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# Short-Range Correlations in Nuclear Matter*

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It is shown that a wave function which is a simple product of two-particle correlation functions gives a smaller short-range three-body correlation energy (calculated as the ground-state expectation value of the kinetic energy and short-range part of the interactions) than the approximate Faddeev wave function used in the recent work of Bethe. Thus for a plausible nucleon-nucleon interaction, the short-range three-body correlations contribute less than 1 MeV per particle to the energy of nuclear matter.

## 1. INTRODUCTION

THERE is still some question concerning the magnitude of short-range three-body correlations in nuclear matter and in particular their contribution to the binding energy. According to the approximation used by Bethe, Brandow, and Petscheck (BBP),[1] such correlations would contribute to the order of +5 to 10 MeV (per particle) to the energy.[2] However, the BBP approximation includes only a part of the important short-range correlation diagrams, mainly the one indicated in Fig. 1(a). Recently Bethe[3] has pointed out that it is also necessary to include three-hole diagrams of higher order, e.g., the one shown in Fig. 1(b). Indeed in order to accurately evaluate the terms proportional to (density)$^2$ in the low-density expansion of the energy (per particle) it is, in principle, necessary to solve the three-body problem exactly.[4] [Similarly, in order to evaluate the terms proportional to $\rho$, we must solve the two-body problem. For example, in the low-density limit, we obtain the well-known result $E/A \sim O(\rho a)$ where $a$ is the zero-energy scattering length of the two-body interaction.] While the three-body problem is analytically soluble only for very special potentials, Faddeev[5] has developed a very powerful method for treating this problem.

In this paper let us consider a greatly simplified three-body problem. We split the assumed two-body interaction into short- and long-range parts $v_S$ and $v_L$ the dividing line being made so that $v_S$ alone gives zero scattering length.[6] The short-range potential contains a repulsion with an attraction outside. As has been shown previously, it does not give any energy shift for a pair of zero relative momentum. Also it follows from the above remarks that with $v_S$ alone the term proportional to $\rho$ in the energy expansion vanishes. However, $v_S$ gives a "wound" in the two-particle wave functions as illustrated in Fig. 2. The wound disappears outside the potential. The difference function $r(\phi-\psi)$ is roughly proportional to $\sin(\frac{1}{2}\pi r/c)$; thus the momenta contained in this function are of order $\frac{1}{2}\pi/c$ for a core radius $c \sim 0.4$ F. On the other hand, the Fermi momentum at normal density is only about 1.35 F$^{-1}$. Thus for the purpose of treating short-range correlations, it is a good approximation to keep only the lowest term of the power-series expansion in $k_F c$. (Of course, the remaining long-range part of the potential mixes in low-momentum components into the wave function and here the low-density expansion fails. However, a conventional perturbation expansion in powers of $v_L$ converges quite rapidly for reasonable potentials.[6] In the present note we consider *only* the effect of $v_S$ and neglect entirely all effect due to $v_L$ including the dispersion correction, which is of the form shown in Fig. 1(a), and which involves both $v_S$ and $v_L$. We will also, following Bethe, assume that $v_S$ depends only on interparticle spacing and is independent of spin and isospin. Under these conditions it follows from the work of Rajaraman[7] that direct and exchange terms add. Each diagram in which all three particles have different $\sigma_z$ and $\tau_z$ quantum numbers is then the same as it would be

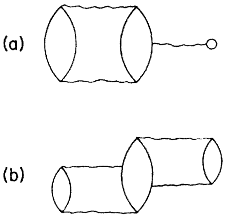

FIG. 1. Two of the infinite series of Goldstone graphs which contribute to the three-body correlation energy; (a) third-order graph involving self-energy insertion, (b) a fourth-order graph containing three hole lines.

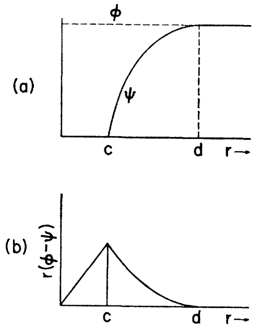

FIG. 2. (a) The unperturbed and correlated two-body wave functions $\phi$ and $\psi$ for zero relative energy and zero scattering length; (b) the corresponding difference function $r(\phi-\psi)$.

