Radiative Corrections. I. High-Energy Bremsstrahlung and Pair Production*

KJELL MORK AND HAAKON OLSEN
National Bureau of Standards, Washington, D. C. and Institute for Theoretical Physics, Norges Tekniske Høgskole, Trondheim, Norway
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The radiative corrections to the high-energy bremsstrahlung and pair-production spectra are calculated in the Weizsäcker-Williams approximation. Results for the soft-photon radiative correction to the spectra are given, and for the case of pair production the soft-plus-hard-photon radiative correction is also obtained. The radiative correction to the total pair-production cross section is found to be practically independent of the photon energy and of the atomic number of the target material. Comparison with available experimental data shows essential agreement within the relatively wide experimental limits.

I. INTRODUCTION

THE radiative correction to pair production and bremsstrahlung has previously been studied by several authors.[1-4] These works have been concerned with the differential cross section, differential in the two final-state particles[1-3] or in one of the pair particles in the pair-production process.[4] On the other hand, recent very accurate measurements of the high-energy gamma absorption coefficient[5,6] seem to indicate the desirability of obtaining accurate theoretical values for the total pair-production cross section. Since most of the high-energy pair production and bremsstrahlung take place for very small angles between the projectile and the secondary particles, we shall in this paper consider the radiative corrections to bremsstrahlung and pair production for small angles and high energies. These cross sections may then be integrated to give spectra and, for the case of pair production, the total cross section.

The form of the differential radiative-correction cross sections obtained directly from the Feynman diagrams is so complicated that an integration becomes virtually impossible. Fortunately it is possible to utilize the Weizsäcker-Williams method[7] to derive the high-energy bremsstrahlung and pair-production radiative-correction cross section from the already known radiative correction to the Compton effect[8] in much the same way as Weizsäcker and Williams originally used the method to obtain the bremsstrahlung and pair-production cross section from the Klein-Nishina formula. The results are expected to be reliable for the case that both the initial and final particles all have energies which are large compared to the rest energy of the electron.

We shall calculate in Sec. III the virtual-photon radiative correction to the bremsstrahlung spectrum $d\sigma_{\rm vir}{}^B(\omega_1)$ corresponding to the diagrams (a)–(d) and (a′)–(d′) of Fig. 1 using the Weizsäcker-Williams method. The virtual-photon radiative correction to the bremsstrahlung spectrum $d\sigma_{\rm vac}{}^B(\omega_1)$ pertaining to vacuum-polarization diagrams v and v' of Fig. 1 cannot be obtained in this way and is calculated directly from the diagrams in Sec. IV. The real soft-photon radiative-correction cross section to the bremsstrahlung spectrum $d\sigma_{\rm real,\,soft}{}^B(\omega_1,\Delta\omega_2)$ is evaluated in Sec. V. This part of the radiative correction has been obtained using the Weizsäcker-Williams method. The total radiative correction to the bremsstrahlung spectrum for the case that the additional photon k_2 with energy ω_2 is soft,

$$d\sigma_{\rm soft}{}^B(\omega_1,\Delta\omega_2)=d\sigma_{\rm vir}{}^B(\omega_1)+d\sigma_{\rm vac}{}^B(\omega_1)+d\sigma_{\rm real,\,soft}{}^B(\omega_1,\Delta\omega_2)\,,\quad (\mathrm{I}.1)$$

is obtained in Sec. VI.

The corresponding radiative corrections to the pair-production spectrum for the positron energy ϵ_+, $d\sigma_{\rm vir}{}^P(\epsilon_+)$, $d\sigma_{\rm vac}{}^P(\epsilon_+)$, $d\sigma_{\rm real,\,soft}{}^P(\epsilon_+,\Delta\omega_2)$, and $d\sigma_{\rm soft}{}^P(\epsilon_+,\Delta\omega_1)$ are obtained in Sec. VII. The radiative correction to the pair spectrum $d\sigma_{\rm soft}{}^P(\epsilon_+,\Delta\omega_2)$ may be looked upon as describing the process of "pair production with bremsstrahlung,"

$$k_1+Z\rightarrow Z+e^++e^-+k_2\,,\quad (\mathrm{I}.2)$$

for the case that all photons k_2 with energy less than $\Delta\omega_2$ are integrated over. It should thus be noted that the cross section $\sigma_{\rm real,\,soft}{}^P(\epsilon_+,\Delta\omega_2)$ for the process Eq. (I.2) cannot be disentangled from the virtual-photon radiative-correction effects due to the infrared divergence of $d\sigma_{\rm real,\,soft}{}^P$. The situation is completely analogous to the case of bremsstrahlung, where the total cross section can be defined only when the radiative corrections to the elastic-scattering cross section are included.

In addition, we also calculate in Sec. VIII the real-photon radiative correction due to the hard photons $d\sigma_{\rm real,\,hard}{}^P(\epsilon_+,\Delta\omega_2)$. The radiative correction to the pair spectrum for the emission of a secondary photon of any

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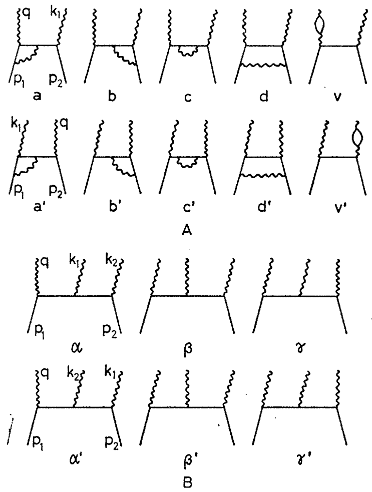

FIG. 1. Bremsstrahlung radiative-correction diagrams. A–virtual-photon radiative corrections, B–real-photon radiative corrections.

energy in the pair production process,

$$d\sigma_{\text{rad}}{}^P(\epsilon_+)=d\sigma_{\text{soft}}{}^P(\epsilon_+,\Delta\omega_2)+d\sigma_{\text{real, hard}}{}^P(\epsilon_+,\Delta\omega_2)\,,\quad \text{(I.3)}$$

is obtained in Sec. IX.

