Generalized Faddeev Integral Equations for Multiparticle Scattering Amplitudes*

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The Faddeev integral equations for nonrelativistic three-body scattering amplitudes are generalized to apply to the multiparticle-scattering problem. The equations obtained are similar in structure to the N-body formulation of Weinberg and have the same desirable feature, namely, that δ-function singularities in the kernel due to disconnected processes have been removed, so that Fredholm theory may be expected to apply. A possible advantage of the present (Faddeev-type) approach to the N-body problem is that the potentials do not appear explicitly; they have been formally eliminated in favor of the two-body scattering amplitudes. In addition to the integral equations for the scattering amplitudes, in all channels, we obtain integral equations for the wave function and for the effective (or optical) potential. A minimum principle for the effective potential obtained previously for $N=3$ is generalized to arbitrary N. To illustrate the possible utility of these equations for a wider class of problems where potentials are not defined, we have shown that the Lee model in the three-particle sector can be reformulated in terms of the Faddeev equations, without introducing field operators.

1. INTRODUCTION

RECENTLY, Weinberg[1,2] has shown how multiparticle scattering theory can be properly formulated in terms of integral equations with square-integrable kernels. The attractive feature of this reformulation is the elimination of δ-function singularities, which are present in the kernel of the ordinary Lippmann-Schwinger equation,[3] and which arise from disconnected processes (where some of the particles do not interact). Thus Fredholm theory may be expected to apply.[4] We wish to point out a slight variation of the Weinberg formulation which has the property that the integral equations reduce, for the three-body problem, to the Faddeev equations[5] and afford a natural multiparticle generalization of these equations. One possible advantage of this alternative form is that while the interparticle potentials appear explicitly in the Weinberg equations they are, in the generalized Faddeev version, formally eliminated in favor of the scattering amplitudes for subsystems of particles. This may be useful in those cases where the potentials are singular while the scattering amplitudes show smoother behavior.[6] On a more speculative level, the appearance of scattering amplitudes rather than potentials would seem to allow a more straightforward generalization to the relativistic domain where the potential model does not apply. In an attempt to give this point additional credence it is shown, in Sec. 2, how the Lee model in the three-body (N-θ-θ, V-θ) sector[7,8] may be reformulated in terms of the Faddeev equations. The N-body integral equations are obtained in Sec. 2 and the complete S matrix is defined in Sec. 3 where, in addition, a proof of the unitarity relation is sketched. In Sec. 4 we obtain integral equations for the scattering and bound-state wave functions. Integral equations which determine the effective potential (and, in particular, the resonance states in the Feshbach picture[9]) are provided. It is shown how the effective potential may be calculated approximately with the aid of a minimum principle, which generalizes previous work on the three-body problem.[10,11]

2. THE INTEGRAL EQUATIONS

The problem under consideration is defined by the Lippmann-Schwinger equation

$$T(E)=V+VG_0(E)T(E), \tag{2.1}$$

where, with K representing the total-kinetic-energy operator, we have

$$G_0(E)=(E+i\eta-K)^{-1}, \quad \eta\to 0+, \tag{2.2}$$

and

$$V=\sum_i V_i, \tag{2.3}$$

the sum running from 1 through $N(N-1)/2$. Here V_i is the interaction potential between the particles in the ith pair; we assume that the $N(N-1)/2$ pairs have been ordered in some manner. It is our object to rewrite Eq. (2.1) in a form in which the kernel is connected and the potentials have been formally eliminated. As a first

* Supported by the National Science Foundation.
[1] S. Weinberg, Phys. Rev. **133**, B232 (1964).
[2] See also, R. Sugar and R. Blankenbecler, Phys. Rev. **136**, B472 (1964).
[3] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[4] It has been shown by W. Hunziker, Phys. Rev. **135**, B800 (1964) that the kernel in the Weinberg integral equation is of the Hilbert-Schmidt type provided the potentials are square-integrable. Presumably an analogous result can be obtained for the generalized Faddeev equations, although we shall not attempt to do so here. For a treatment of the three-body case see Lovelace, Ref. 16.
[5] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].
[6] K. A. Brueckner, Phys. Rev. **97**, 1353 (1955).
[7] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[8] G. Källen and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd **30**, No. 7 (1955).
[9] H. Feshbach, Ann. Phys. (N. Y.) **5**, 357 (1958); **19**, 287 (1962).
[10] L. Rosenberg, Phys. Rev. **138**, B1343 (1965).
[11] Minimum principles closely related to the version given in Ref. 10 were presented earlier by Sugar and Blankenbecler (Ref. 2) and by Y. Hahn, T. F. O'Malley, and L. Spruch, Phys. Rev. **134**, B911 (1964). We find it convenient to refer to the formalism of Ref. 10 in discussing the N-body generalization.

step we introduce operators ${}^{(i)}T(E)$ defined by

$$ {}^{(i)}T = V_i + V_i G_0 T, \quad i=1, 2, \cdots, N(N-1)/2. \tag{2.4} $$

We may conclude, after summing over i and comparing the result with Eq. (2.1), that

$$ \sum_i {}^{(i)}T = T. \tag{2.5} $$

Our next step is to eliminate the potentials in favor of the two-body T operators defined as the solutions of

$$ T_i = V_i + V_i G_0 T_i. \tag{2.6} $$

In fact we assert that if the ${}^{(i)}T$ satisfy the set of equations

$$ {}^{(i)}T = T_i + T_i G_0 \sum_{j\neq i} {}^{(j)}T, \quad i=1,2,\cdots,N(N-1)/2, \tag{2.7} $$

then T, determined from Eq. (2.5) will satisfy the Lippmann-Schwinger equation.[12] This is easily verified by writing Eq. (2.7) as

$$ \begin{aligned} {}^{(i)}T &= V_i + V_i G_0 T_i + (V_i + V_i G_0 T_i) G_0 \sum_{j\neq i} {}^{(j)}T \\ &= V_i + V_i G_0 (T_i + T_i G_0 \sum_{j\neq i} {}^{(j)}T + \sum_{j\neq i} {}^{(j)}T) \\ &= V_i + V_i G_0 T, \end{aligned} \tag{2.8} $$

in agreement with Eqs. (2.4). Clearly, we could also have obtained the set of equations

$$ T^{(i)} = T_i + \sum_{j\neq i} T^{(j)} G_0 T_i \tag{2.9} $$

with $T = \sum_i T^{(i)}$. For $N=3$, Eqs. (2.7) are just the Faddeev equations.[5] However, for $N>3$ this form is no longer satisfactory since the disconnected processes have not all been separated out. Consequently, the kernel contains δ-function singularities which prevent it from being square-integrable, and additional partial summations must be introduced.[12a]