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[1] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).

[2] K. A. Brueckner and K. S. Masterson, Phys. Rev. **128**, 2267 (1962); M. Razavy, Phys. Rev. **130**, 1091 (1958).

[3] H. A. Bethe, Phys. Rev. **138**, B804 (1965).

[4] R. Rajaraman, Phys. Rev. **131**, 1244 (1963).

[5] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].

[6] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) **11**, 65 (1960).

[7] R. Rajaraman, Phys. Rev. **129**, 265 (1963).

if the particles obeyed Bose statistics. Thus the three-body problem considered here is essentially the problem of three zero-momentum bosons interacting via two-body zero-scattering-length potentials.

## 2. THE APPROXIMATE FADDEEV WAVE FUNCTION

In the three-body problem[3], there are three pairs of interacting particles, namely 12, 13, and 23. Interactions can occur between any pair, any number of times. However, in the reaction matrix formalism we define, say $g_{12}$, which sums all $v_{12}$ occurring in immediate succession; i.e.,

$$g_{12}=v_{12}-v_{12}e^{-1}v_{12}+v_{12}e^{-1}v_{12}e^{-1}v_{12}+\cdots, \qquad (1)$$

where $e$ is an appropriate energy denominator. Thus the two-body wave function can be written as

$$\psi_{12}=\phi-e^{-1}g_{12}\phi. \qquad (2)$$

$\phi$ is the unperturbed wave function. Apart from a normalization factor, $\phi=1$ for zero momentum and we shall use this value from now on. Clearly no $g_{ij}$ can occur twice in succession without a different interaction line intervening. Faddeev[5] defines a three-body wave function $\psi^{(i)}$ which includes, besides the unperturbed wave function, all diagrams in which the *last* interaction line involves particle $i$. Thus for $\psi^{(1)}$, for example, it is $g_{12}$ or $g_{13}$. Suppose it is $g_{12}$. Then the preceding interaction line must be either $g_{13}$ or $g_{23}$ (or no interaction at all). This describes precisely the wave function $\psi^{(3)}$ in our notation. Thus Faddeev's equations can be written as

$$\psi^{(1)}=1-e^{-1}g_{12}\psi^{(3)}-e^{-1}g_{13}\psi^{(2)}, \qquad (3)$$

and two others which can be obtained from Eq. (3) by cyclic permutation of the indices. Once the $\psi^{(i)}$ are obtained, the complete wave function is given by

$$(\psi^{(1)}-\phi)+(\psi^{(2)}-\phi)+(\psi^{(3)}-\phi)=2(\psi-\phi). \qquad (4)$$

All diagrams in which the last interaction is $g_{ij}$ are included in *both* $\psi^{(i)}$ and $\psi^{(j)}$. This is the origin of the factor 2 on the right-hand side of Eq. (4).

The terms $e^{-1}g_{ij}$ are of course integral operators; thus we have three coupled integral equations which must, in general, be solved numerically. Since this is a difficult task, it is reasonable to look for approximate solutions. One very plausible approximation is to replace the $e^{-1}g_{ij}$ by a simple function of the interparticle coordinates:

$$e^{-1}g_{ij}=\eta(r_{ij})=\eta_{ij}=1-f_{ij}. \qquad (5)$$

This procedure is exact for the two-body problem and thus it should be satisfactory also for the three-body problem. With the above approximation, the Faddeev equations become

$$\psi^{(1)}=1-\eta_{12}\psi^{(3)}-\eta_{13}\psi^{(2)}, \qquad (6)$$

etc. The three linear algebraic equations (6) can be solved explicitly. Thus we obtain

$$\psi^{(1)}=(1-\eta_{12})(1-\eta_{13})Q^{\mathrm{AF}}, \qquad (7)$$

where

$$Q^{\mathrm{AF}}=(1-\eta_{12}\eta_{13}-\eta_{12}\eta_{23}-\eta_{13}\eta_{23}+2\eta_{12}\eta_{13}\eta_{23})^{-1}. \qquad (8)$$