The radiative correction to the total pair cross section obtained from Eq. (I.3) by integrating over the pair spectrum,

$$\sigma_{\text{rad}}{}^P=\sigma_{\text{soft}}{}^P+\sigma_{\text{real, hard}}{}^P\,,\quad \text{(I.4)}$$

is calculated in Sec. X. The cross section $\sigma_{\text{rad}}{}^P$ represents the radiative correction to the gamma absorption coefficient. Relations to experiments are given in Sec. XI.

In this paper all energies and momenta are measured in units of mc^2 and mc, respectively.

II. THE WEIZSÄCKER-WILLIAMS METHOD

When the diagrams for radiative corrections to bremsstrahling, Figs. 1 (a)–(d) and (a′)–(d′), are compared to the diagrams for radiative corrections to the Compton effect, Figs. 2 (a)–(d) and (a′)–(d′), it is seen that the bremsstrahlung diagrams, except the vacuum-polarization diagrams v and v′, are similar to the Compton-effect diagrams, the only difference being the appearance of the Coulomb interaction q in the case of bremsstrahlung, while in the Compton effect an interaction with the incident photon k_0 occurs. Thus it is clear that since at high energies in the rest system of the electron the field of the nucleus will look like the transverse field of a photon, the contribution to the radiative correction to bremsstrahlung from all the diagrams not involving vacuum polarization may be obtained from the radiative correction to Compton scattering. The vacuum-polarization diagrams which correspond in the Weizsäcker-Williams picture to the change of the number of virtual photons in the Coulomb field of the nucleus due to vacuum polarization will be considered separately.

Before we consider the radiative corrections to bremsstrahlung, we discuss briefly the Weizsäcker-Williams method for the bremsstrahlung process without radiative corrections.

The bremsstrahlung cross section for the emission of a photon of momentum and energy $\mathbf{k}_1$ and ω_1 in the laboratory system is in the Weizsäcker-Williams approximation[7]

$$d\sigma_0{}^B(\mathbf{k}_1)=\int_1^R 2\pi\rho d\rho d\omega_0{}'F(\omega_0{}')d\sigma^C(\omega_0{}',\mathbf{k}_1{}')\quad \text{(II.1)}$$

where primed quantities refer to the rest system of the electron and unprimed to the laboratory system.

$F(\omega_0{}')d\omega_0{}'$ is the number of virtual photons with energy $\omega_0{}'$ in the field of the nucleus in the rest system of the electron

$$F(\omega_0{}')d\omega_0{}'=\alpha\left(\frac{Z}{\pi\rho}\right)^2\frac{d\omega_0{}'}{\omega_0{}'}\,,\quad \text{for}\quad \omega_0{}'<\epsilon_1/\rho$$

$$=0\,,\qquad \text{for}\quad \omega_0{}'>\epsilon_1/\rho\,,\quad \text{(II.2)}$$

where $\alpha=e^2/\hbar c$, and ρ is the impact parameter.

The Compton-scattering cross section may be written

$$d\sigma^C(\omega_0{}',\mathbf{k}_1{}')=\tfrac{1}{2}r_0{}^2(\omega'/\omega_0{}')^2U^C d\Omega_1{}'\,,\quad \text{(II.3)}$$

with $r_0=e^2/mc^2$ and

$$U^C=4\left(\frac{1}{\kappa}+\frac{1}{\tau}\right)^2+4\left(\frac{1}{\kappa}+\frac{1}{\tau}\right)-\frac{\kappa}{\tau}-\frac{\tau}{\kappa}\,,\quad \text{(II.4)}$$

where $\kappa=-2\omega_0{}'$ and $\tau=2\omega_1{}'$.

The maximum impact parameter R is given by

$$R=2\epsilon_1\epsilon_2/\omega_1\quad \text{for no screening}$$

$$=183Z^{-1/3}\quad \text{for complete screening.}\quad \text{(II.5)}$$

The relation between the energy of the scattered photon $\omega_1{}'$ in the rest system of the electron and $\omega_0{}'$ is

$$\omega_1{}'=\omega_0{}'[1+\omega_0{}'(1-\cos\theta_1{}')]^{-1}\,,\quad \text{(II.6)}$$

where $\theta_1{}'$ is the scattering angle of the virtual photon.

Since in this work we are interested in energy spectra, the cross sections will always be integrated over angles.

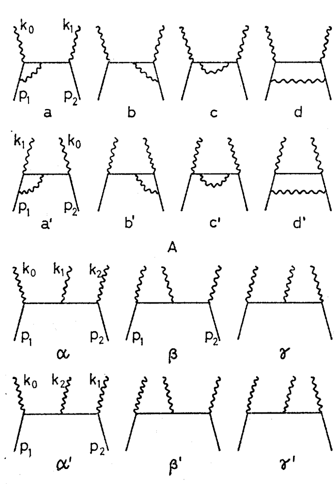

FIG. 2. Compton-effect radiative-correction diagrams. A–virtual-photon radiative corrections, B–real-photon radiative corrections.