Before discussing the general case it will be convenient to cast the three-body problem into a form which facilitates generalization to higher N. In addition, the new form has some practical advantages which will become apparent. We seek an integral equation to determine the matrix operator $\mathbf{T}$, with elements ${}^{(i)}T^{(j)}$, such that

$$ {}^{(i)}T \equiv \sum_{j=1}^{3} {}^{(i)}T^{(j)}, \quad i=1, 2, 3, \tag{2.10} $$

satisfies Eqs. (2.7). The elements ${}^{(i)}T^{(j)}$ may, roughly speaking, be viewed as the sum of all multiple scattering processes which involve an interaction between the pair j initially and the pair i finally. We look for $\mathbf{T}$ in the form

$$ \mathbf{T} = \mathbf{T}^D + \mathbf{T}^C, \tag{2.11} $$

where the disconnected part $\mathbf{T}^D$ is defined as a diagonal matrix with elements ${}^{(i)}T^{(j)D} = T_i \delta_{ij}$. In addition, we define matrices $\mathbf{I}$ and $\mathbf{B}$ according to

$$ {}^{(i)}I^{(j)} = T_i(1-\delta_{ij}), \tag{2.12} $$

and

$$ \mathbf{B} = \mathbf{I}\mathbf{G}_0\mathbf{T}^D, \tag{2.13} $$

with $\mathbf{G}_0 = G_0\mathbf{1}$. If we rewrite Eq. (2.7) in terms of ${}^{(i)}T^C = {}^{(i)}T - T_i$ we get

$$ {}^{(i)}T^C = T_i G_0 \sum_{j\neq i} T_j + T_i G_0 \sum_{j\neq i} {}^{(j)}T^C. \tag{2.7'} $$

The integral equation for the matrix $\mathbf{T}^C$ is then seen to be

$$ \mathbf{T}^C = \mathbf{B} + \mathbf{I}\mathbf{G}_0\mathbf{T}^C, \tag{2.14} $$

since it gives rise to Eq. (2.7') upon summation over all columns, for the ith row. According to Eqs. (2.11) and (2.13) we also have

$$ \mathbf{T}^C = \mathbf{I}\mathbf{G}_0\mathbf{T}. \tag{2.15} $$

In a similar way, starting with Eqs. (2.9), we obtain

$$ \mathbf{T}^C = \mathbf{T}\mathbf{G}_0\mathbf{I}^{\dagger(-)}, \tag{2.16} $$

where

$$ T^{(i)} = \sum_{j=1}^{3} {}^{(j)}T^{(i)}. \tag{2.17} $$

The dagger in Eq. (2.16) implies that the matrix adjoint, as well as the operator adjoint of each element, is to be taken; the $(-)$ symbol instructs us to allow the energy to approach the real axis from below rather than from above. In writing Eq. (2.16) in the form shown we have used the reciprocity relation $T_i^{\dagger(-)} = T_i$, which implies, furthermore, that

$$ \mathbf{B} = \mathbf{T}^D\mathbf{G}_0\mathbf{I}^{\dagger(-)}. \tag{2.18} $$

Thus, Eq. (2.16) may be written as

$$ \mathbf{T}^C = \mathbf{B} + \mathbf{T}^C\mathbf{G}_0\mathbf{I}^{\dagger(-)}. \tag{2.19} $$

Equations (2.14) and (2.16) lead to the representation

$$ \mathbf{T}^C = \mathbf{B} + \mathbf{I}\mathbf{G}_0\mathbf{T}\mathbf{G}_0\mathbf{I}^{\dagger(-)}. \tag{2.20} $$

It is seen that the three-body reciprocity relation takes the form

$$ \mathbf{T}^{\dagger(-)} = \mathbf{T}. \tag{2.21} $$

Equations (2.15), (2.16), and (2.20) have the nice feature that initial- and final-state interactions are displayed explicitly. We will see below that the N-body equations can be cast in precisely the same form.

In order to illustrate the application of the above formalism to a simple case suppose that each of the two-body scattering operators have momentum-space

[12] No attempt is made to show a one-to-one correspondence between solutions of the Lippmann-Schwinger equation and the generalized Faddeev equation. Indeed, solutions of the Lippmann-Schwinger equation will not be unique (see Ref. 5).

[12a] Of course, Eqs. (2.7) will still be useful, as they stand, in a wide variety of circumstances. Indeed, an elaborate theory of multiple scattering processes, based on equations formally equivalent to Eqs. (2.7), has been developed and applied by K. M. Watson. In particular, see the second of Eqs. (26) in K. M. Watson, Phys. Rev. **103**, 489 (1956).

matrix elements (in the two-body center-of-mass system) of the separable form

$$t_i(\mathbf{k},\mathbf{k}';E)=g_i(\mathbf{k})[S_i(E)/(E+\epsilon_i)]g_i(\mathbf{k}'), \quad (2.22)$$

where $S_i(E)$ contains the branch cut demanded by unitarity, and $S_i(-\epsilon_i)=1$, with $g_i(\mathbf{k})$ chosen so that the residue at the bound-state pole at $E=-\epsilon_i$ is given correctly. We then write, for $\{i,j,k\}$ some permutation of $\{1, 2, 3\}$,

$$\langle \mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3|\,{}^{(i)}T^{(j)C}(E)\,|\,\mathbf{p}_1',\mathbf{p}_2',\mathbf{p}_3'\rangle = g_i(\mathbf{k}_{jk})\frac{S_i(E-k_i^2/2\mu_i)}{E-k_i^2/2\mu_i+\epsilon_i}h_{ij}(\mathbf{k}_i,\mathbf{k}_j';E) \times\frac{S_j(E-k_j'^2/2\mu_j)}{E-k_j'^2/2\mu_j+\epsilon_j}g_j(\mathbf{k}_{ik}')\delta(\textstyle\sum_i \mathbf{p}_i-\sum_i \mathbf{p}_i'), \quad (2.23)$$

where

$$\mathbf{k}_i=\mu_i\left[\frac{\mathbf{p}_i}{m_i}-\frac{\mathbf{p}_j+\mathbf{p}_k}{m_j+m_k}\right]; \quad \mu_i=\frac{m_i(m_j+m_k)}{m_i+m_j+m_k}, \quad (2.24)$$

and

$$\mathbf{k}_{jk}=\mu_{jk}[\mathbf{p}_j/m_j-\mathbf{p}_k/m_k]; \quad \mu_{jk}=m_jm_k/(m_j+m_k). \quad (2.25)$$

The matrix element h_{ij} can be identified with the continuation off the energy shell of the amplitude for scattering in which the pair j is bound initially and the pair i is bound finally. This identification follows from the rule, derived in Sec. 3, for obtaining the bound-state scattering amplitudes by taking the residues at the poles in the two-body initial- and final-state interactions.[13] We introduce a representation similar to that of Eq. (2.23) for matrix elements of the Born term **B**, with **h** replaced by **b** where, for $i\neq j$,

$$b_{ij}(\mathbf{k}_i,\mathbf{k}_j';E)=\frac{g_i(\mathbf{k}_j'+[m_j/(m_j+m_k)]\mathbf{k}_i)g_j(-\mathbf{k}_i-[m_i/(m_i+m_k)]\mathbf{k}_j')}{E-(k_j')^2/2m_j-(\mathbf{k}_i+\mathbf{k}_j')^2/2m_k-k_i^2/2m_i}, \quad (2.26)$$

and $b_{ij}=0$ for $i=j$. Then Eq. (2.14) simplifies to

$$h_{ij}(\mathbf{k}_i,\mathbf{k}_j';E)=b_{ij}(\mathbf{k}_i,\mathbf{k}_j';E)+\sum_{k=1}^{3}\int d\mathbf{k}b_{ik}(\mathbf{k}_i,\mathbf{k};E)\frac{S_k(E-k^2/2\mu_k)}{E-k^2/2\mu_k+\epsilon_k}h_{kj}(\mathbf{k},\mathbf{k}_j';E). \quad (2.27)$$

Thus the scattering problem has reduced, with the assumption of Eq. (2.22), to a multichannel two-body problem [with an effective potential **b** and a propagator modified by the factor $S(E)$] which can be easily solved numerically. If the two-body T matrix is written as a sum of terms of the type shown in Eq. (2.22), taking into account a number of bound states and resonances, the approximation retains this simplifying feature. The only additional complexity introduced is that the number of coupled equations to be solved is increased.

