Density-Matrix Analysis for Spin Determination

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The general properties of the density matrix of an unstable particle produced in a two-body collision are analyzed with a view toward deriving useful tests for the spin of the particle. Spin tests are obtained for production on polarized and on unpolarized targets, and special situations which do not allow for a complete determination of the density matrix are examined. Production of the unstable particle from a spinless boson incident on a spin-$\frac{1}{2}$ fermion is discussed in particular, and a special decomposition of the density matrix for that case appears to be quite useful for formulating tests of spin. Inequalities that must be satisfied from the elements of the density matrix are expressed in terms of directly measurable quantities and shown to provide general sets of spin tests.

1. INTRODUCTION

In this paper we present general spin tests for unstable particles, which are based on the analysis of the density matrix of the unstable particle as constructed from its decay distributions. We shall discuss in more detail the properties of the density matrix of a particle produced from a spinless boson incident on a spin-$\frac{1}{2}$ fermion, polarized or unpolarized. For unpolarized initial fermion, invariance under the operation R (defined as the reflection with respect to the production plane[1]) permits a decomposition of the density matrix into two submatrices, whose ranks must satisfy an upper limit as a consequence of a general theorem that we give in Sec. 2. This result generalizes one that had been previously derived for a particular case.[2] We present, in Sec. 3, general tests based on inequalities that must be satisfied from the density matrix. In particular we discuss tests that can be applied in cases of incomplete determination of the density matrix and we show how a theorem of Peshkin[3] can be made substantially stronger if used in conjunction with R invariance. Tests for production from polarized targets are considered in Sec. 4. We find a number of constraints for the density matrix of the unstable particle and some inequalities for the spin in terms of directly measurable quantities. In Sec. 5 we briefly discuss the Adair limit. Section 6 contains a general summary of the main results from the view point of their practical use.

2. PRODUCTION PROCESS

We consider the production process

$$b+f\to F+B\,, \tag{1}$$

where b is a spinless boson, f a spin-$\frac{1}{2}$ fermion, B a boson of spin j, and F a fermion of spin j'. We make no assumptions about the polarization of the initial fermion. We represent its polarization state by a density matrix $\rho^{(f)}$.

[1] A. Bohr, Nucl. Phys. **10**, 486 (1959).
[2] M. Ademollo and R. Gatto, Phys. Rev. **133**, B531 (1964).
[3] M. Peshkin, Phys. Rev. **123**, 637 (1961).

We call

$$\langle jm,j'\mu|M|\tfrac{1}{2}\alpha\rangle=\beta_{m\mu}{}^{(\alpha)}(\vartheta)$$

the transition matrix element, at given energy and production angle ϑ, for production of $B+F$. The spin components of B and F are m and μ, respectively, quantized along the normal $\mathbf{n}$ to the production plane, and α is the spin index of the initial fermion. We can identify the index α with the helicity quantum number of the initial fermion or, if we choose a different representation, with its magnetic quantum number along the normal $\mathbf{n}$.

In terms of the initial density matrix $\rho^{(f)}$, we may write the final density matrix as

$$\rho_{m\mu,m'\mu'}=\sum_{\alpha\alpha'}\beta_{m\mu}{}^{(\alpha)}(\vartheta)\rho_{\alpha\alpha'}{}^{(f)}\beta_{m'\mu'}{}^{(\alpha')*}(\vartheta)\Big/\sum_{\alpha\alpha'\mu m}\beta_{m\mu}{}^{(\alpha)}(\vartheta)\rho_{\alpha\alpha'}{}^{(f)}\beta_{m\mu}{}^{(\alpha')*}(\vartheta)\,. \tag{2}$$

Invariance under the operation R, reflection through the production plane, allows one to write for the matrix elements of M the relation

$$\langle m\mu|M|\alpha,\text{hel}\rangle=(-1)^{j-m+j'-\mu}\langle m\mu|M|-\alpha,\text{hel}\rangle\,, \tag{3}$$

which simply states the connection between the transition amplitudes from the two possible helicity states of the initial fermion.[4] Alternatively, if we quantize the spin of the initial fermion along the normal $\mathbf{n}$, we obtain the selection rule:

$$\beta_{m\mu}{}^{(\alpha)}=0\quad\text{unless}\quad\pi_i\pi_f e^{i(m+\mu-\alpha)\pi}=1\,, \tag{4}$$

where α is now identified with the spin component of the initial fermion along $\mathbf{n}$, and π_i, π_f are the initial- and final-state intrinsic parities.

When the initial fermion is unpolarized, its density matrix is, in the helicity basis,

$$\rho_{\alpha\alpha'}{}^{(f)}=\tfrac{1}{2}\delta_{\alpha\alpha'}\,. \tag{5}$$

When the initial fermion is completely polarized, its density matrix may be written, when we quantize its

[4] M. Peshkin, Phys. Rev. **129**, 1864 (1963).

spin along $\mathbf{n}$, as

$$\rho_{\alpha\alpha'}{}^{(f)}=\tfrac{1}{2}(1\pm\sigma_n)_{\alpha\alpha'}, \tag{5'}$$

where σ_n is the familiar Pauli spin matrix.