It is then not necessary to transform the angle θ_1' to the laboratory system; the angular integration may be performed in the rest system of the electron. This procedure is in fact preferable from the point of view of calculations, since it turns out that the high-energy cross sections considered in this work are all sharply peaked functions of angle in the laboratory system, while in the rest system the cross sections are slowly varying functions of θ_1'. This means that the cross section in the rest system of the electron does not become singular for any angle even if we put $\beta_1=1$, and we may simplify our expressions considerably by putting $\beta_1=1$ in expressions of the type $1-\beta_1\cos\theta_1'$. The energy of the scattered photon in the laboratory system ω_1 is given by the Lorentz transformation with transformation velocity equal to the velocity of the electron in the laboratory system, β_1, where $\beta_1\approx 1$:

$$\omega_1=\epsilon_1\omega_1'(1-\cos\theta_1'). \tag{II.7}$$

From Eqs. (II.6) and (II.7) we find

$$\begin{aligned}\kappa&=-(2\omega_1/\epsilon_2)(1-\cos\theta_1')^{-1},\\ \tau&=(2\omega_1/\epsilon_1)(1-\cos\theta_1')^{-1},\\ d\omega_0'&=(\epsilon_1 d\omega_1/\epsilon_2^2)(1-\cos\theta_1')^{-1}.\end{aligned} \tag{II.8}$$

Inserting these expressions into Eqs. (II.1) and (II.3), we get, after performing the integration over the impact parameter ρ,

$$d\sigma_0^B(k_1)=(\alpha r_0^2Z^2/\pi)\ln R(\epsilon_2 d\omega_1/\epsilon_1\omega_1)U^C d\Omega_1', \tag{II.9}$$

where

$$U^C=(\omega_1^2/\epsilon_1\epsilon_2)+1+\cos^2\theta_1'. \tag{II.10}$$

The bremsstrahlung spectrum is obtained by integrating Eq. (II.9) over the angle θ_1'. The integration is extremely simple and yields

$$d\sigma_0^B(\omega_1)=4\alpha Z^2r_0^2\ln R(d\omega_1/\epsilon_1^2\omega_1)\times\{\epsilon_1^2+\epsilon_2^2-\tfrac{2}{3}\epsilon_1\epsilon_2\}. \tag{II.11}$$

III. THE VIRTUAL-PHOTON RADIATIVE CORRECTION TO BREMSSTRAHLUNG

From the discussion in Sec. II it follows that the virtual-photon radiative-correction cross section to bremstrahlung corresponding to the diagrams (a)–(d) and (a′)–(d′) of Fig. 1 is obtained from Eq. (II.1) by substituting for $d\sigma^C$ the virtual-photon radiative-correction cross section to Compton scattering $d\sigma_{\text{vir}}^C$ corresponding to diagrams (a)–(d) and (a′)–(d′) of Fig. 2,

$$d\sigma_{\text{vir}}^B(\mathbf{k}_1)=\int_1^R 2\pi\rho d\rho d\omega_0' F(\omega_0')d\sigma_{\text{vir}}^C(\omega_0',\mathbf{k}_1'). \tag{III.1}$$

The cross section $d\sigma_{\text{vir}}^C$ has been calculated by Brown and Feynman[8] with the result

$$d\sigma_{\text{vir}}^C=-(\alpha r_0^2/2\pi)(\tau/\kappa)^2\,\mathrm{Re}U_{\text{vir}}^C(\kappa,\tau)d\Omega_1'. \tag{III.2}$$

It is convenient for the subsequent integrations to rewrite Brown and Feynman's expression for U_{vir}^C in the form

$$U_{\text{vir}}^C(\kappa,\tau)=\sum_{n=1}^{8} i_n(\kappa,\tau)+i_9(\kappa,\tau)\ln\lambda, \tag{III.3}$$

where

$$i_n(\kappa,\tau)=i_n'(\kappa,\tau)+i_n'(\tau,\kappa), \tag{III.4}$$

with

$$\begin{aligned}
i_1'&=4\left(\frac{1}{\kappa}+\frac{1}{\tau}\right)^2-\frac{12}{\kappa}-\frac{3\kappa}{2\tau}-2\frac{\kappa}{\tau^2}+\frac{1}{\kappa-1}\left(\frac{\kappa}{\tau}+\frac{1}{2}\right),\\
i_2'&=G_0(\kappa)\left(\frac{\kappa^2}{\tau}+\frac{\kappa}{\tau}+\kappa+\frac{\tau}{2}+\frac{2}{\kappa}-\frac{3}{\tau}-1+\frac{\tau}{\kappa^2}\right),\\
i_3'&=\ln\kappa\left(\frac{3\tau}{2\kappa^2}+\frac{3\tau}{2\kappa}+\frac{3}{\tau}+1-\frac{7}{\kappa\tau}+\frac{8}{\kappa}-\frac{8}{\kappa^2}\right.\\
&\qquad\left.+\frac{2\kappa-\tau^2-\kappa^2\tau}{2\kappa^2\tau(\kappa-1)}-\frac{1}{2\tau}\frac{2\kappa^2+\tau}{(\kappa-1)^2}\right),\\
i_4'&=4y\tanh y\left(\frac{1}{\kappa}-\frac{1}{2}\right),\\
i_5'&=y^2\operatorname{csch}^2 y\left(\frac{2}{\kappa}-\frac{7\kappa}{4}-\frac{3\tau^2}{4\kappa}\right),\\
i_6'&=[-4y\sinh y(\kappa\tau)^{-1}(2-\cosh 2y)+2y\coth]h(y),\\
i_7'&=-2y\coth 2y(2h(y)-h(2y))U^C,\\
i_8'&=(\ln\kappa)4y\coth 2y\left[\frac{4}{\kappa\tau}\cosh^2 y+\frac{\kappa-6}{2\tau}\operatorname{sech}2y\right.\\
&\qquad\left.+\frac{4}{\kappa^2}-\frac{1}{\kappa}-\frac{\tau}{2\kappa}-\frac{\kappa}{\tau}-1\right],\\
i_9'&=(1-2y\coth 2y)U^C.
\end{aligned} \tag{III.5}$$