It has been noted that potentials which are separable in momentum space lead to T matrices of the type shown in Eq. (2.22), with a consequent simplification of the three-body equations.[14–18] Now an energy-dependent effective potential of the form

$$v(\mathbf{k},\mathbf{k}';E)=[g_0^2/(2\pi)^3][\Gamma(\mathbf{k})\Gamma(\mathbf{k}')/(E-E^{(0)})], \quad (2.28)$$

which arises from the simplest type of resonance approximation,[9] is in the above-mentioned class. It may be of some interest to observe that it is possible to reformulate the Lee model, in the N-θ and V-θ sectors, in terms of the two-body potential, Eq. (2.28), and the three-body Faddeev equations. Thus, the scattering amplitude associated with this potential is

$$t(\mathbf{k},\mathbf{k}';E)=[g_0^2/(2\pi)^3][u(\omega_k)u(\omega_{k'})/(2\omega_k2\omega_{k'})^{1/2}] \times\left[E-E^{(0)}-\frac{g_0^2}{4\pi^2}\int_\mu^\infty\frac{d\omega(\omega^2-\mu^2)^{1/2}u^2(\omega)}{E-\omega}\right]^{-1}, \quad (2.29)$$

where we have taken

$$\omega_k^2=k^2+\mu^2, \quad (2.30)$$

and

$$\Gamma(\mathbf{k})=u(\omega_k)/(2\omega_k)^{1/2}. \quad (2.31)$$

We now assume that a solution E_V of

$$E_V=E^{(0)}+\frac{g_0^2}{4\pi^2}\int_\mu^\infty\frac{d\omega(\omega^2-\mu^2)^{1/2}u^2(\omega)}{E_V-\omega} \quad (2.32)$$

exists with $E_V<\mu$. Then Eq. (2.29) can be written as

$$t(\mathbf{k},\mathbf{k}';E)=[g^2/(2\pi)^3][1/(2\omega_k2\omega_{k'})^{1/2}] \times[u(\omega_k)S(E)u(\omega_{k'})/(E-E_V)], \quad (2.33)$$

with

$$[S(E)]^{-1}=1-\frac{g^2}{(2\pi)^2}(E-E_V) \times\int_\mu^\infty\frac{d\omega(\omega^2-\mu^2)^{1/2}u^2(\omega)}{(E_V-\omega)^2(E-\omega)}. \quad (2.34)$$

[13] After arriving at this rule we noticed a remark by Lovelace (see Ref. 16) to the effect that Faddeev (in a work not available to us) has proposed the same rule. We give the N-body generalization in Sec. 3.

[14] A. N. Mitra, Nucl. Phys. **32**, 529 (1962). The Faddeev equations were not used here.

[15] R. D. Amado, Phys. Rev. **132**, 485 (1963). Amado's results were obtained using field-theoretic methods, and extend beyond the model of pure potential scattering. These more general results can nevertheless be obtained directly from the Faddeev equations, as we show below.

[16] C. Lovelace, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Plenum Press, Inc., New York, 1964); Phys. Rev. **135**, B1225 (1964).

[17] L. Rosenberg, Phys. Rev. **134**, B937 (1964).

[18] L. Rosenberg, Phys. Rev. **135**, B715 (1964).

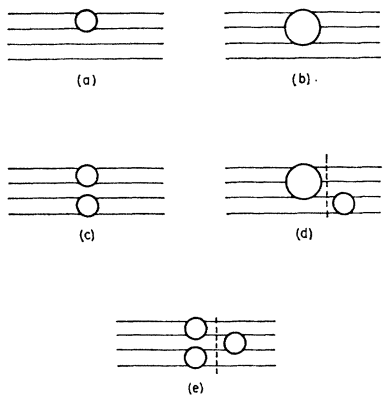

FIG. 1. Diagrams (a), (b), and (c) contribute to the disconnected part of the four-body amplitude, and (d) and (e) show leading contributions to the connected part. The portions of diagrams (d) and (e) which lie to the left of the vertical dashed lines contribute to the "almost connected" amplitude defined by Eq. (2.42). The circles denote T operators for two- and three-particle systems.

Here $g^2 \equiv Zg_0^2$ with

$$Z=1-Z\frac{g_0^2}{(2\pi)^2}\int_\mu^\infty \frac{d\omega(\omega^2-\mu^2)^{1/2}u^2(\omega)}{(E_V-\omega)^2}. \quad (2.35)$$

Equation (2.33) is just the N-θ scattering amplitude in the Lee model. Proceeding to the N-θ-θ sector, and taking the θ particles to be identical bosons, we find that the appropriate sum, $h=h_{11}+h_{12}$, of direct and exchange amplitudes satisfies a single-channel version of Eq. (2.27). Specifically, with

$$b(\mathbf{k},\mathbf{k}';E)=[g^2/(2\pi)^3][1/(2\omega_k 2\omega_{k'})^{1/2}] \times[u(\omega_k)u(\omega_{k'})/(E-\omega_k-\omega_{k'})], \quad (2.36)$$

the Faddeev equations reduce to

$$h(\mathbf{k},\mathbf{k}';E)=b(\mathbf{k},\mathbf{k}';E)+\int d\mathbf{k}''b(\mathbf{k},\mathbf{k}'';E) \times[S(E-\omega_{k''})/(E-\omega_{k''}-E_V)]h(\mathbf{k}'',\mathbf{k}';E). \quad (2.37)$$

The amplitude h, which would correspond to bound-state scattering in a potential model ($Z=0$[15,19]), is now identified with the V-θ amplitude in this generalized model. In fact, Eq. (2.37) is equivalent to the Källen-Pauli[8] integral equation for the state vector and has recently been solved directly in closed form.[20] We wish to emphasize what seems to us to be the significant point of this exercise, namely, that T-matrix integral equations which do not involve the potential explicitly may be useful even when the potential picture breaks down.[15,16,18] To explore the utility of this approach in describing strongly interacting particles it is necessary to obtain the N-body generalization of the Faddeev equations and this we now proceed to do.