A. For an unpolarized initial fermion, Eqs. (2) and (5) give

$$\rho_{m\mu,m'\mu'}=\sum_\alpha \beta_{m\mu}{}^{(\alpha)}\beta_{m'\mu'}{}^{(\alpha)*}. \tag{6}$$

The matrix elements $\beta_{m\mu}{}^{(\alpha)}$ are normalized according to

$$\sum_{am\mu}|\beta_{m\mu}{}^{(\alpha)}|^2=1. \tag{7}$$

From Eq. (3) one obtains

$$\rho_{m\mu,m'\mu'}=\tfrac{1}{2}[1+(-1)^{m-m'+\mu-\mu'}]\beta_{m\mu}\beta_{m'\mu'}{}^* \tag{8}$$

(we have dropped the superscript $\frac{1}{2}$ which is now obviously inessential). From this expression we have for the density matrix $\rho^{(B)}$ of the final boson B, when we do not observe the spin state of the final fermion F,

$$\rho_{mm'}{}^{(B)}=\tfrac{1}{2}[1+(-1)^{m-m'}]\sum_{\mu=-j'}^{j'}\beta_{m\mu}\beta_{m'\mu}{}^*. \tag{9}$$

Similarly for the density matrix $\rho^{(F)}$ of the final fermion, when the spin state of B is not observed, we have

$$\rho_{\mu\mu'}{}^{(F)}=\tfrac{1}{2}[1+(-1)^{\mu-\mu'}]\sum_{m=-j}^{j}\beta_{m\mu}\beta_{m\mu'}{}^*. \tag{10}$$

A density matrix of the form (9) and (10) has the following properties:

(1) The only nonzero matrix elements are those with $j-m$ and $j-m'$ both even or both odd (we denote here by j the spin of the observed particle and by j' the spin of the other produced particle whose spin is not observed). We can therefore write ρ as the direct sum of two submatrices of lower order,

$$\rho=\rho'\oplus\rho'', \tag{11}$$

where ρ' contains only the elements of ρ with $j-m$ and $j-m'$ both even, and ρ'' contains only those with $j-m$ and $j-m'$ both odd. The dimensions of ρ' and ρ'' are both $j+\frac{1}{2}$ if ρ describes a fermion of spin j. The dimensions are $j+1$ and j, respectively, if ρ describes a spin-j boson.

(2) The ranks of ρ' and ρ'' are at most $2j'+1$ as we will show in the next section.

B. Let us consider the case of a completely polarized initial fermion. The final density matrix is

$$\rho_{m\mu,m'\mu'}{}^{(\pm1/2)}=\beta_{m\mu}{}^{(\pm1/2)}\beta_{m'\mu'}{}^{(\pm1/2)*}/\sum_{m\mu}|\beta_{m\mu}{}^{(\pm1/2)}|^2, \tag{12}$$

where $\pm\frac{1}{2}$ is the spin component of the initial fermion along $\mathbf{n}$. From this expression we obtain

$$\rho_{mm'}{}^{(B,\pm1/2)}=\sum_\mu \beta_{m\mu}{}^{(\pm1/2)}\beta_{m'\mu}{}^{(\pm1/2)*}/\sum_{m\mu}|\beta_{m\mu}{}^{(\pm1/2)}|^2 \tag{13}$$

and

$$\rho_{\mu\mu'}{}^{(F,\pm1/2)}=\sum_m \beta_{m\mu}{}^{(\pm1/2)}\beta_{m\mu'}{}^{(\pm1/2)*}/\sum_{m\mu}|\beta_{m\mu}{}^{(\pm1/2)}|^2. \tag{14}$$

We remark that, from the selection rule (4), we can again split ρ into the sum of two submatrices, constructed as in A, Eq. (11):

$$\rho=\rho'\oplus\rho''.$$

3. DENSITY MATRIX FOR PRODUCTION FROM UNPOLARIZED TARGET

We have seen in the preceding section that, if B and F are produced on an unpolarized target, their density matrices can both be split into two submatrices of the form [see Eqs. (9) and (10)]

$$\tilde{\rho}=\sum_{\lambda=-j'}^{j'}\beta_\lambda\beta_\lambda{}^\dagger. \tag{15}$$

In Eq. (15) β_λ are vectors in a space of the same dimension as the submatrix. The components of β_λ are the coefficients $\beta_{m\lambda}$ when the particle observed is the final boson, or the coefficients $\beta_{\lambda\mu}$ when the particle observed is the final fermion F. The following theorem holds:

Theorem. The necessary and sufficient condition for a Hermitian matrix $\tilde{\rho}$ of order n to be of the form

$$\tilde{\rho}=\sum_{k=1}^{r}\beta_k\beta_k{}^\dagger,$$

where $\beta_1,\beta_2,\cdots\beta_r(r\leq n)$ are r linearly independent vectors of the n-dimensional space, is that the rank of $\tilde{\rho}$ be r. The rank of $\tilde{\rho}$ is less than r if the vectors $\beta_1\cdots\beta_r$ are linearly dependent. The proof can be given as follows:

The condition is necessary. In fact in the n-dimensional space we can find $n-r$ linearly independent vectors $\gamma_1,\gamma_2,\cdots\gamma_{n-r}$ which are all orthogonal to each of the vectors $\beta_1,\beta_2\cdots\beta_r$. It follows that $\tilde{\rho}\gamma_l=0$ $(l=1,2,\cdots,n-r)$. In a basis formed from all the vectors $\beta_1,\beta_2\cdots\beta_r$, $\gamma_1\cdots\gamma_{n-r}$, the rank of $\tilde{\rho}$ is evidently r: since it is invariant, it is r in any representation.