The quantities κ and τ are defined in Eq. (II.8); U^C is given by Eq. (II.4); and y is defined by

$$\sinh^2 y=(\gamma/2)(1-\cos\theta_1')^{-1}, \tag{III.6}$$

with

$$\gamma=\omega_1^2/\epsilon_1\epsilon_2. \tag{III.7}$$

The functions $G_0(\kappa)$ and $h(y)$ are given by

$$G_0(\kappa)=-(2/\kappa)\int_{1-\kappa}^{1}\ln(1-u)\,du/u, \tag{III.8}$$

and

$$h(y)=y^{-1}\int_0^y u\,du\coth u. \tag{III.9}$$

Equations (III.4) and (III.2) lead to a formula which is completely analogous to Eq. (II.9) where now

$-(\alpha/\pi)\,\mathrm{Re}U_{\mathrm{vir}}{}^C$ appears instead of U^C:

$$d\sigma_{\mathrm{vir}}{}^B(k_1)=-\frac{\alpha Z^2r_0{}^2}{\pi}\ln R\frac{\epsilon_2d\omega_1}{\epsilon_1\omega_1}\frac{\alpha}{\pi}\mathrm{Re}U_{\mathrm{vir}}{}^Cd\Omega_1{}'. \quad \text{(III.10)}$$

The cross section here is the virtual-photon radiative-correction cross section, which contains the infrared divergence and thus the "photon mass" λ. To this cross section has to be added the soft-photon double-bremsstrahlung cross section in order to remove λ in the usual way. We shall here proceed with the evaluation of $d\sigma_{\mathrm{vir}}{}^B$ and treat the soft-photon cross section later in Sec. V.

Equation (III.10) when integrated over the angle $\theta_1{}'$ gives the virtual-photon radiative correction to the bremsstrahlung spectrum

$$d\sigma_{\mathrm{vir}}{}^B(\omega_1)=-\frac{2}{\pi}(\alpha Zr_0)^2\ln R\frac{\epsilon_2d\omega_1}{\epsilon_1\omega_1}\{\sum_{n=1}^{8}I_n{}^B+I_9{}^B\ln\lambda\}, \quad \text{(III.11)}$$

where

$$I_n{}^B=I_n{}'^B(\epsilon_1,\epsilon_2)+I_n{}'^B(-\epsilon_2,-\epsilon_1), \quad \text{(III.12)}$$

$$I_n{}'^B(\epsilon_1,\epsilon_2)=\int_{-1}^{+1}d(\cos\theta_1{}')\,\mathrm{Re}i_n{}'(\kappa,\tau), \quad \text{(III.13)}$$

where $i_n{}'$ is given in Eq. (III.5). We have used the fact that interchanging κ and τ is equivalent to interchanging ϵ_1 with $-\epsilon_2$ and ϵ_2 with $-\epsilon_1$ as seen from Eq. (II.8). Therefore

$$\int_{-1}^{+1}d(\cos\theta_1{}')\,\mathrm{Re}i_n{}'(\tau,\kappa)=I_n{}'^B(-\epsilon_2,-\epsilon_1),$$

which is used in Eq. (III.12).

The evaluation of the integrals $I_n{}^B$ is straightforward but tedious. The results are listed in Appendix I. Equation (III.11) together with these expressions for $I_n{}^B$ will be used in Sec. VI.

IV. THE VACUUM-POLARIZATION RADIATIVE CORRECTION TO BREMSSTRAHLUNG

Since in high-energy small-angle bremsstrahlung the momentum transfer q is never larger than of the order 1, it is to be expected that the contribution to the radiative correction from the vacuum-polarization effects, which are important only for large momentum transfers, is rather small.

Using standard theory of quantum electrodynamics[9] we find that when the vacuum polarization diagrams v and v′ of Fig. 1 are included, the Bethe-Heitler matrix element M_{BH} is replaced by

$$M=M_{\mathrm{BH}}(1-\Pi(q^2)(1/q^2)), \quad \text{(IV.1)}$$

where $\Pi(q^2)$ is given in Eqs. (9)-(41) of Ref. 10. Renormalization has the effect of replacing $\Pi(q^2)$ by $q^4\Pi_f(q^2)$ where $\Pi_f(q^2)$ is given by[9]

$$\Pi_f(q^2)=\frac{\alpha}{3\pi q^2}\left[\frac{5}{3}-\zeta-\left(1-\frac{\zeta}{2}\right)(1+\zeta)^{1/2}\ln\frac{(1+\zeta)^{1/2}+1}{(1+\zeta)^{1/2}-1}\right], \quad \text{(IV.2)}$$

with $\zeta=4/q^2$.

Thus, when vacuum polarization is included, the bremsstrahlung cross section $d\sigma_0{}^B$ is replaced by

$$d\sigma_0{}^B+d\sigma_{\mathrm{vac}}{}^B=d\sigma_0{}^B[1-2q^2\Pi_f(q^2)]. \quad \text{(IV.3)}$$

The vacuum polarization thus has the effect of introducing a vacuum-polarization form factor $F_{\mathrm{vac}}=1-[1-2q^2\Pi_f(q^2)]^{1/2}=q^2\Pi_f(q^2)$ to lowest order in α. It should be noted that in the Weizsäcker-Williams method the vacuum polarization would correspond to an increase in the number of virtual photons in the field of the nucleus, since when vacuum polarization is included the electrostatic field of the nucleus is stronger than $-Ze^2/r$.

