which were omitted in the body of the paper.

The components of the baryon octet are denoted by $B_\nu{}^\mu$ (where $B_\lambda{}^\lambda=0$) and those of its charge conjugate by $\bar{B}_\nu{}^\mu$, where

$$B_\mu{}^\nu \equiv \bar{B}_\nu{}^\mu. \quad \text{(A1)}$$

From the product of these two octets, the following unitary multiplets can be formed:

Singlet:
$$(\bar{B}\cdot B)\equiv \bar{B}_\rho{}^\lambda B_\lambda{}^\rho. \quad \text{(A2)}$$

D-type octet:
$$D_\nu{}^\mu=\tfrac{1}{2}[\bar{B}_\nu{}^\lambda B_\lambda{}^\mu+\bar{B}_\lambda{}^\mu B_\nu{}^\lambda -\tfrac{2}{3}\delta_\nu{}^\mu(\bar{B}\cdot B)]. \quad \text{(A3)}$$

F-type octet:
$$F_\nu{}^\mu=\tfrac{1}{2}[\bar{B}_\nu{}^\lambda B_\lambda{}^\mu-\bar{B}_\lambda{}^\mu B_\nu{}^\lambda]. \quad \text{(A4)}$$

Decuplet:
$$[10]_{\alpha\beta}{}^{\mu\nu}=(1+P_{\alpha\beta})(1-P^{\mu\nu}) \times[\bar{B}_\alpha{}^\mu B_\beta{}^\nu-\tfrac{2}{3}\delta_\alpha{}^\mu F_\beta{}^\nu]. \quad \text{(A5)}$$

Conjugate decuplet:
$$[10^*]_{\alpha\beta}{}^{\mu\nu}=(1-P_{\alpha\beta})(1+P^{\mu\nu}) \times[\bar{B}_\alpha{}^\mu B_\beta{}^\nu+\tfrac{2}{3}\delta_\alpha{}^\mu F_\beta{}^\nu]. \quad \text{(A6)}$$

(27)-plet:
$$[27]_{\alpha\beta}{}^{\mu\nu}=(1+P_{\alpha\beta})(1+P^{\mu\nu}) \times[B_\alpha{}^\mu B_\beta{}^\nu-\tfrac{2}{5}\delta_\alpha{}^\mu D_\beta{}^\nu-\tfrac{1}{12}\delta_\alpha{}^\mu\delta_\beta{}^\nu(B\cdot B)]. \quad \text{(A7)}$$

The permutation symmetries of the decuplets and (27)-plet[7] are clearly indicated by the way in which the permutation operators $P_{\alpha\beta}$ and $P^{\mu\nu}$ have been introduced. All of these multiplets are traceless:

$$D_\lambda{}^\lambda=F_\lambda{}^\lambda=[10]_{\alpha\lambda}{}^{\mu\lambda}=[10^*]_{\alpha\lambda}{}^{\mu\lambda}=[27]_{\alpha\lambda}{}^{\mu\lambda}=0. \quad \text{(A8)}$$

Equations (13) and (15) in Sec. 2 follow from (A8) and the permutation symmetries of (A5)–(A7).

Following Okubo,[13] we introduce a totally antisymmetric, traceless tensor

$$S_{\alpha\beta}{}^{\mu\nu}=(1-P_{\alpha\beta})(1-P^{\mu\nu})[\bar{B}_\alpha{}^\mu B_\beta{}^\nu+2\delta_\alpha{}^\mu D_\beta{}^\nu +\tfrac{1}{6}\delta_\alpha{}^\mu\delta_\beta{}^\nu(\bar{B}\cdot B)]. \quad \text{(A9)}$$

Adding Eqs. (A5), (A6), (A7), (A9), and using the fact that all components of $S_{\alpha\beta}{}^{\mu\nu}$ are identically zero,[13] we find

$$\begin{aligned}\bar{B}_\alpha{}^\mu B_\beta{}^\nu=&\tfrac{1}{4}\{[10]_{\alpha\beta}{}^{\mu\nu}+[10^*]_{\alpha\beta}{}^{\mu\nu}+[27]_{\alpha\beta}{}^{\mu\nu}\}\\ &+\tfrac{1}{3}(1-P_{\alpha\beta}P^{\mu\nu})\delta_\beta{}^\mu F_\alpha{}^\nu\\ &+\tfrac{1}{5}(1+P_{\alpha\beta}P^{\mu\nu})(3\delta_\beta{}^\mu D_\alpha{}^\nu-2\delta_\alpha{}^\mu D_\beta{}^\nu)\\ &-(1/24)[\delta_\alpha{}^\mu\delta_\beta{}^\nu-3\delta_\beta{}^\mu\delta_\alpha{}^\nu](\bar{B}\cdot B). \quad \text{(A10)}\end{aligned}$$

Equations (9) and (10) in the body of the paper are special cases of Eq. (A10).

Under charge conjugation

$$B_\nu{}^\mu \leftrightarrow \bar{B}_\mu{}^\nu, \quad \text{(A11)}$$

the components of the decuplets and (27)-plet transform according to

$$[10]_{\alpha\beta}{}^{\mu\nu}\to-[10^*]_{\mu\nu}{}^{\alpha\beta}, \quad \text{(A12)}$$

$$[27]_{\alpha\beta}{}^{\mu\nu}\to+[27]_{\mu\nu}{}^{\alpha\beta}. \quad \text{(A13}$$

The negative sign in (A12) accounts for the fact that, with derivative coupling in the effective Hamiltonian, the Hermitian conjugate of $[10]_2{}^3\pi$ is $-[10^*]_3{}^2\pi$ [see Eq. (21), Sec. 3].

Finally, we note that the properties of the multiplets under R conjugation can be derived from Eq. (17) and the definitions in (A2)–(A7).

[12] Other arguments against the (35) have been given by S. P. Rosen and S. Pakvasa, Phys. Rev. Letters **13**, 773 (1964). The one given in the text, however, is the most attractive.

[13] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).