The condition is sufficient. The matrix $\tilde{\rho}$ can be reduced to diagonal form: we denote by $\lambda_i(i=1,2,\cdots,r)$ the nonzero (and positive) eigenvalues of $\tilde{\rho}$ and by S the unitary matrix which diagonalizes $\tilde{\rho}$. We can write

$$\tilde{\rho}_{\mu\nu}=\sum_{i=1}^{r}S_{\mu i}{}^\dagger\lambda_i S_{i\nu},$$

which coincides with Eq. (15) if $\beta_{\mu k}=\lambda_k{}^{1/2}S_{k\mu}{}^*$.

It follows from this theorem that the rank of ρ' and ρ'' is at most $2j'+1$. If $2j'+1$ is less than the dimension of ρ' or ρ'' we have relations among matrix elements which express the vanishing of the minors of order greater than $2j'+1$.

For $j'=0$ we have shown[2] that the condition that ρ' and ρ'' be of rank one, and the condition that the trace of ρ be non-negative, together provide a set of necessary and sufficient conditions for a given spin assignment.

We shall now present some tests based on general inequalities that must be satisfied by the elements of the density matrix. These tests are in the form of necessary conditions for a given spin assignment; they can provide both upper and lower bound for the spin value.

The density matrix ρ is a Hermitian non-negative definite matrix. From a theorem of matrix algebra[5] we learn that the necessary and sufficient condition for ρ to be non-negative definite is that all the principal minors of ρ have non-negative determinants. Any minor on the diagonal (that is, any minor of ρ whose diagonal elements are diagonal elements of ρ) can be reduced to a principal minor by a unitary transformation. We thus have the redundant condition that all minors on the diagonal have non-negative determinant (in particular, all diagonal matrix elements of ρ must be non-negative, as is physically obvious).

The condition that the rank of ρ be at most equal to the number of pure states r contributing to the statistical mixture described by ρ (an obvious generalization of our preceding discussion), together with the normalization of ρ to unit trace, provides upper limits to the determinants of the minors on the diagonal.

Let us denote by $\rho^{(n)}$ a minor of order n of ρ, situated on the diagonal. We obtain, from the above conditions, the set of inequalities:

$$0\leq \det[\rho^{(n)}]\leq (1/n)^n \quad \text{for} \quad n\leq r, \qquad (16)$$
$$\det[\rho^{(n)}]=0 \qquad\qquad n>r.$$

In fact, by reducing $\rho^{(n)}$ to diagonal form, $\det[\rho^{(n)}]$ is given by the product of the n eigenvalues of $\rho^{(n)}$. The sum of eigenvalues being fixed ($\mathrm{Tr}[\rho^{(n)}]\leq 1$), the maximum of their product is reached when they are all equal, and therefore

$$\det[\rho^{(n)}]\leq [(1/n)\ \mathrm{Tr}\rho^{(n)}]^n\leq (1/n)^n.$$

One can also easily derive the following inequalities:

$$(1/r)^{m-1}\leq \mathrm{Tr}[\rho^m]\leq 1 \quad (m \text{ integer}). \qquad (17)$$

The proof of this relation is similar to the preceding proof. On reducing ρ to diagonal form, $\mathrm{Tr}[\rho^m]$ is given by the sum of the mth powers of the nonzero eigenvalues of ρ, which are at most r. Since the sum of the eigenvalues is normalized to unity, the minimum of $\mathrm{Tr}[\rho^m]$ is obtained when the nonzero eigenvalues are equal to $1/r$ and therefore

$$\mathrm{Tr}[\rho^m]\geq r(1/r)^m=(1/r)^{m-1},$$

whereas the maximum is obtained when one eigenvalue is unity and the others are zero.

The inequalities (17) can provide an upper value for the spin j of the unstable particle from its decay distribution. For example: suppose the angular distribution is isotropic; the density matrix ρ is in general $(2j+1)^{-1}$ times the unit matrix and (17) implies $(2j+1)\leq r$. This is an intuitive result; in fact the isotropic distribution implies that the decaying mixture is completely unpolarized and it is described by an incoherent mixture of $2j+1$ pure states. Such a mixture cannot be realized if the available number of pure states is less than $2j+1$.

When ρ is expressible as in Eq. (11), in the form of a direct sum of ρ' and ρ'', the limitations (16) and (17) apply separately to ρ' and ρ''. In this case r is the number of pure states when the initial fermion is in a given helicity state. (For the total density matrix ρ the rank is twice as large.)

To use such tests a complete knowledge of the density matrix is, in general, required. Unfortunately such a complete knowledge of the density matrix is not always obtainable. Furthermore, the test functions, i.e., the determinants and the traces appearing in the inequalities, are generally nonlinear functions of the experimental averages. The tests must then be applied to events with identical production kinematics. The only test functions linear in the experimental averages are the diagonal elements of ρ, which satisfy the limitations[6]

$$0\leq \rho_{kk}\leq 1 \qquad (18)$$

[a special case of (16) for $n=1$]. Linear test functions (as opposed to nonlinear test functions) can be averaged over the production kinematics and must still satisfy their original inequalities. The test can thus be performed on a large sample of events.