The problem of generalizing the Faddeev equations resolves itself into a determination of the matrix operators $\mathbf{T}^D$ and $\mathbf{I}$ introduced above. For the sake of orientation we first consider the case $N=4$. In Figs. 1(a)–(c) we have pictured the three types of disconnected contributions to the four-body amplitude. Types (a) and (b) have already been defined in our discussion of the three-body problem. For example, a diagram of type (b) will arise as the sum of a subclass of terms in the iterative solution of Eqs. (2.7) in which the interactions involving one of the four particles do not appear. These terms may be summed formally by solving a set of equations which reduce to the three-body equations already considered after the momentum-conserving δ function for the noninteracting particle has been factored out. Diagrams of type (c) are new and we must give a rule for calculating them. Consider all the ways one can partition the system S into two subsystems S_1 and S_2 such that each subsystem contains two particles. For each partition we define an amplitude $T(S_1;S_2)$ as the solution of a set of integral equations obtained from Eqs. (2.7) by dropping the interactions between those pairs of particles which have one member of the pair in S_1 and the other member in S_2; the two systems are disconnected. Now $T(S_1;S_2)$ itself contains disconnected diagrams, of the type shown in Fig. 1(a). When these are subtracted off we obtain an amplitude $T^C(S_1;S_2)$ which is connected, relative to the particular partition, in the sense that only the factorizable δ function associated with conservation of the momentum of the total system S and of each subsystem S_1 and S_2 appear in the matrix element of the kernel. When these δ functions are factored out the resulting integral equation for $T^C(S_1;S_2)$ is of the proper type and we may consider these amplitudes to be well defined.[4] More explicitly, suppose S_1 consists of the pair 1 and S_2 consists of the pair 2. Then $T(S_1;S_2)={}^{(1)}T(S_1;S_2)+{}^{(2)}T(S_1;S_2)$ with

$$\begin{aligned}{}^{(1)}T(S_1;S_2)&=T_1+T_1G_0{}^{(2)}T(S_1;S_2),\\ {}^{(2)}T(S_1;S_2)&=T_2+T_2G_0{}^{(1)}T(S_1;S_2).\end{aligned} \quad (2.38)$$

The proper integral equations for $T^C(S_1;S_2)$ are obtained by replacing ${}^{(i)}T(S_1;S_2)$ with

$${}^{(i)}T(S_1;S_2)={}^{(i)}T^C(S_1;S_2)+T_i, \quad i=1,2, \quad (2.39)$$

in Eqs. (2.38). Clearly, there are three such connected amplitudes, corresponding to the three different ways of pairing off the four particles, which make up the class of diagrams of Fig. 1(c). There are four different three-body connected amplitudes [Fig. 1(b)] which also arise from partitions of the system S into two subsystems (with three particles in one subsystem and one particle in the other subsystem) and six different two-body amplitudes [Fig. 1(a)] arising from a parti-

[19] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

[20] C. M. Sommerfield, Bull. Am. Phys. Soc. **10**, 61 (1965); T. L. Trueman, Phys. Rev. **137**, B1566 (1965). These solutions are equivalent to the earlier results, obtained with different techniques, by R. D. Amado, Phys. Rev. **122**, 696 (1961); R. P. Kenschaft and R. D. Amado, J. Math. Phys. **5**, 1340 (1964).

tion of S into three disconnected subsystems. The sum of these 13 amplitudes is denoted by $T^D(S)$. Note that the use of the *connected* parts of the amplitudes of Figs. 1(b) and (c) is required in order to avoid overcounting contributions of the type shown in Fig. 1(a).

A quite similar cluster-decomposition analysis can be made for the N-body amplitude provided we have first solved the $(N-1)$-body problem. Let us assume this has been done and argue inductively. We write the N-body amplitude as

$$T(S)=T^D(S)+T^C(S), \tag{2.40}$$

where, as in the four-body case just discussed, $T^D(S)$ can be written as a sum of amplitudes describing interactions within disconnected subsystems. To avoid overcounting, the sum is expressed in terms of the connected parts of these amplitudes. In an obvious generalization of the notation introduced above for $N=4$ the expanded form of Eq. (2.40) becomes

$$T(S)=\sum_{m=1}^{N-1}\sum_{S_1,S_2,\cdots,S_m}^{(S)} T^C(S_1;S_2;\cdots;S_m). \tag{2.41}$$

The sum is over all distinct partitions of the system S into subsystems S_i such that each contains at least one particle. For each partition the amplitude is defined such that particles within S_i interact with each other, but not with particles in other subsystems. The term with $m=1$ is just $T^C(S)$, and we have omitted the term $m=N$ since this diagram contributes to the S matrix but clearly not to the T matrix.

One class of contributions to $T^D(S)$ which is important in the following analysis is the set of "almost connected" amplitudes. Thus, we let the index k denote the pair of particles i and j and define

$$I^{(k)}=\sum_{S(i),S(j)}^{(S)} T^C[S(i);S(j)], \tag{2.42}$$

where the sum is over all distinct partitions of S into two subsystems, such that particle i belongs to one subsystem and particle j to the other. Contributions to $T^C(S)$ arise when a process described by $I^{(k)}$ is followed by an interaction between particles i and j. For example, Figs. 1(d) and (e) show leading contributions to the four-body connected amplitude. The "almost connected" amplitude is pictured to the left of the vertical dashed line.

To verify the cluster-decomposition formula, Eq. (2.41), and to deduce the proper integral equation for $T^C(S)$ we study the form of an iterative solution of Eq. (2.7). That is, we consider a sequence of transformations obtained from the repeated replacement

$${}^{(i)}T\rightarrow T_i+T_iG_0\sum_{j\neq i}{}^{(j)}T \tag{2.43}$$

on the right-hand side of the integral equation. At each stage in this sequence the inhomogeneous part of the integral equation will contain a set of disconnected terms belonging to $T^D(S)$. In fact it is easy to see that each one of these terms can be *uniquely* associated with an identical term in the multiple-scattering expansion of one of the amplitudes $T^C(S_1;S_2;\cdots;S_m)$ of Eq. (2.41). That is, the validity of the cluster-decomposition formula for $T^D(S)$ is verified in the sense that it leads to the correct multiple-scattering expansion. Presumably this result could be obtained without resorting to multiple-scattering expansions.

Now suppose we have gone out sufficiently far in the sequence of transformations such that terms belonging to the almost connected amplitudes $I^{(k)}$ appear in the inhomogeneous part. Then the replacement, Eq. (2.43), is to be made for a particular term only if it adds to the representation of $T^D(S)$ in the inhomogeneous part. Otherwise that term is to be left alone; it contributes to the connected part of the amplitude. In this way we arrive at the representation

$${}^{(i)}T^C(S)=\sum_k {}^{(i)}I^{(k)}(S)G_0{}^{(k)}T(S). \tag{2.44}$$

At any stage in the sequence, of course, ${}^{(i)}I^{(k)}$ will be represented by a finite number of terms in its multiple-scattering expansion. We shall assume that Eq. (2.44), as well as Eq. (2.41), is rigorously valid. By combining Eqs. (2.44) and (2.40) we obtain an integral equation of the form shown in Eq. (2.14), as well as the representations, Eqs. (2.16) and (2.19), where now the matrices have dimension $N(N-1)/2$. According to our inductive argument the amplitudes $T^C(S_1;S_2;\cdots;S_m)$ are to be determined by integral equations similar in form to Eq. (2.14) for $T^C(S)$ itself, with interactions between particles in different subsystems dropped. The basic input is the set of two-body amplitudes T_i defined by Eq. (2.6).