Using Eq. (4) we obtain the complete wave function

$$\psi^{\mathrm{AF}}=(1-\eta_{12})(1-\eta_{13})(1-\eta_{23})Q^{\mathrm{AF}}, \qquad (9)$$

where AF stands for "approximate Faddeev." This wave function clearly has the right qualitative physical behavior, i.e., if particle $i$ is far away from $j$ and $k$, then $\eta_{ij}=\eta_{ik}=0$ and the wave function reduces to the two-particle correlation function. On the other hand, if all particles are at the same point, then all $\eta$ are equal $[=\eta(0)]$ and

$$\psi^{\mathrm{AF}}=(1-\eta(0))/(1+2\eta(0)). \qquad (10)$$

For a hard-core potential $\eta(0)=1$ and $\psi^{\mathrm{AF}}$ vanishes as expected. Using a wave function of essentially the form[8] of Eq. (9) and a two-body potential of the form proposed by Wong,[9] Bethe estimates a three-body correlation energy of only ~0.6 MeV/nucleon at normal density ($k_F\sim1.35$ F$^{-1}$). Wong's potential has a rather small core $e\sim0.30$ F. Even with a larger core radius of $c\sim0.4$ F, Bethe's estimate would give only about 1 MeV/nucleon for the three-body correlation energy. In any case, the short-range three-body correlation energy is quite small compared with other sources of error or uncertainties, e.g., the parameters of the two-body interaction.

## 3. THE PRODUCT WAVE FUNCTION

While the AF wave function may be quite accurate for describing short-range correlations, we propose a slightly different wave function which is simpler and may be even more accurate, namely a simple product wave function

$$\psi^{p}=f_{12}f_{13}f_{23}, \qquad (11)$$

where

$$f_{12}=1-\eta_{12}. \qquad (12)$$

The physical assumption underlying Eq. (11) is that the two-particle correlations act independently. Thus

$$\psi^{\mathrm{AF}}=\psi^{p}Q^{\mathrm{AF}}. \qquad (13)$$

Clearly $\psi^{p}$ has the correct behavior in the limits $r_{ij}, r_{ik}\to\infty$ and $r_i=r_j=r_k$. In the latter case,

$$\psi^{p}\to f^{3}(0), \qquad (14)$$

while

$$\psi^{\mathrm{AF}}\to\tfrac{1}{3}f(0); \qquad (15)$$

i.e., as $f(0)\to0$, the product wave function goes to zero faster than does the AF wave function.

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[8] Bethe uses an improved treatment of the energy denominators. His calculations are, however done directly with $\psi^{(1)}$, $\psi^{(2)}$, and $\psi^{(3)}$ and not with $\psi^{\mathrm{AF}}$.

[9] C. W. Wong, Nucl. Phys. **56**, 213 (1964).

## 4. THREE-BODY CORRELATION ENERGY WITH PRODUCT WAVE FUNCTION

While neither $\psi^{p}$ nor $\psi^{\mathrm{AF}}$ is an exact solution of the general three-body problem, it is useful to compare these two wave functions to see which is more accurate.

The short-range three-body correlation energy may be estimated using the variational expression

$$E=\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle\,, \tag{16}$$

where $H$ is the assumed three-body Hamiltonian.

$$H=-(\hbar^2/2m)(\nabla_1^2+\nabla_2^2+\nabla_3^2)+v_S(r_{12})+v_S(r_{13})+v_S(r_{23})\,. \tag{17}$$

The more accurate the three-body wave function, the lower should be the energy expectation value. In this section we calculate the latter using a product wave function. As before, we assume that $f_{ij}$ is the exact solution of the two-body Schrödinger equation:

$$(\hbar^2/m)\,\nabla_{ij}^2 f_{ij}=v_S(r_{ij})f_{ij}\,. \tag{18}$$

It is readily verified that the energy of a triplet is given by

$$E=\tfrac{3}{4}\frac{\hbar^2}{m\Omega^3}\int\!\!\int\!\!\int\boldsymbol{\nabla}_1(f_{12}^2)\cdot\boldsymbol{\nabla}_1(f_{13}^2)f_{23}^2d^3r_1d^3r_2d^3r_3\,. \tag{19}$$