To see better how these tests are applicable in practice, let us expand the density matrix ρ in terms of irreducible tensor operators according to

$$\rho=\sum_{L=0}^{2j}\sum_{M=-L}^{L}\rho(L,M)T_L^M, \qquad (19)$$

where T_L^M are tensor operators, whose matrix elements are given, as usual, by

$$\langle j\mu|T_L^M|j\mu'\rangle=(\hat{L}/\hat{j})(j\mu',LM|j\mu), \quad [\hat{L}\equiv(2L+1)^{1/2}]. \qquad (20)$$

They satisfy

$$T_L^{M\dagger}=(-1)^M T_L^{-M}, \qquad (21)$$

$$\mathrm{Tr}(T_L^M T_{L'}^{M'\dagger})=\delta_{LL'}\delta_{MM'}. \qquad (21')$$

The coefficients $\rho(L,M)$ satisfy the relations

$$\rho(L,M)=\mathrm{Tr}(\rho T_L^{M\dagger}), \qquad (22)$$

$$\rho(0,0)=\hat{j}^{-1}, \qquad (22')$$

$$\rho(L,M)^*=(-1)^M\rho(L,-M). \qquad (22'')$$

They are related to the "moments" of the angular distributions of the decay products. Precisely, for a two-

[5] R. Bellman, *Introduction to Matrix Analysis*, (McGraw-Hill Book Co., New York, 1960), p. 74.

[6] T. D. Lee and C. N. Yang, Phys. Rev. **109**, 1755 (1959).

body decay, we have

$$I(\vartheta,\varphi)=\sum_{LM} a(L,M)Y_L{}^M(\vartheta,\varphi)\,,$$
$$a(L,M)=C(j,L)\rho(L,M)\,, \tag{23}$$

where $I(\vartheta,\varphi)$ is the final angular distribution and the coefficients $C(j,L)$ depend on the particular decay process. For example, in the following three cases, we have (where j^P denotes a particle of spin j and parity P):

(1) $j^P\to 0^-+0^-$; $P=(-1)^j$ (see Ref. 3),

$$C(j,L)=(4\pi)^{-1/2}\hat{\jmath}(j0,L0|j0)\,. \tag{24}$$

(2) $j^P\to 1^-+0^-$; (see Ref. 7),

$$C(j,L)=(-1)^{j+1}(4\pi)^{-1/2}(2j+1)\hat{L}^{-1}\times\sum_{ll'} T_lT_{l'}{}^*\hat{l}\hat{l}'(l0,l'0|L0)W(ljl'j;1L)\,, \tag{25}$$

with $l=l'=j$ for $P=(-1)^j$ and $l,l'=j\mp1$ for $P=(-1)^{j+1}$.

(3) $j\to\frac{1}{2}+0$ (without parity conservation) (see Ref. 2),

$$C(j,L)=-g_L(4\pi)^{-1/2}(j\tfrac{1}{2},L0|j\tfrac{1}{2})\,, \tag{26}$$

where

$$g_L=1 \quad \text{for even } L\,,$$
$$g_L=\alpha \quad \text{for odd } L\,.$$

From the preceding relations we can see that, from

$$\rho(L,0)=\frac{\hat{L}}{\hat{\jmath}}\sum_{\mu=-j}^{j}\rho_{\mu\mu}(j\mu,L0|j\mu)\,,$$

the conditions (18), as can easily be seen, give

$$|\rho(L,0)|\leq(\hat{L}/\hat{\jmath})\mathop{\rm Max}_{\mu}|(j\mu,L0|j\mu)|\,. \tag{27}$$

We shall now extend our discussion to situations in which the density matrix cannot be determined completely. The determination of the density matrix cannot be complete, for instance, in a two-body parity-conserving decay, if only the final angular distribution is observed. This is always the case for a boson decaying into two spinless bosons. We shall make use in our discussion of some results of Peshkin.[3]

It is convenient to split ρ into the even-L part ρ_e and the odd-L part ρ_o according to (19). It is easy to see that, while ρ_e can be determined in any case, ρ_o cannot be determined from Eq. (23) if parity is conserved in the decay process. In fact, the averages $\langle Y_L{}^M\rangle$ and the coefficients $C(j,L)$ will vanish for odd L, since partial waves of different parity cannot interfere.

It is also easy to see that ρ_e and ρ_o are, respectively, even and odd under the operation of time reversal (as defined by Edmonds for angular momentum states).[8]

$$T|j\mu\rangle=(-1)^{j+\mu}\langle j-\mu|\,. \tag{28}$$

From (20) we have

$$TT_L{}^MT^{-1}=(-1)^{L+M}T_L{}^{-M} \tag{29}$$

and from (19), (22), and (29) we get

$$T\rho_eT^{-1}=\rho_e\,, \tag{30}$$

$$T\rho_oT^{-1}=-\rho_o\,. \tag{31}$$

Since ρ is a non-negative definite matrix, also $T\rho T^{-1}=\rho_e-\rho_o$ must be non-negative; therefore ρ_e has also the same property. The test of non-negativeness of ρ_e can be used in place of the tests on the whole ρ matrix, described at the beginning of this section, whenever ρ_o cannot be determined. We have no significant limitation on the rank of ρ_e, but we still have the condition $\mathrm{Tr}\rho_e=1$. In fact ρ_o is traceless, as a consequence of (31). In analogy with Eq. (16) we thus have the inequalities

$$0\leq\det[\rho_e{}^{(n)}]\leq(n)^{-n} \quad (n=1,\cdots,N)\,, \tag{32}$$

where $\rho_e{}^{(n)}$ denotes any minor on the principal diagonal of order n of ρ_e and $N=2j+1$ is the dimension of ρ_e.