3. CONSTRUCTION OF THE UNITARY S-MATRIX

Momentum-space matrix elements of the operator $T(S)$ provide the amplitudes for scattering between initial and final states in which none of the particles are bound. To obtain the complete S matrix we must define the operators which describe bound-state scattering, i.e., scattering in which groups of particles are bound in initial and final states. One way of proceeding would be to construct the full Green's function

$$G=G_0+G_0TG_0, \tag{3.1}$$

from which all the elements $T_{\alpha\beta}$ of the center-of-mass T matrix may be obtained using

$$T_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta;E)=(\Phi_\alpha,[H-E]\Phi_\beta)+([H-E]\Phi_\alpha,G(E)[H-E]\Phi_\beta). \tag{3.2}$$

Here H is the Hamiltonian of the system and Φ_α is the "plane wave" describing the free relative motion of the

bound subsystems in channel α. Specifically, we have

$$\Phi_\alpha(\varrho_\alpha,\mathbf{r}_\alpha)=X_\alpha(\varrho_\alpha)\exp(i\mathbf{k}_\alpha\cdot\mathbf{r}_\alpha), \quad (3.3)$$

where $X_\alpha(\varrho_\alpha)$ is the product of bound-state wave functions. We use the center-of-mass hyperspace coordinates defined previously.[17] In particular, $\mathbf{k}_\alpha=k_\alpha\hat{k}_\alpha$ is the hyperspace momentum vector conjugate to $\mathbf{r}_\alpha$, and related to the total energy through

$$E=-\epsilon_\alpha+(\hbar^2/2m_\alpha)k_\alpha^2. \quad (3.4)$$

The mass parameter m_α is chosen, for each channel α, such that the transformation to hyperspace coordinates[17] has unit Jacobian; ϵ_α is the sum of the binding energies of the subsystems. When identical particles are present the physical amplitudes are obtained by taking the appropriate linear combinations of the $T_{\alpha\beta}$, in the usual way.

We observe that an alternative procedure for obtaining the amplitudes $T_{\alpha\beta}$ can be devised which does not involve the potentials explicitly and is therefore more in keeping with the spirit of the present approach. Thus, let $\Psi_\beta^{(+)}$ represent a solution of the Schrödinger equation corresponding to an incoming "plane wave" in channel β. Then the amplitude of the outgoing scattered wave in channel α, appropriately normalized, gives us the (on-shell) scattering amplitude $T_{\alpha\beta}^{(+)}$ of interest.[17,21] Now Ψ_β may be constructed from[22]

$$\Psi_\beta=\Phi_\beta+G(H-E)\Phi_\beta, \quad (3.5)$$

which may be written in the form

$$\Psi_\beta=\lim_{\eta\to 0+} i\eta G(E+i\eta)\Phi_\beta, \quad (3.6)$$

with G given by Eq. (3.1). Consequently, the amplitudes $T_{\alpha\beta}$ may be computed once the operators ${}^{(i)}T^{(j)}$ (introduced in Sec. 2) are known; no further integral equations need be solved and the potentials are not involved explicitly. The unitarity relations satisfied by the $T_{\alpha\beta}$ can be derived directly from the identity

$$0=(\Psi_\alpha^{(+)},[H-E]\Psi_\beta^{(+)})-(\Psi_\beta^{(+)},[H-E]\Psi_\alpha^{(+)})^*. \quad (3.7)$$

The right-hand side may be reduced to a surface integral using the hyperspace generalization of Green's theorem. The methods of Ref. 17 allow us to evaluate these surface terms. We are then led to the relations

$$T_{\alpha\beta}(\mathbf{k}_\alpha,\mathbf{k}_\beta)-T_{\beta\alpha}^*(\mathbf{k}_\beta,\mathbf{k}_\alpha)=-2\pi i\sum_\gamma\int d\mathbf{k}_\gamma \times T_{\gamma\alpha}^*(\mathbf{k}_\gamma,\mathbf{k}_\alpha)T_{\gamma\beta}(\mathbf{k}_\gamma,\mathbf{k}_\beta)\delta(E_\gamma-E), \quad (3.8)$$

where $\sum_\gamma$ is a sum over *all* possible intermediate states. It should be emphasized that had we used the usual form of the Lippmann-Schwinger equation,[3] rather than Eq. (2.14), the asymptotic form of the wave function Ψ_β would *not* show outgoing waves in all possible exit channels and the unitarity relation would *not* have been obtained in the complete form displayed in Eq. (3.8). From Eq. (3.8) it follows that the S matrix, with elements

$$S_{\alpha\beta}=\delta_{\alpha\beta}-2\pi i\delta(E_\alpha-E_\beta)\delta(\mathbf{K}_\alpha-\mathbf{K}_\beta)T_{\alpha\beta} \quad (3.9)$$

($\mathbf{K}$ is the total momentum) is guaranteed to be unitary, as is easily demonstrated.

To put the above rule for calculating the amplitudes $T_{\alpha\beta}$ in more specific form we consider the three-body case in greater detail. It will then be possible to give a concise prescription for treating the general case. Let $\Phi_{j(n)}$, $j=1, 2, 3$ be the "plane wave" in which the pair j is bound in a state denoted by the index n; Φ_0 represents the state in which all three particles are free. We also define

$$G_j=[E-K-V_j+i\eta]^{-1}, \quad j=0, 1, 2, 3, \quad (3.10)$$

with $V_0\equiv 0$. The relation

$$\lim_{\eta\to 0+} i\eta G_j\Phi_i=\Phi_i\delta_{ij}, \quad i,j=0, 1, 2, 3, \quad (3.11)$$

then follows directly from the eigenfunction expansion of G_j and the observation that for $i\neq j$ $G_j\Phi_i$ is not singular in the limit so that there is no contribution.[5] With $i=j=0$ Eq. (3.6) leads, in conjunction with Eqs. (3.1) and (3.11), to

$$\Psi_0=\Phi_0+G_0T\Phi_0. \quad (3.12)$$

For $i\neq 0$ we use Eq. (2.9) to write

$$G_0TG_0=G_0\sum_k[T_k+\sum_{j\neq k}T^{(j)}G_0T_k]G_0. \quad (3.13)$$

We may replace $G_0T_kG_0=G_k-G_0$ by G_k since the neglected term G_0 does not contribute in the limit. We then have

$$\Psi_i=\Phi_i+\sum_{j\neq i}G_0T^{(j)}\Phi_i, \quad (3.14)$$

which, in view of Eq. (3.12), holds for $i=0, 1, 2, 3$.

As mentioned above, we can study the asymptotic form of Ψ_i in various regions of configuration space to identify the scattering amplitudes. Consider first the contribution coming from the region where all three interparticle distances tend to infinity (exit channel $j=0$). In the notation of Ref. 17 we have, with $\mathbf{r}$ denoting the hyperspace position vector in the center-of-mass system,

$$\langle\mathbf{r}|G_0(E)|\mathbf{r}'\rangle=(2m/\hbar^2)C(k,p)r^{-p/2}H_{p/2}^{(1)}(k|\mathbf{r}-\mathbf{r}'|) \underset{r\to\infty}{\longrightarrow}(2m/\hbar^2)C(k,p)r^{-(p+1)/2}(2/k\pi)^{1/2} \times\exp\{i[kr-\tfrac{1}{4}(p+1)\pi]\}\exp(-ik\hat{r}\cdot\mathbf{r}') \quad (3.15)$$

[21] E. Gerjuoy, Phys. Rev. **109**, 1806 (1958); Ann. Phys. (N. Y.) **5**, 58 (1958).