The cross section Eq. (IV.3) for high energies may easily be integrated over angles using Bethe's method[11] which is applicable to any form factor.

The integration is performed in Appendix II with the result

$$d\sigma_{\mathrm{vac}}{}^B(\omega_1)=\frac{2(\alpha Zr_0)^2}{81\pi}\frac{d\omega_1}{\epsilon_1{}^2\omega_1}[123(\epsilon_1{}^2+\epsilon_2{}^2)-62\epsilon_1\epsilon_2]. \quad \text{(IV.4)}$$

This result will be used in Sec. VI.

V. THE REAL-SOFT-PHOTON RADIATIVE CORRECTION TO BREMSSTRAHLUNG

The infrared divergence occurring in $d\sigma_{\mathrm{vir}}{}^B$ Eq. (III.11) through the term $\ln\lambda$ is removed in the usual way by adding to the virtual-photon radiative correction the real-soft-photon radiative correction. This correction, which is the cross section for the emission of an additional soft photon k_2 besides the hard photon k_1 in the bremsstrahlung process, is most easily obtained from the formula

$$d\sigma_{\mathrm{real,\,soft}}{}^B(\mathbf{k}_1,\mathbf{k}_2)=\frac{\alpha}{4\pi^2}d\sigma_0{}^B(\mathbf{k}_1)\frac{d^3k_2}{\omega_2}\left(\frac{p_1}{p_1\cdot k_2}-\frac{p_2}{p_2\cdot k_2}\right)^2, \quad \text{(V.1)}$$

where p_1, p_2, and k_1 occurring in the last factor are four-vectors and $d\sigma_0{}^B$ is the Bethe-Heitler cross section. Formula (V.1) follows from general considerations given by Jauch and Rohrlich.[12]

[9] Reference 10, p. 194, Eq. (9.66). Note that Eq. (9.66) contains a misprint; the minus sign in front of $k^2\Pi_f$ should be deleted. Equation (9.66) as it stands gives a positive $k^2\Pi_f$, while it is easily seen from Eq. (9.65) that $k^2\Pi_f$ is always negative. Equations (9.67) and (9.68) are, however, correct.

[10] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Cambridge, 1955) pp. 189–195.

[11] H. A. Bethe, Proc. Cambridge Phil Soc. **30**, 524 (1934).

[12] Reference 10, pp. 390–392. Note Eq. (16.3).

For the Bethe-Heitler cross section we use the Weizsäcker-Williams approximation Eq. (II.11):

$$d\sigma_0^B(\omega_1)=4\alpha Z^2 r_0^2 \ln R(d\omega_1/\epsilon_1^2\omega_1)\times\{\epsilon_1^2+\epsilon_2^2-\tfrac{2}{3}\epsilon_1\epsilon_2\}. \quad \text{(V.2)}$$

The cross section for the emission of an additional photon k_2 of energy less than $\Delta\omega_2$ in the bremsstrahlung process is obtained by integrating Eq. (V.1) over the part of phase space for which $\omega_2<\Delta\omega_2$, where $\Delta\omega_2\ll\epsilon_1$:

$$d\sigma_{\text{real, soft}}^B(\mathbf{k}_1,\Delta\omega_2)=(-\alpha/\pi)d\sigma_0^B I(\mathbf{p}_1,\mathbf{p}_2,\Delta\omega_2), \quad \text{(V.3)}$$

with

$$I(\mathbf{p}_1,\mathbf{p}_2,\Delta\omega_2)=\frac{-1}{4\pi}\int_{\omega_2<\Delta\omega_2}\frac{d^3k_2}{\omega_2}\left(\frac{p_1}{p_1\cdot k_2}-\frac{p_2}{p_2\cdot k_2}\right)^2. \quad \text{(V.4)}$$

Since this integral occurs in any process in which an emission of a soft photon takes place we have in Appendix III calculated I for general vectors $\mathbf{p}_1$ and $\mathbf{p}_2$. In our case of small angles and high energies we have according to Eq. (A3.9)

$$I(\mathbf{p}_1,\mathbf{p}_2,\Delta\omega_2)=2(1-2y\coth 2y)\ln\frac{\Delta\omega_2}{\lambda(\epsilon_1\epsilon_2)^{1/2}}+\tfrac{1}{2}\coth 2y\left\{2y\ln\frac{(1-\zeta^2)(\zeta^2\eta^2-1)}{4\zeta^2(\eta^2-1)}+\ln\frac{\eta+1}{\eta-1}\ln\frac{\zeta\eta+1}{\zeta\eta-1}\right.$$
$$\left.+L_2\left(\frac{(1+\zeta)(\zeta\eta+1)}{2\zeta(\eta+1)}\right)-L_2\left(\frac{(1-\zeta)(\zeta\eta-1)}{2\zeta(\eta+1)}\right)+L_2\left(\frac{(1+\zeta)(\zeta\eta-1)}{2\zeta(\eta-1)}\right)-L_2\left(\frac{(1-\zeta)(\zeta\eta+1)}{2\zeta(\eta-1)}\right)\right\}, \quad \text{(V.5)}$$

where

$$\zeta=\tanh y=\frac{\omega_1}{(\epsilon_1^2+\epsilon_2^2-2\epsilon_1\epsilon_2\cos\theta_1')^{1/2}}, \quad \text{(V.6)}$$

L_2 is the Euler dilogarithm

$$L_2(x)=-\int_0^x \ln(1-t)\,dt/t, \quad \text{(V.7)}$$

and

$$\eta=(\epsilon_1+\epsilon_2)/\omega_1. \quad \text{(V.8)}$$

When the variable ζ is introduced instead of θ_1' the cross section Eq. (V.3) integrated over the angle θ_1' becomes

$$d\sigma_{\text{real, soft}}^B(\omega_1,\Delta\omega_2)=-\frac{2}{\pi}(\alpha Z r_0)^2\ln R\frac{\omega_1 d\omega_1}{\epsilon_1^2\epsilon_2}\int_{\eta^{-1}}^{1}\frac{d\zeta}{\zeta^3}U^C I(\mathbf{p}_1,\mathbf{p}_2,\Delta\omega_2),$$

where U^C is given in Eq. (II.4).