It is possible to obtain an upper limit for $\mathrm{Tr}[\rho^m]$ in terms of the eigenvalues of ρ_e only, which we call μ_i. For $m=2$ we have, because of (30) and (31)

$$\mathrm{Tr}[\rho^2]=\mathrm{Tr}[\rho_e{}^2]+\mathrm{Tr}[\rho_o{}^2]\,. \tag{33}$$

Peshkin[3] was able to derive an upper limit for $\mathrm{Tr}[\rho_o{}^2]$ in terms of the eigenvalues μ_i. This is possible because of the condition that both $\rho_e+\rho_o$ and $\rho_e-\rho_o$ must be non-negative. The limitations are the following:

$$\mathrm{Tr}[\rho^2]\leq\sum_{i=1}^{\frac{1}{2}N}(\mu_{2i}+\mu_{2i-1})^2 \qquad \text{for even } N\,, \tag{34}$$

$$\mathrm{Tr}[\rho^2]\leq\sum_{i=1}^{\frac{1}{2}(N-1)}(\mu_{2i}+\mu_{2i-1})^2+\mu_N{}^2 \quad \text{for odd } N\,, \tag{34'}$$

where the μ_i are taken in decreasing order of magnitude and $N=2j+1$ is the dimension of ρ.

This result can be extended to $m>2$ but the expressions become rather complicated.

The inequalities (34) and (34′) can be replaced by the weaker inequality

$$\mathrm{Tr}[\rho^2]\leq 2\,\mathrm{Tr}[\rho_e{}^2]\,. \tag{35}$$

From (19), (21′), (23), and (17) we obtain

$$\sum_{\text{even}\,L} C(j,L)^{-2}\sum_M|\langle Y_L{}^M\rangle|^2\geq 1/2r\,. \tag{36}$$

Inequalities of this kind have been also considered by Eberhard and Good.[9]

As an example, let us consider the production of a boson B in reaction (1), assuming that F has spin $\frac{1}{2}$ and its polarization is observed. Therefore in (36), $r=2$. Suppose further that $P_B=(-1)^j$ and that the decay

[7] M. Ademollo, R. Gatto and G. Preparata, Phys. Rev. Letters **12**, 462 (1964).

[8] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957), p. 29.

[9] P. Eberhard and M. L. Good, Phys. Rev. **120**, 1442 (1960).

angular distribution is found to contain only $L=0,2$. From Eq. (24) we find that, if $\sum_M |\langle Y_2{}^M\rangle|^2$ is less than $1/(56\times 4\pi)$, the spin j must be 1; if this quantity is less than $9/(80\times 4\pi)$ the spin can only be 1 or 2.

We conclude this section with some remarks about consequences of R invariance. We can split ρ into ρ' and ρ'' according to Eq. (11). However, some care is needed for extending the preceding results. We must distinguish between integer j and half-integer j. For integer j, Eqs. (30) and (31) hold for even-L and odd-L parts of ρ' and ρ'' separately. For half-integer j we have

$$T\rho_e'T^{-1}=\rho_e'', \qquad (37)$$

$$T\rho_o'T^{-1}=-\rho_o''. \qquad (37')$$

We see that ρ_e' and ρ_e'' can be transformed into each other: they must thus have the same eigenvalues and

$$\mathrm{Tr}[\rho_e']=\mathrm{Tr}[\rho_e'']=\tfrac{1}{2}. \qquad (38)$$

On the other hand, ρ_o' and ρ_o'' have opposite eigenvalues and in general

$$\mathrm{Tr}[\rho_o']=-\mathrm{Tr}[\rho_o'']\neq 0.$$

For j an integer the non-negativity condition for ρ_e' and ρ_e'' gives rise to conditions similar to (32), where N is the relevant dimension. For j a half-integer, because of (38) we have

$$0\leq \det[\rho_e'^{(n)}]\leq (2n)^{-n}, \quad 1\leq n\leq j+\tfrac{1}{2} \qquad (39)$$

and similarly for ρ_e''. In particular for the diagonal matrix elements we have

$$(\rho_e)_{kk}\leq \tfrac{1}{2}. \qquad (40)$$

Let us now consider the limitations (34) and (34′). For integer j we can apply to ρ' and ρ'' separately the same argument we have used for ρ. Since the rank of ρ' and ρ'' is at most r, we have, instead of (17), for $m=2$:

$$\mathrm{Tr}[\rho'^2]\geq (1/r)(\mathrm{Tr}\rho_e')^2, \qquad (41)$$

$$\mathrm{Tr}[\rho''^2]\geq (1/r)(\mathrm{Tr}\rho_e'')^2. \qquad (41')$$

Furthermore, for each of $\mathrm{Tr}[\rho'^2]$ and $\mathrm{Tr}[\rho''^2]$ there are limitations analogous to (34) and (34′) in terms of the eigenvalues of ρ_e' and ρ_e'', respectively. The relations we obtain in this way are of course stronger than (34) and (34′) as applied to the whole matrix ρ. In fact, (i) in $\mathrm{Tr}[\rho^2]$, expressed as the sum of $\mathrm{Tr}[\rho'^2]$ and $\mathrm{Tr}[\rho''^2]$, the μ_i are not altogether in decreasing order of magnitude, and (ii) in the sum of the right-hand sides of (41) and (41′) we have $\mathrm{Tr}[\rho_e'^2]+\mathrm{Tr}[\rho_e''^2]\geq\tfrac{1}{2}$.