[22] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

with $p=4$ in the case at hand and $p=3n-5$ in the general case where there are n noninteracting systems, $n\geq 2$. The total kinetic energy is $E=(\hbar^2/2m)k^2$, where m is the scaling parameter defined below Eq. (3.4). The normalization factor $C(k,p)$ guarantees the unit source condition for G_0.[23] From an examination of the amplitude of the scattered wave we may infer that

$$T_{0i}=(\Phi_0,\mathcal{T}_{0i}\Phi_i), \tag{3.16}$$

with

$$\mathcal{T}_{0i}=\sum_{j\neq i} T^{(j)}. \tag{3.17}$$

The amplitude for the inverse reaction can be obtained from the reciprocity relation[22] which in general form is[17]

$$T_{\beta\alpha}^{(-)*}(-\mathbf{k}_\beta,-\mathbf{k}_\alpha)=T_{\alpha\beta}^{(+)}(\mathbf{k}_\alpha,\mathbf{k}_\beta). \tag{3.18}$$

We infer that $\mathcal{T}_{i0}=\mathcal{T}_{0i}^{\dagger(-)}$, so that

$$\mathcal{T}_{i0}=\sum_{j\neq i} {}^{(j)}T, \tag{3.19}$$

using Eq. (2.21). Accordingly, we will only concern ourselves with entrance channels $i=1$, 2, or 3 in the following.

We now examine the form of Ψ_i in the asymptotic region where the relative position vector ϱ_j for the particles in the pair j is of finite length. Owing to the $r^{-5/2}$ dependence of the outgoing wave in channel 0 we may neglect the overlap of this channel and consider only the contribution from exit channel j, with $j=1$, 2, or 3. We therefore write Eq. (3.14) as

$$\Psi_i=\Phi_i+G_0\{T_j(1-\delta_{ij})+\sum_{k\neq i} {}^{(j)}T^{(k)C}+\cdots\}\Phi_i, \tag{3.20}$$

where the terms not explicitly included do not contribute to the scattering into exit channel j. Consider first the term

$$G_0T_j\Phi_i=G_jV_j\Phi_i, \quad i\neq j, \tag{3.21}$$

which is the Born contribution to the rearrangement amplitude. We require the representation

$$\langle\mathbf{r}_j,\varrho_j|G_j(E)|\mathbf{r}_j',\varrho_j'\rangle = \mathbf{S}_n\langle\mathbf{r}_j|G_0(E-\epsilon_{j(n)})|\mathbf{r}_j'\rangle X_{j(n)}(\varrho_j)X_{j(n)}^*(\rho_j'), \tag{3.22}$$

where $\mathbf{S}_n$ indicates a sum over discrete states and an integration over continuum states, and $\mathbf{r}_j$ is the distance between the center of mass of the pair j and the third particle. The generalization of Eq. (3.22) to the case where there are n noninteracting subsystems (rather than two, as above) is directly obtained with the aid of Eq. (3.15). Passing to the limit $r_j\to\infty$, with $\rho_j<\infty$, we immediately obtain the Born contribution

$$b_{ji}=(\Phi_j,V_j\Phi_i). \tag{3.23}$$

If one makes use of the Schrödinger equation for the bound-state wave function $X_{j(n)}$ the Born term can be put in a form which has no explicit dependence on the potential V_j [see Eq. (2.26)]. To obtain the remainder of the amplitude we use Eqs. (2.15) and (3.21) to write

$$G_0\sum_{k\neq i} {}^{(j)}T^{(k)C}\Phi_i=G_jV_jG_0\sum_{k\neq i}\sum_{l\neq j} {}^{(l)}T^{(k)}\Phi_i. \tag{3.24}$$

The representation given in Eq. (3.22), along with the eigenvalue equation

$$G_0V_j\Phi_j=\Phi_j \tag{3.25}$$

(we actually require the adjoint of this equation) which is valid on the energy shell, enables us to deduce the result

$$T_{ji}=b_{ji}+(\Phi_j,[\sum_{k\neq i}\sum_{l\neq j} {}^{(l)}T^{(k)}]\Phi_i) \tag{3.26}$$

for the rearrangement amplitude. With b_{ji} defined to be zero for the remaining elements of the T matrix, Eq. (3.26) gives the complete set of transition amplitudes.

It is possible to state the above prescription for obtaining the T matrix in a more concise but equivalent manner which holds as well for the general case where N is arbitrary. Consider the connected part of the N-body scattering operator in the representation given by Eq. (2.20). The connected part of the on-shell T-matrix element for the case where none of the particles are bound in initial or final states is obtained by taking a matrix element of this form with respect to states in momentum space corresponding to the energy E. (The general rule for continuing the T-matrix elements off the energy shell is quite simple and will be stated below.) Now suppose we are interested in the amplitude for scattering into an exit channel, denoted by the subscript α, in which subsystems S_1, S_2, $\cdots$, S_m are bound with total eigen-energy $-\sum_{i=1}^m \epsilon_{i(n)}=-\epsilon_\alpha$. In accordance with Eq. (3.4) we express the total energy in the final state as

$$E=s_\alpha+(\hbar k_\alpha')^2/2m_\alpha, \tag{3.27}$$

where s_α is the total internal energy of the m subsystems and which takes on the value $-\epsilon_\alpha$ in the final state under consideration. Let us recall that $\mathbf{I}(S)$, as given by Eq. (2.42), is a sum of amplitudes each of which is to be evaluated in the same manner as $\mathbf{T}^C(S)$ itself, and will therefore have a representation similar in form to that shown in Eq. (2.15). Clearly this process of decomposition may be carried out to a stage where a class of terms in the expansion of $\mathbf{I}(S)$ contain as their left-hand factor the operator $T^C(S_1;S_2;\cdots;S_m)$ corresponding to the final-state interaction which can lead to the exit channel α of interest. In fact, this class

[23] We take this opportunity to point out a misprint in A. Sommerfeld, *Partial Differential Equations* (Academic Press Inc., New York, 1949). Equation (2f) on page 227 should read

$$\Omega_p=2(\pi)^{(p+1)/2}/\Gamma\left(\frac{p+1}{2}\right).$$

Unfortunately, this error was carried over to Eq. (A10) of Ref. 17. The corrected value of the normalization constant is

$$C(k,p)=(4i)^{-1}(k/2\pi)^{p/2},$$

which should replace Eq. (A11) of Ref. 17.

of terms will contribute to the residue of a pole at $s_\alpha = -\epsilon_\alpha$ in the momentum-space representation of $T^C(S)$, due to intermediate states in the scattering process described by $T^C(S_1; S_2; \cdots; S_m)$, with energy

$$E' = -\epsilon_\alpha + (\hbar k_\alpha')^2/2m_\alpha. \tag{3.28}$$

The existence of the pole may be deduced by comparing Eqs. (3.27) and (3.28) and noting that from momentum conservation the matrix element of $T^C(S)$ vanishes unless $\mathbf{k}_\alpha' = \mathbf{k}_\alpha^f$. The rule for obtaining the T-matrix element for scattering into exit channel α may now be stated. We take the residue at the pole at $s_\alpha = -\epsilon_\alpha$ in the momentum-space matrix element of $T^C(S)$. As usual, the residue at a bound-state pole factorizes [as exemplified by Eq. (2.22) for the two-body amplitude]. The left-hand factor arises from the ket, and the right-hand factor arises from the bra, in the dyadic representation of the Green's function. Our rule is to discard the left-hand factor, leaving us with the physical amplitude, constrained by the conservation condition

$$E = -\epsilon_\alpha + (\hbar k_\alpha^f)^2/2m_\alpha. \tag{3.29}$$

To continue the amplitude off the energy shell we simply drop this condition. If one or more subsystems are bound in entrance channel β we proceed in a similar fashion. We write

$$E = s_\beta + (\hbar k_\beta^i)^2/2m_\beta, \tag{3.30}$$

and take the residue at the pole at $s_\beta = -\epsilon_\beta$, discarding the right-hand factor of the residue. It is easily seen that for $N=3$ this rule leads to Eq. (3.26) for the T-matrix. Note that our earlier identification of the h_{ij} of Eq. (2.23) with the bound-state scattering amplitudes is now justified. For $N>3$ the validity of the above rule may be verified by formulating the N-body generalization of the arguments which led to Eq. (3.26). Since this is straightforward, but notationally cumbersome, we omit further details here.