The integration gives the result

$$d\sigma_{\text{real, soft}}^B(\omega_1,\Delta\omega_2)=-(2/\pi)(\alpha Z r_0)^2\ln R(\epsilon_2 d\omega_1/\epsilon_1\omega_1)\{I_9(\ln(\Delta\omega_2/\epsilon_1\lambda)-\tfrac{1}{2}\xi)$$
$$-(1/105)[\tfrac{2}{3}\pi^2(2\gamma^3+28\gamma^2+35\gamma)+4(4\gamma^3+48\gamma^2+87\gamma+46)\xi\eta(h(\xi)-\tfrac{1}{2}\ln\gamma)]$$
$$+[2/(105)^2][4(74\gamma^3-175\gamma-829)+(148\gamma^3+10787\gamma^2+22750\gamma+9660)\ln\gamma$$
$$+(148\gamma^3+11331\gamma^2+21069\gamma+8002)\xi\eta]\}. \quad \text{(V.9)}$$

Here γ, $h(\xi)$ and η are defined in Eqs. (III.7), (III.9), and (V.8), respectively, and

$$\xi=\ln(\epsilon_2/\epsilon_1). \quad \text{(V.10)}$$

VI. THE SOFT-PHOTON RADIATIVE CORRECTION TO THE BREMSSTRAHLUNG SPECTRUM

The high-energy bremsstrahlung spectrum including radiative corrections is, according to the discussion in Sec. I,

$$d\sigma^B(\omega_1,\Delta\omega_2)=d\sigma_0^B(\omega_1)+d\sigma_{\text{soft}}^B(\omega_1,\Delta\omega_2), \quad \text{(VI.1)}$$

where $d\sigma_0^B(\omega_1)$ is the bemsstrahlung spectrum without radiative corrections, the Bethe-Heitler spectrum, or our approximate cross section Eq. (II.11). The radiative correction to the spectrum is

$$d\sigma_{\text{soft}}^B(\omega_1,\Delta\omega_2)=d\sigma_{\text{vir}}^B(\omega_1)+d\sigma_{\text{vac}}^B(\omega_1)+d\sigma_{\text{real, soft}}^B(\omega_1,\Delta\omega_2), \quad \text{(VI.2)}$$

where the virtual-photon contribution $d\sigma_{\text{vir}}^B(\omega_1)$ is given by Eq. (III.11) and Appendix I, the vacuum-polarization contribution $d\sigma_{\text{vac}}^B(\omega_1)$ by Eq. (IV.4) and the soft-real-photon contribution $d\sigma_{\text{real, soft}}^B(\omega_1,\Delta\omega_2)$ by Eq. (V.9) for $\Delta\omega_2\ll\epsilon_1$.

We will write Eq. (VI.1) in the form

$$d\sigma^B(\omega_1,\Delta\omega_2)=d\sigma_0^B(\omega_1)[1+\delta_{\text{soft}}^B(\omega_1,\Delta\omega_2)], \quad \text{(VI.3)}$$

where

$$\delta_{\text{soft}}^B(\omega_1,\Delta\omega_2)=(d\sigma_{\text{vir}}^B+d\sigma_{\text{vac}}^B+d\sigma_{\text{real, soft}}^B)/d\sigma_0^B. \quad \text{(VI.4)}$$

TABLE I. The functions $F_1(\omega_1/\epsilon_1)$, $F_2(\omega_1/\epsilon_1)$ and $F_{\text{vac}}(\omega_1/\epsilon_1)$ occurring in the soft-photon bremsstrahlung radiative corrections Eq. (VI.5).

ω_1/ϵ_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$F_1\times 10^2$	−0.773	−0.340	−0.097	0.123	0.380	0.750	1.405	2.78	5.18
$F_2\times 10^2$	0.014	0.048	0.095	0.171	0.240	0.360	0.530	0.686	1.05
$F_{\text{vac}}\times 10^2$	0.198	0.196	0.194	0.192	0.190	0.188	0.185	0.183	0.180

The sum Eq. (VI.4) eliminates the infrared divergence $\ln\lambda$ since the coefficient of $\ln\lambda$ in $d\sigma_{\text{vir}}^B$, Eq. (III.11), is $-(2/\pi)(\alpha Zr_0)^2\ln R(\epsilon_2 d\omega_1/\epsilon_1\omega_1)I_9$, while it is $+(2/\pi)\times(\alpha Zr_0)^2\ln(\epsilon_2 d\omega_1/\epsilon_1\omega_1)I_9$ in $d\sigma_{\text{real, soft}}^B$, Eq. (V.9).