For half-integer j the Peshkin argument does not hold for ρ' and ρ'' because ρ_o' and ρ_o'' are not antisymmetric in a basis in which ρ_e is diagonal. In this case, since the eigenvalues of ρ_e always occur in pairs, we obtain from (34)

$$\mathrm{Tr}[\rho^2]\leq 4\,\mathrm{Tr}[\rho_e'^2]=4\,\mathrm{Tr}[\rho_e''^2]. \qquad (42)$$

4. DENSITY MATRIX FROM POLARIZED TARGET

In Sec. 2 we have derived the general form of the density matrix of a particle produced in a two-body reaction on a polarized target. From Eqs. (13) and (14) we see that such a density matrix can still be decomposed into a direct sum of two lower order submatrices in the the same way as for unpolarized targets. The implications that follow from this property are therefore valid also in this case. There are however further special results that we shall derive here.

A. Let us begin our discussion with the case of a spin-j fermion produced in a reaction (1) together with a spinless boson. The density matrix (14) becomes in the usual symbols

$$\begin{aligned}\rho^{(F,1/2)}&=\rho^{(1/2)'} \quad \text{if} \quad \pi=(-1)^{j+1/2} \quad (\pi=\pi_i\pi_f),\\ &=\rho^{(1/2)''} \quad \text{if} \quad \pi=(-1)^{j-1/2},\end{aligned} \qquad (43)$$

and similarly

$$\begin{aligned}\rho^{(F,-1/2)}&=\rho^{(-1/2)'} \quad \text{if} \quad \pi=(-1)^{j-1/2},\\ &=\rho^{(-1/2)''} \quad \text{if} \quad \pi=(-1)^{j+1/2}.\end{aligned} \qquad (44)$$

In the notation of the preceding section we may write in a compact way

$$\begin{aligned}\rho_e^{(\nu)'}&=\omega(-1)^{1/2+\nu}\rho_o^{(\nu)'},\\ &\qquad [\omega=\pi(-1)^{j-1/2},\ \nu=\tfrac{1}{2},\ -\tfrac{1}{2}]\\ \rho_e^{(\nu)''}&=\omega(-1)^{1/2-\nu}\rho_o^{(\nu)''}.\end{aligned} \qquad (45)$$

This is a set of $(j+\frac{1}{2})^2$ relations of a linear type which may be usefully taken into account to test a given spin assignment for the fermion F. Furthermore ρ' and ρ'', whenever different from zero, must have rank one. This condition gives a number of additional relations among the matrix elements of ρ.

We may apply an argument due to Peshkin[10] on the maximum complexity of the angular distribution of decay products. For the Clebsch-Gordan coefficient $(j\mu,2j0|j\mu)$, we use the simple expression

$$(j\mu,2j0|j\mu) =(-1)^{j-\mu}\left[\frac{2j+1}{(4j+1)!}\right]^{1/2}\frac{[(2j)!]^2}{(j+\mu)!(j-\mu)!}. \qquad (46)$$

We can then write for the multipole coefficients $\rho(2j,0)$ the expression

$$\rho(2j,0)=\left[\frac{1}{(4j)!}\right]^{1/2}[(2j)!]^2\sum_\mu\frac{(-1)^{j-\mu}\rho_{\mu\mu}{}^{(F)}}{(j+\mu)!(j-\mu)!}. \qquad (47)$$

From (43) and (44) it is easily seen that the sum on the right-hand side cannot vanish. Actually we may set a lower bound

$$|\rho(2j,0)|\geq[(4j)!]^{-1/2}(2j)!. \qquad (48)$$

This maximum-complexity test has the attractive

[10] M. Peshkin, Phys. Rev. **133**, B428 (1964).

feature that it is linear and may be averaged over all events at all production angles (in particular over a certain range of angles).

A lower bound to the spin value of F may be provided by using a limitation given in a work by Shapiro.[11] We can define

$$P'=\frac{I^{(1/2)}(\theta)-I^{(-1/2)}(\theta)}{I(\theta)}=\omega\sum_{\mu=-j}^{j}(-1)^{j-\mu}\rho_{\mu\mu}^{(F)},\quad[\omega=\pi(-1)^{j-1/2}]\tag{49}$$

where $I^{(\pm 1/2)}(\theta)=\sum_{m\mu}|\beta_{m\mu}^{(\pm 1/2)}|^2$, as we see from Eq. (14). The expression (49) may be cast in the form

$$P'=\omega\sum_{\text{odd}L}\rho(L,0)\Delta(L,0),\tag{50}$$

where $\Delta(L,0)=(\hat{L}/\hat{\jmath})\sum_\mu(-1)^{j-\mu}(j\mu,L0|j\mu)$. As is clear from Eq. (23) and (26), $\rho(L,0)$ is a quantity which may be deduced from the observation of the two-body decay of F produced from an unpolarized target; thus we may, by measuring P', test (50) for a given spin assignment. Shapiro has shown that a lower bound for the spin is provided by