4. INTEGRAL EQUATIONS FOR THE WAVE FUNCTION AND FOR THE EFFECTIVE POTENTIAL

While all physically relevant information concerning the scattering can be obtained, in principle, from the T operator defined in Sec. 2 it is sometimes useful to have an integral equation for the scattering wave function. For example, such an integral equation facilitates the construction of an effective potential, as discussed below. As a first step we derive an integral equation for the unperturbed wave in channel α, denoted, as above, by Φ_α. (Of course no integral equation is required for entrance channel $\alpha=0$ where none of the particles interact.) For notational convenience we write the cluster-decomposition formula for $T^D(S)$ [see Eq. (2.41)] in the abbreviated form

$$T^D(S) = \sum_\alpha T_\alpha(S_\alpha), \tag{4.1}$$

where S_α denotes a particular partition of the system S into subsystems. If $G(S_\alpha)$ represents the Green's function obtained from $G(S)$ by dropping interactions between disconnected subsystems we have

$$\begin{aligned}\Phi_\alpha &= \lim_{\eta\to 0+} i\eta G(S_\alpha)\Phi_\alpha \\ &= \lim_{\eta\to 0+} i\eta G_0 T_\alpha(S_\alpha) G_0 \Phi_\alpha,\end{aligned} \tag{4.2}$$

where in going from the first form of Eq. (4.2) to the second form we have used the appropriate version of Eq. (3.1) and have dropped terms which vanish in the limit. Thus we have the representation

$$\Phi_\alpha = \sum_i {}^{(i)}\Phi_\alpha \tag{4.3}$$

with

$${}^{(i)}\Phi_\alpha = \lim_{\eta\to 0+} i\eta G_0 [{}^{(i)}T_\alpha(S_\alpha)] G_0 \Phi_\alpha. \tag{4.4}$$

If only one pair is bound in entrance channel α, with the other particles noninteracting, the sum in Eq. (4.3) reduces to a single term, with index corresponding to the bound pair. For all other channels we obtain coupled, homogeneous, integral equations for the ${}^{(i)}\Phi_\alpha$ by noting that $T_\alpha(S_\alpha)$ satisfies an integral equation of the form shown in Eq. (2.14). When combined with Eq. (4.4) we obtain the set of equations

$${}^{(i)}\Phi_\alpha = G_0 \sum_k {}^{(i)}I^{(k)}(S_\alpha)\,{}^{(k)}\Phi_\alpha \tag{4.5}$$

which determine Φ_α. The integral equation for the scattering wave function Ψ_α is now constructed by writing, for $\alpha \neq 0$,[24]

$$\Psi_\alpha = \sum_i {}^{(i)}\Psi_\alpha. \tag{4.6}$$

Then, by successive applications of Eqs. (3.6), (3.1), (2.40), (4.1), (4.4), and (2.15), we obtain

$${}^{(i)}\Psi_\alpha = {}^{(i)}\Phi_\alpha + G_0 \sum_k {}^{(i)}I^{(k)}(S)\,{}^{(k)}\Psi_\alpha. \tag{4.7}$$

The discrete N-body bound states satisfy the homogeneous version of Eq. (4.7), namely,

$${}^{(i)}X = G_0 \sum_k {}^{(i)}I^{(k)}(S)\,{}^{(k)}X, \tag{4.8}$$

which is of course of the same form as Eq. (4.5).

As an application of this formalism we complete a derivation, initiated earlier, of a minimum principle for the effective (or optical) potential.[10] The earlier discussion was incomplete in the sense that the projection operator formalism of Feshbach[9] was employed, and these projection operators have not been explicitly constructed for the general case. However, for the three-body problem the formal results were turned into a practical minimum principle, of the Rayleigh-Ritz type, with the aid of the Faddeev representation of the wave function. This representation enabled us to replace the effects of the projection operators by simple orthogonality constraints on the trial function. It is

[24] The derivation for $\alpha=0$ proceeds along slightly different lines and can be constructed, as in Ref. 5 for $N=3$, without difficulty.

now possible to obtain the N-body generalization of the minimum principle with the aid of the generalized Faddeev equations. Recall that a knowledge of the effective potential is essentially equivalent to a solution of the scattering problem involving two-body reactions of the type $A+B\rightarrow C+D$, since the reaction matrix can be obtained from the effective potential matrix by numerical solution of a set of coupled two-body Lippmann-Schwinger equations [similar in form to Eqs. (2.27)].

Consider a set of two-body channels denoted collectively by P; all other channels belong to the set Q, say. To each channel in P space there corresponds a particular partition of the system S into two subsystems; these are the subsystems which are bound in the asymptotic region. We now introduce the decomposition

$$T^{C}(S_1;S_2)=T^{C(P)}(S_1;S_2)+T^{C(Q)}(S_1;S_2), \quad (4.9)$$

where $T^{C(P)}$ consists of the pole contributions arising from the discrete bound states associated with the set P. In analogy with Eq. (2.42) we define an operator $\mathbf{I}^{(Q)}$ by

$$I^{(k)(Q)}(S)=\sum_{S(i),S(j)}^{(S)} T^{C(Q)}(S(i);S(j)). \quad (4.10)$$

Thus the wave function, which in intermediate stages of the scattering process describes propagation only in states belonging to Q space, is defined as the solution of Eq. (4.7) with $\mathbf{I}$ replaced by $\mathbf{I}^{Q}$. Upon summation over the index i the solution takes the form

$$\Psi_{\alpha}^{(Q)}=\Phi_{\alpha}+G_0\,\Sigma_k\,[I^{(k)(Q)}]^{(k)}\Psi_{\alpha}^{(Q)}. \quad (4.11)$$

It is noted that solutions of the homogeneous form of the integral equation for Ψ^{Q} provide the discrete states which, in the Feshbach picture,[9] give rise to resonances in the scattering. We thus have a well-defined dynamical scheme for describing these resonances directly.