Adding Eqs. (III.11), (IV.4), and (V.9), we may write the radiative correction δ^B in the convenient form

$$\delta_{\text{soft}}^B(\omega_1,\Delta\omega_2)=F_1(\omega_1/\epsilon_1)+F_2(\omega_1/\epsilon_1)\ln(\Delta\omega_2/\epsilon_1)+F_{\text{vac}}(\omega_1/\epsilon_1)(\ln R)^{-1}, \quad \text{(VI.5)}$$

where F_1, F_2, and F_{vac} are functions of ω_1/ϵ_1 only and are given by

$$\begin{aligned}F_1(\omega_1/\epsilon_1)=-(\alpha/\pi a_0)[&a_1+(a_2+a_3\xi)\xi\\&+(a_4+a_5D)\eta+(a_6+a_7\xi+a_8\ln\gamma)\ln\gamma\\&+(a_9+a_{10}\xi+a_{11}\ln\gamma+a_{12}h(\xi/2)\\&+4a_{10}h(\xi))\xi\eta], \quad \text{(VI.6)}\end{aligned}$$

$$F_2(\omega_1/\epsilon_1)=(2\alpha/\pi a_0)(a_2+a_7\ln\gamma+a_{10}\xi\eta),$$

$$F_{\text{vac}}(\omega_1/\epsilon_1)=(\alpha/\pi a_0)(123\gamma+184)/81.$$

The quantities γ, η and ξ are given by Eqs. (III.7), (V.8), and (V.10), respectively, and D is defined by

$$D=L_2(\epsilon_1/\omega_1)-L_2(-\epsilon_2/\omega_1)=-\int_{-\epsilon_2/\omega_1}^{\epsilon_1/\omega_1}\ln(l-t)dt/t.$$

The coefficients in Eq. (VI.6) are

$$a_0=2\gamma+8/3,$$

$$a_1=\frac{8\pi^2}{315}\gamma^3+\left(-\frac{32}{105}+\frac{581}{630}\pi^2\right)\gamma^2+\left(\frac{59941}{105^2}+\frac{7\pi^2}{6}\right)\gamma+4\frac{12229}{105^2},$$

$$a_2=-\left(\frac{16}{105}\gamma^2+\frac{40}{21}\gamma+8\frac{23}{105}\right),$$

$$a_3=-\left(\frac{16}{105}\gamma^3+\frac{157}{84}\gamma^2+\frac{1763}{420}\gamma+\frac{424}{105}+\frac{132}{35\gamma}\right),$$

$$a_4=(\gamma-6)\frac{\pi^2}{6},$$

$$a_5=-3\gamma^2-(5/3)\gamma+2,$$

$$a_6=\frac{9653}{105^2}\gamma^2+\frac{102655}{2\times 105^2}\gamma+\frac{28}{105},$$

$$a_7=-\left(\frac{8}{105}\gamma^3+\frac{16}{15}\gamma^2+\frac{4}{3}\gamma\right),$$

$$a_8=3\gamma^2+5\gamma,$$

$$a_9=\frac{11333}{105^2}\gamma^2-\frac{9443}{2\times 105^2}\gamma+\frac{6608}{105^2},$$

$$a_{10}=-\left(\frac{8}{105}\gamma^3+\frac{32}{35}\gamma^2+\frac{58}{35}\gamma+\frac{92}{105}\right),$$

$$a_{11}=-\frac{16}{105}\gamma^2-\frac{1}{10}\gamma+\frac{119}{105},$$

$$a_{12}=\frac{32}{105}\gamma^3+\frac{142}{35}\gamma^2+8\frac{94}{105}\gamma-\frac{16}{21}. \quad \text{(VI.7)}$$

We note that δ_{soft}^B is of the order α/π when ω_1/ϵ_1 is of the order one, this is also true for the lower part of the spectrum where $\omega_1\to 0$. At the upper end of the spectrum the correction becomes of the order $(\alpha/\pi)\ln^2(\epsilon_1/\epsilon_2)$.

The functions F_1, F_2, and F_{vac} are given in Table I.

The contribution to δ^B from the vacuum polarization $F_{\text{vac}}(\ln R)^{-1}$ is small in all cases. When this small contribution is neglected δ_{soft}^B becomes a function of ω_1/ϵ_1 and $\Delta\omega_2/\epsilon_1$ only, independent of the initial energy ϵ_1 and of the atomic number of the bremsstrahlung target. The curves for δ_{soft}^B given in Fig. 3 for some values of $\Delta\omega_2/\epsilon_1$ show that the effect of the radiative corrections on the bremsstrahlung spectrum is small unless $\Delta\omega_2/\epsilon_1$ is extremely small. In the application of the present theory the energies of both the photon and secondary electron should be determined and then $\Delta\omega_2$ should be set equal to the maximum value of the energy imbalance $\epsilon_1-\epsilon_2-\omega_1$.

VII. THE SOFT-PHOTON RADIATIVE CORRECTION TO THE PAIR-PRODUCTION SPECTRUM

The radiative correction δ_{soft}^P to the pair spectrum when a soft secondary photon k_2 with energy less than $\Delta\omega_2$ is emitted in the pair-production process is given analogously to Eqs. (VI.3) and (VI.4) by

$$d\sigma^P(\epsilon_+,\Delta\omega_2)=d\sigma_0^P(\epsilon_+)[1-\delta_{\text{soft}}^P(\epsilon_+,\Delta\omega_2)], \quad \text{(VII.1)}$$

where

$$\delta_{\text{soft}}^P(\epsilon_+,\Delta\omega_2)=\{d\sigma_{\text{vir}}^P(\epsilon_+)+d\sigma_{\text{vac}}^P(\epsilon_+)+d\sigma_{\text{real, soft}}^P(\epsilon_+,\Delta\omega_2)\}/d\sigma_0^P(\epsilon_+). \quad \text{(VII.2)}$$

Here ϵ_+ is the positron energy. In all the following formulas the electron energy ϵ_- should be substituted for ϵ_+ when the energy spectrum of the electron is desired.