$$|2jP-\omega P'|\leq(2j-1),\tag{51}$$

where P is the polarization of F measured in the decay from an unpolarized target. We can also define

$$D=\frac{I^{(1/2)}P^{(1/2)}-I^{(-1/2)}P^{(-1/2)}}{I(\theta)}=\omega\sum_{\text{even}L}\rho(L,0)\Delta'(L,0),\tag{52}$$

where

$$\Delta'(L,0)=\sum_\mu(-1)^{j-\mu}(\mu/j)(j\mu,L0|j\mu).$$

This relation again provides a test for spin assignment if we measure the polarization of F in the two cases of target polarization and observe the angular distribution in the decay of F from an unpolarized target. The limitations on D, slightly different from those obtained by Shapiro, are listed below (with the usual meaning of symbols):

$$j=\tfrac{3}{2},\quad|\omega D-\tfrac{1}{3}|^2\leq(4/9)[1+2P'^2-(9/5)P^2],\tag{53}$$

$$j=\tfrac{5}{2},\quad|\omega D-\tfrac{1}{5}|^2\leq(64/150)\times[2+3P'^2-(15/7)P^2],\tag{53'}$$

$$j=\tfrac{7}{2},\quad|\omega D-\tfrac{1}{7}|^2\leq(20/49)[3+4P'^2=(7/3)P^2].\tag{53''}$$

B. We now consider the case of a spin-j boson produced in reaction (1) together with a spin-$\frac{1}{2}$ fermion. From Eq. (13) we have

$$\begin{aligned}\rho'_{mm'}{}^{(B,\pm1/2)}&=\beta_{m,\pm1/2}{}^{(\pm1/2)}\beta_{m',\pm1/2}{}^{(\pm1/2)*}/\sum_m|\beta_{m,\pm1/2}{}^{(\pm1/2)}|^2\text{ if }\pi=(-1)^j,\\&=\beta_{m,\mp1/2}{}^{(\pm1/2)}\beta_{m',\mp1/2}{}^{(\pm1/2)*}/\sum_m|\beta_{m,\mp1/2}{}^{(\pm1/2)}|^2\quad\text{if}\quad\pi=(-1)^{j+1},\end{aligned}\tag{54}$$

$$\begin{aligned}\rho''_{mm'}{}^{(B,\pm1/2)}&=\beta_{m,\mp1/2}{}^{(\pm1/2)}\beta_{m'\mp1/2}{}^{(\pm1/2)*}/\sum_m|\beta_{m,\mp1/2}{}^{(\pm1/2)}|^2\quad\text{if}\quad\pi=(-1)^j,\\&=\beta_{m,\pm1/2}{}^{(\pm1/2)}\beta_{m',\pm1/2}{}^{(\pm1/2)*}/\sum_m|\beta_{m,\pm1/2}{}^{(\pm1/2)}|^2\quad\text{if}\quad\pi=(-1)^{j+1},\end{aligned}\tag{54'}$$

and we immediately see that both ρ' and ρ'' are of rank one. They must satisfy the constraints that we have examined in the preceding section, Eq. (16), for $r=1$.

Another piece of information on the spin of B may be obtained by observing the produced fermion. We define the measurable quantity

$$R=[I^{(1/2)}(\theta)P^{(1/2)}-I^{(-1/2)}(\theta)P^{(-1/2)}]/I(\theta),\tag{55}$$

determined from experiment by measuring the polarization of the outgoing spin-$\frac{1}{2}$ fermion for spin-up target $P^{(1/2)}$, and spin-down target $P^{(-1/2)}$. We have

$$(-1)^j\pi R-1/(2j+1)=\sum_{L\geq 2;\ L\text{ even}}\rho(L,0)\Delta(L,0),\tag{56}$$

where $\Delta(L,0)=(\hat{L}/\hat{\jmath})\sum_m(-1)^{j-m}(jm,L0|jm)$ and $\rho(L,0)$ are the multipole parameters of the density matrix of B for production from unpolarized target. We note that (55) has the same structure as (52). A possibly useful limitation, which can be deduced from (55) is

$$\left|(-1)^j\pi R-\frac{1}{2j+1}\right|^2\leq\left[\frac{2j(i+1)-3jP^2}{j+1}\right]^2,\tag{57}$$

where P is the vector polarization of the boson produced from an unpolarized target.

It is to be noted that in Eq. (56) the sum must be extended on even L. The result can be used also when, owing to parity conservation, one can obtain $\rho(L,M)$ from experiment only for even L.

5. ADAIR'S LIMIT

Let us finally consider Adair's limit[12] of forward and backward production.

A. We consider first the case of a fermion F produced in reaction (1), together with a spinless boson. The cylindrical symmetry guarantees that $M=0$ in the expansion (19), so that, by (19) and (20), ρ becomes

[11] G. Shapiro, Phys. Rev. **134**, B1393 (1964).

[12] R. K. Adair, Phys. Rev. **100**, 1540 (1955).

diagonal and generally the above tests assume a simpler form. Furthermore, from R invariance, only the even-L terms appear in (19) and $\rho_o=0$. In fact, if we quantize the spin along the direction of the momenta, we must have $\rho_{\mu\mu}=\rho_{-\mu,-\mu}$ which implies, by (20), L even. The whole ρ matrix can thus be measured.