The effective potential is essentially the scattering operator modified such that propagation in intermediate states belonging to the set P has been projected out. When the channels in Q space are closed the effective potential has the momentum space representation[10]

$$V_{\alpha\beta}(\mathbf{k}_{\alpha},\mathbf{k}_{\beta};E)=(\Psi_{\alpha}^{(Q)},[H-E]\Psi_{\beta}^{(Q)}). \quad (4.12)$$

The minimum principle is contained in the statement that the error in $V_{\alpha\beta}$ which is made by the replacement of $\Psi_{\alpha}^{(Q)}$ and $\Psi_{\beta}^{(Q)}$ with trial functions $\Psi_{\alpha t}^{(Q)}$ and $\Psi_{\beta t}^{(Q)}$, respectively, is the matrix element of a negative operator. (When resonances exist below the scattering energy E the above formulation of the minimum principle must be modified slightly.[10]) There still remains the question of how to choose the trial functions $\Psi_{\alpha t}^{(Q)}$ such that intermediate states in P space are projected out. The answer is obtained by generalizing the argument given for $N=3$, and we only sketch it here. We define the Green's function

$$G^{Q}(S_{\beta})=G_0+G_0T^{Q}(S_{\beta})G_0. \quad (4.13)$$

It is a simple matter to rewrite Eq. (4.11) in terms of the $G^{Q}(S_{\beta})$, rather than the $G_0I^{(k)(Q)}$. The equation then takes the form

$$\Psi_{\alpha}^{(Q)}=\Phi_{\alpha}+\textstyle\sum_{\beta}\Psi_{\beta\alpha}^{(Q)}, \quad (4.14)$$

where $\Psi_{\beta\alpha}^{(Q)}$ is the contribution to the wave function generated by the operator $G^{(Q)}(S_{\beta})$, with β labelling the particular partition of the system into subsystems. Now $G^{(Q)}(S_{\beta})$ may be represented by the hyperspace generalization of Eq. (3.22), with contributions from states in P space omitted in the summation over discrete states. This implies the representation

$$\Psi_{\beta\alpha}^{(Q)}=\mathbf{S}_n^{(Q)}X_{\beta(n)}(\boldsymbol{\varrho}_{\beta})f(\mathbf{r}_{\beta}), \quad (4.15)$$

the superscript (Q) on the generalized sum $\mathbf{S}_n$ indicating that only states in Q space are included. The function $f(\mathbf{r}_{\beta})$ vanishes as $r_{\beta}\rightarrow\infty$; the specific form of the function needn't concern us here. Let $X_{\beta(n)}^{(P)}$, $n=1,2,\cdots,N_{\beta}^{(P)}$, be one of the bound-state functions belonging to P space, and to the partition β. The essential constraint on the function $\Psi_{\alpha}^{(Q)}$ may be formulated by the condition

$$(X_{\beta(n)}^{(P)},\Psi_{\beta\alpha}^{(Q)})=0, \quad n=1,2,\cdots,N_{\beta}^{(P)}, \quad (4.16)$$

and this is the meaning we shall give to the statement that $\Psi_{\alpha}^{(Q)}-\Phi_{\alpha}$ has no projection onto P-space. The trial function $\Psi_{\alpha t}^{(Q)}$ is then to be constructed in the form

$$\Psi_{\alpha t}^{(Q)}=\Phi_{\alpha}+\textstyle\sum_{\beta}\Psi_{\beta\alpha t}^{(Q)} \quad (4.17)$$

with the constraint

$$(X_{\beta(n)}^{(P)},\Psi_{\beta\alpha t}^{(Q)})=0, \quad n=1,2,\cdots,N_{\beta}^{(P)}. \quad (4.18)$$

Now the spectrum of the Hamiltonian in the space of functions with no projection onto P space has a continuum threshold E_c determined by the opening of Q-space channels. For scattering energy $E<E_c$ application of the Rayleigh-Ritz principle leads directly to the minimum principle for the effective potential. The Feshbach projection-operator formalism has been eliminated entirely in favor of the orthogonality conditions shown in Eq. (4.18).

5. DISCUSSION

One feature of the generalized Faddeev equations presented here which differs from previous formulations of multiparticle-scattering theory[1,2] is the absence of any explicit dependence on the potentials. This feature may provide a significant advantage in dealing with a more general class of scattering problems generated, e.g., by the assumption that all particles are in some sense to be considered as composites of other particles.[25]

[25] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).

An example of this increased flexibility was provided in Sec. 2 where the scattering of a θ particle by a V particle was calculated as bound-state scattering even though the V particle is not a bound N-θ state in the usual potential-theoretic sense. Another example has been given previously in a discussion of peripheral $\pi-N$ collisions using the Faddeev equations with the nucleon treated as a bound pion-nucleon state.[18,26] Lovelace[16] has treated π-N scattering in the static model in the same spirit. In this more general class of problems we can envision the calculation of $(N$-$m)$-body scattering by deduction from the N-body amplitude, i.e., by taking residues at the initial- and final-state interaction poles as described in Sec. 3. At the same time the N-body amplitude is to be built up by induction on the lower order amplitudes as described in Sec. 2. It is tempting to speculate that requirements of self-consistency may be sufficient to determine the dynamics completely.[27] This would require, in order that the calculation be finite, that the self-consistent solutions for the lower order amplitudes be insensitive to an increase in N for sufficiently large N. The integral equations presented here, or rather their relativistic counterpart,[28] would provide a unitary framework for a detailed examination of this idea.

[26] L. Rosenberg, Phys. Rev. **131**, 874 (1963).

[27] Speculations along these lines have been advanced previously in Ref. 16.

[28] A relativistic extension of the three-body Faddeev equations has been given by V. A. Alessandrini and R. L. Omnes, Phys. Rev. **139**, B167 (1965).

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Coulomb Correlations in Inhomogeneous Many-Particle Systems*

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The fluctuation-dissipation theorem is applied to the study of the pair correlation functions of the system containing charged particles which is not invariant under spatial translation. The results are expressed in terms of the longitudinal response functions and involve correlation functions which depend only on screened short-range Coulomb interactions S_0. In the case of the electron gas, the density-density correlation function is

$$S(\mathbf{r}\mathbf{r}';\omega)=[\delta(\mathbf{r}-\mathbf{R})-K(r\mathbf{R}'';\omega)v(\mathbf{R}''\mathbf{R})]S_0(\mathbf{R}\mathbf{R}';\omega)[\delta(\mathbf{R}'-\mathbf{r}')-v(\mathbf{R}'\mathbf{R}''')\tilde{K}(\mathbf{R}'''\mathbf{r}';-\omega)],$$

where K is the response function of the electron density to an external potential field, and $\tilde{K}$ its conjugate; v represents the Coulomb interaction; and the repeated spatial coordinates are integrated over the volume. It is shown that this expression implies the principle of superposition of dressed particles (the bare particles plus their associated screening clouds); for the calculations of the correlation functions, the dressed particles may be superposed without further consideration of the long-range parts of Coulomb interactions. Similar reductions are carried out for the various correlation functions in the electron-ion system, and the superposition principle is confirmed. A possible extension of the results to a nonequilibrium stationary system is suggested and discussed. The interrelationship among various longitudinal response functions, which is characteristic of the inhomogeneous system, is discussed and clarified. Simplifications arising from the homogeneity and discussions from diagrammatic considerations are given in Appendices.

I. INTRODUCTION

THE purpose of the present paper is the theoretical study of the pair correlation functions of a system containing charged particles which is not invariant under spatial translations. We derive the density-density correlation functions expressed in terms of the correlation functions with screened Coulomb interactions and the longitudinal response functions. The results are rigorous for a system in thermodynamic equilibrium; the arguments are extended to provide an appropriate treatment of certain nonequilibrium stationary systems. In the course of the investigation, we clarify the interrelationship among the various longitudinal response functions, which is characteristic of the inhomogeneous system.

For the study of the properties of a many-particle system in thermodynamic equilibrium, the fluctuation-dissipation theorem is used to provide a rigorous link between the pair correlation functions and the linear response functions.[1] The theorem relates the canonically or grand-canonically averaged commutator and anti-commutator of any pair of Hermitian operators; the

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[1] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951); R. Kubo, J. Phys. Soc. Japan **12**, 570 (1957); L. P. Kadanoff and P. C. Martin, Ann. Phys. (N. Y.) **24**, 419 (1963); and some others.