The quantity $\Omega$ denotes the normalization volume and,

$$\langle\psi|\psi\rangle=\Omega^3[1+O(\Omega^{-1})\cdots]\,. \tag{20}$$

Note that the energy includes both kinetic and potential contributions. Some of the kinetic-energy terms involve an integral over $(\boldsymbol{\nabla}_1 f_{12})^2$. Consequently, if $f$ is a step function, then the kinetic and potential energies are both infinite; however, the sum of Eq. (19) is finite. For a many-particle system, $N\gg1$, the number of interacting triplets is given by $\frac{1}{6}N^3\varphi$. Here $\varphi$ denotes the fraction of triplets in which all three particles have different $\sigma_z$ and $\tau_z$ quantum numbers. For a total $S$, $T$ degeneracy $g$, we have

$$\varphi=(1-g^{-1})(1-2g^{-1})\,. \tag{21}$$

Thus for nuclear matter, $g=4$ and $\varphi=\frac{3}{8}$. (A boson gas corresponds to $g=\infty$ and $\varphi=1$, while for liquid He$^3$ or a neutron gas $g=2$ and $\varphi=0$; i.e., for the latter, it is not possible to form triplets of the kind discussed here.) Since $\psi$ depends only on interparticle spacings and not the absolute coordinates we can keep one of the coordinates fixed, and integrate first over the other two. The integral over the remaining coordinate, say $r_1$, is done last and gives a factor $\Omega$. We obtain

$$E^p/A=(\hbar^2/m)\tfrac{1}{8}\varphi\rho^2I^p\,, \tag{22}$$

where

$$\rho=A/\Omega=(2/3\pi^2)k_F^3 \tag{23}$$

is the particle density, and

$$I^p=-\int\!\!\int_{r_1=0}\boldsymbol{\nabla}_2(f_{12}^2)\cdot\boldsymbol{\nabla}_3(f_{13}^2)f_{23}^2d^3r_2d^3r_3\,. \tag{24}$$

This result coincides with the lowest term in the cluster expansion of Iwamoto and Yamada,[10] [when we assume that the two-particle correlations satisfy Eq. (18)]. If the $f$ are functions of interparticle spacings alone, the integral over angles can be done very simply and we obtain

$$I^p=4\pi^2\int_0^\infty\int_0^\infty\int_{|r_2-r_3|}^{r_2+r_3}\frac{df^2(r_2)}{dr_2}\frac{df^2(r_3)}{dr_3}f^2(r_{23})\times(r_{23}^2-r_2^2-r_3^2)r_{23}dr_{23}dr_2dr_3\,. \tag{25a}$$

For our calculations, we chose a correlation function of the form

$$\begin{aligned} f&=0\,, && r<c\\ &=(r-c)/(d-c)\,, && c<r<d\\ &=1\,, && d<r\,, \end{aligned} \tag{25b}$$

which is a reasonably good approximation to the correct correlation function.

In the limit that $d\to c$, i.e., if $f$ is a step function, the integration in Eq. (24) is trivial and we find

$$I_0^p=3\pi^2c^4\,. \tag{26}$$

For $d>c$, it is still convenient to express the integral $I^p$ in terms of $I_0^p$; thus

$$I^p=I_0^pR(\epsilon)\,, \tag{27}$$

where

$$d=c(1+\epsilon)\,. \tag{28}$$

The ratio $R$ can be evaluated analytically and it has a simple form if $\epsilon<1$, $(d\leqslant 2c)$, namely

$$R(\epsilon)=1+\frac{8}{3}\epsilon+\frac{23}{9}\epsilon^2+\frac{16}{15}\epsilon^3+\frac{1}{6}\epsilon^4\,. \tag{29}$$

The function $R$ was also calculated numerically, and it is found that an excellent approximation valid at least up to $\epsilon=10$ is

$$R_0(\epsilon)=(1+0.65\epsilon)^4\,. \tag{30}$$

Substituting into Eq. (22) we obtain the short-range three-body correlation energy $\mathcal{E}$ of nuclear matter (per particle):

$$\mathcal{E}_0^p=\frac{E^p}{A}=\frac{\hbar^2}{2m}k_F^2\frac{k_F^4}{8\pi^2}(0.35c+0.65d)^4\,. \tag{31}$$

For Wong's potential, a plausible choice of parameters to be chosen is $c=0.3$ F, $d=0.9$ F. Since the correlation function of Eq. (25a) has a discontinuous derivative at the separation distance, the latter must be slightly smaller than the usual separation distance $\sim$1.1 F in order to give a best fit to the exact short-range correla-

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[10] F. Iwamoto and M. Yamada, Progr. Theoret. Phys. (Kyoto) **19**, 345 (1957).