From these considerations and from the results on the ranks of ρ' and ρ'' one concludes that the only nonzero elements of ρ are

$$\rho_{1/2,\,1/2}=\rho_{-1/2,\,-1/2}=\tfrac{1}{2} \tag{58}$$

and the multipole coefficients in the expansion (19) are

$$\rho(L,M)=(\hat{L}/\hat{\jmath})(j\mu,LM\,|\,j\mu)\quad(\text{even } L,\, M=0). \tag{59}$$

The angular distribution (23) and (26), for $j\rightarrow\frac{1}{2}+0$, is

$$a(L,M)=(4\pi)^{-1/2}(\hat{L}/\hat{\jmath})(j\tfrac{1}{2},L0\,|\,j\tfrac{1}{2})^2, \tag{60}$$

as noted by Adair.

Let us now consider a spin-j boson produced in (1) together with a spin-$\frac{1}{2}$ fermion. The density matrix has only three nonzero elements, that is

$$\rho_{11}=\rho_{-1-1}=\tfrac{1}{2}|b|^2 \quad\text{and}\quad \rho_{00}=|a|^2, \tag{61}$$

with

$$|a|^2+|b|^2=1,$$

so that the multipole coefficients take the form

$$\rho(L,0)=(\hat{L}/\hat{\jmath})\{|a|^2(j0,L0\,|\,j0)+|b|^2(j1,L0\,|\,j1)\}\quad(\text{even } L). \tag{62}$$

The particularly simple form of $\rho(L,0)$ makes it possible to obtain detailed information on the spin. Suppose for instance that the experimental averages become zero for $L>2t$. One can thus assume $j\geq t$ from the structure of the multipole coefficients $\rho(L,M)$. If actually $j>t$ from (62) one easily sees that the only compatible assignment is $j=t+1$ and

$$|a|^2=(t+1)/(2t+3),\quad |b|^2=(t+2)/(2t+3). \tag{63}$$

The density matrix thus is completely determined and one can test the experimental consistency of the spin assignment $j=t+1$.

It is remarkable that the ambiguity of spin assignment is reduced in Adair's limit to the two possibilities $j=t,\ t+1$.

6. CONCLUSIONS

We summarize the results of the preceding sections for the different cases that we have considered.

A. General Case

The density matrix ρ is completely determined from the angular and polarization distributions of the decay products. The conditions that ρ has to satisfy are (1) that its rank cannot exceed the number r of pure states of the mixture describing the unstable particle; (2) the non-negativeness condition, Eq. (16); and (3) the limitations, Eq. (17), on the trace of ρ^m, $m>1$.

B. Incomplete Determination of the Density Matrix

Only the even-L part ρ_e in Eq. (19) can be determined. The conditions that ρ_e has to satisfy are (1) the non-negativeness condition, Eq. (32); and (2) the limitations on $\mathrm{Tr}[\rho^m]$ as in the general case discussed above. The upper limit of $\mathrm{Tr}[\rho^m]$ must, however, be expressed in terms of ρ_e only. For $m=2$ we have mentioned, in Eqs. (34), (34′), a result of Peshkin which allows one to put an upper limit on $\mathrm{Tr}[\rho^2]$ in terms of the eigenvalues of ρ_e.

C. Production on Unpolarized Target

We consider production of the unstable particle from a spinless boson on an unpolarized spin-$\frac{1}{2}$ fermion. The density matrix is the direct sum of the two submatrices ρ' and ρ'' according to Eq. (11). When the density matrix is completely determined from the angular and polarization distributions of the decay products, the same conditions derived for the general case A for the matrix ρ, apply here separately to ρ' and ρ''. The number r of pure states, which appears in the expression of the conditions (1) and (3) of the general case A, is now the number of pure states for given initial helicity, i.e., its value is one-half of the value in the general case A. On the other hand, when, as for case B discussed above, the density matrix is not completely determined, the limitations depend on whether the unstable particle whose spin is being determined is a fermion or a boson. For bosons the same limitations obtained for case B above apply separately to ρ' and ρ'', providing us with a set of stronger conditions than for case B. For fermions a non-negativeness condition, similar to that valid for case B, but stronger, must be satisfied. This condition is expressed by Eq. (39). The limitations that must be satisfied in case B by $\mathrm{Tr}[\rho^m]$ of course still apply here.

D. Production on Polarized Target

We consider production of the unstable particle from a spinless boson on a polarized spin-$\frac{1}{2}$ fermion. The density matrix ρ can again be split into the direct sum of ρ' and ρ''.

For a spin-j fermion produced together with a spinless boson one has the following conditions: (1) according to the parities, either ρ' or ρ'' has to be zero, Eqs. (43) and (44); (2) the non-zero matrix (ρ' or ρ'') must have rank one; (3) the $L=2j$ coefficient of the multipole expansion of ρ cannot vanish, Eq. (44): the angular distribution of the decay products must therefore exhibit maximum complexity consistent with spin j; and (4) the relations (51), (52), and (53), all among measurable quantities, must be satisfied.

For a spin j boson produced together with a spin-$\frac{1}{2}$ fermion we have: (1) both ρ' and ρ'' must have rank one; and, (2) the relation (56) among measurable quantities has to be satisfied for spin j.