TABLE I. Various contributions to three-body correlation energy expressed in units of the energy defined by Eq. (31).

| $d/c$ | $I^p/I_0{}^p$ | $I_1{}^{AF}/I_0{}^p$ | $I_2{}^{AF}/I_0{}^p$ | $I^{AF}/I_0{}^p$ |
|---|---|---|---|---|
| 1 | 1.00 | 1.42 | 0.21 | 1.63 |
| 1.5 | 1.01 | 1.22 | 0.20 | 1.42 |
| 2 | 1.01 | 1.11 | 0.21 | 1.32 |
| 3 | 0.99 | 0.97 | 0.23 | 1.20 |
| 6 | 0.99 | 0.93 | 0.24 | 1.17 |

tion function.[11] At normal density, we obtain $\mathcal{E}\sim 0.4$ MeV which is fairly close to the value 0.6 MeV for the three-body correlation energy obtained by Bethe[3,8] using his improved approximate Faddeev wave function, and a different method of calculation.

## 6. VARIATIONAL METHOD

The form of the energy expectation value of Eq. (16) for a more general three-body wave function of the form

$$\psi=\psi^p Q(r_{12},r_{13},r_{23}) \tag{32a}$$

can be obtained without much difficulty, provided we assume $f$ and $Q\rightarrow 1$ at large interparticle spacing.

We obtain, instead of Eq. (24),

$$I^{AF}=I_1{}^{AF}+I_2{}^{AF}\,, \tag{32b}$$

where

$$I_1{}^{AF}=-\iint_{r_1=0} \nabla_2(f_2{}^2)\cdot\nabla_3(f_3{}^2)f_{23}{}^2Q^2d^3r_2d^3r_3 \tag{33a}$$

and

$$I_2{}^{AF}=2\iint_{r_1=0} f_2{}^2f_3{}^2f_{23}{}^2(\nabla_1 Q)^2d^3r_2d^3r_3\,. \tag{33b}$$

All cross terms involving $\nabla f\cdot\nabla Q$ vanish. For the product wave function, $Q=1$ and $I_2{}^{AF}=0$. However, for any other wave function, the integrand in Eq. (33b) is positive definite and thus $I_2$ must be positive. In addition, for the AF wave function we have $Q\geq 1$ everywhere. Thus the integrand in Eq. (33a) must be everywhere larger in magnitude than for the product wave function. Presumably then $I_1{}^{AF}$ should be also larger, though this is not necessarily the case, since the integrand can be either positive or negative.

A numerical evaluation shows, in fact, that $I_1{}^{AF}$ is larger than $I^p$ only for $\epsilon\leq 1$. However, the total $I^{AF}$ including both $I_1{}^{AF}$ and $I_2{}^{AF}$ is appreciably larger than $I^p$ for the whole range of values considered. Table I shows the various contributions to the three-body energy in units of the energy defined by Eq. (31).

It is clear that the approximate Faddeev wave function gives a substantially larger trial energy than the product wave function and thus it is presumably less accurate. This conclusion has been verified only for a correlation function of the form of Eq. (25b) but we see no reason why it should not hold in general.

Of course it cannot be concluded from the above argument that the product wave function gives the lowest possible variational energy. Note that Bethe uses an improvement over the simple AF wave function which implies a better treatment of the energy denominator. It is not inconceivable that this improved wave function might give a lower trial energy than the simple product wave function. However, this wave function, like the simple AF wave function, implies a $Q$ function which is larger than the product wave function when several particles are closer together. In contrast, more detailed calculations now in progress in collaboration with G. Thomas indicate that in some cases an even lower energy can be obtained assuming a $Q$ function which is *less* than unity when several particles are close together.

In any case the results of this paper support Bethe's conclusion that the short-range three-body correlation energy is quite small in nuclear matter and that it can be safely neglected in most nuclear structure calculations.

## ACKNOWLEDGMENTS

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[11] See Ref. 6, p. 76.