The cross section $d\sigma_{\text{soft}}{}^P(\epsilon_+,\Delta\omega_2)$ is the cross section for pair production with soft bremsstrahlung

$$Z+k_1 \rightarrow Z+e^+ + e^- + k_2\,, \qquad \text{(VII.3)}$$

when all secondary photons k_2 with energies less than $\Delta\omega_2$ are included.

The pair-production cross section $d\sigma_0{}^P(\epsilon_+)$, the virtual-photon, vacuum-polarization, and soft-photon radiative-correction cross sections $d\sigma_{\text{vir}}{}^P(\epsilon_+)$, $d\sigma_{\text{vac}}{}^P(\epsilon_+)$, and $d\sigma_{\text{real, soft}}{}^P(\epsilon_+,\Delta\omega_2)$, respectively, are obtained from the corresponding cross sections for bremsstrahlung, Eqs. (II.11), (III.11), (IV.4), and (V.9), by the substitutions $\epsilon_1 \rightarrow -\epsilon_+$, $\epsilon_2 \rightarrow \epsilon_-$, and $\omega_1 \rightarrow -\omega_1$ and by multiplication with the statistical factor ratio $\epsilon_+{}^2 d\epsilon_+/(\omega_1{}^2 d\omega_1)$. Furthermore, as discussed by Harris and Brown[13] for the case of two-quantum positron annihilation, the definition of y, Eq. (III.6), must be changed by substituting for y a new variable x by $y=x-i\pi/2$. The quantities i_n', Eq. (III.5), for the case of pair production may be shown to get the following changes besides the substitutions $\epsilon_1 \rightarrow -\epsilon_+$, $\epsilon_2 \rightarrow \epsilon_-$ and $\omega_1 \rightarrow -\omega_1$[13]:

$$\begin{aligned} \sinh^2 y &\rightarrow -\cosh^2 x\,, \\ \mathrm{Re}[y h(y)] &\rightarrow x[h(2x)-h(x)]\,, \qquad \text{(VII.4)} \\ \mathrm{Re}[y h(2y)] &\rightarrow x[h(2x)-\pi^2/4|x|]\,, \end{aligned}$$

where x is defined by

$$\cosh^2 x = \tfrac{1}{2}\gamma_p(1-\cos\theta_1')^{-1}\,, \qquad \text{(VII.5)}$$

with

$$\gamma_p = \omega_1{}^2/\epsilon_+\epsilon_-\,. \qquad \text{(VII.6)}$$

In this way we obtain $I_n{}^P$, where

$$I_n{}^P(\epsilon_+,\epsilon_-) = I_n{}'^P(\epsilon_+,\epsilon_-) + I_n{}'^P(-\epsilon_+,-\epsilon_-)\,,$$

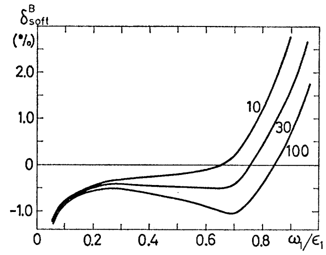

FIG. 3. Soft-photon radiative correction to the bremsstrahlung spectrum, $\delta_{\text{soft}}{}^B = F_1 + F_2 \ln(\Delta\omega_2/\epsilon_1)$. Numbers affixed to the curves give the values of $\epsilon_1/\Delta\omega_2$.

with

$$I_n{}'^P(\epsilon_+,\epsilon_-) = \int_{-1}^{+1} d(\cos\theta_1')\, \mathrm{Re}\, i_n{}'^P(\kappa,\tau)\,, \qquad \text{(VII.7)}$$

where $i_n{}'^P$ is obtained from i_n' of Eq. (III.5) by the substitutions Eq. (VII.4). The results are given in Appendix I.

For the virtual part of the radiative correction we find from Eq. (III.11)

$$d\sigma_{\text{vir}}{}^P(\epsilon_+) = -\frac{2}{\pi}(\alpha Z r_0)^2 \ln R \frac{\epsilon_+\epsilon_- d\epsilon_+}{\omega_1{}^3}\Big\{\sum_{n=1}^{8} I_n{}^P + I_9{}^P \ln\lambda\Big\}\,, \qquad \text{(VII.8)}$$

where the integrals $I_n{}^P$ are given in Appendix I.

The vacuum-polarization part of the virtual-photon radiative correction is from Eq. (IV.4):

$$d\sigma_{\text{vac}}{}^P(\epsilon_+) = \frac{2(\alpha Z r_0)^2}{81\pi}\frac{d\epsilon_+}{\omega_1{}^3} \times [123(\epsilon_+{}^2+\epsilon_-{}^2)+62\epsilon_+\epsilon_-]\,. \qquad \text{(VII.9)}$$

We obtain the real-soft-photon radiative correction from Eq. (V.9), noting that since the cross section is a first-order Born-approximation cross section for a real process no complications of the form discussed above Eq. (VII.4) may occur. We may thus directly substitute $\epsilon_1 \rightarrow \epsilon_+$, $\epsilon_2 \rightarrow \epsilon_-$, $\omega_1 \rightarrow -\omega_1$ and $\ln\gamma \rightarrow \ln\gamma_p$. Rewriting the expression in such a way that $\ln(\Delta\omega_2/\omega_1)$ appears explicitly, we obtain

$$\begin{aligned} d\sigma_{\text{real, soft}}{}^P(\epsilon_+,\Delta\omega_2) = -(2/\pi)(\alpha Z r_0)^2 \ln R(\epsilon_+\epsilon_- d\epsilon_+/\omega_1{}^3)\{&I_9 \ln(\Delta\omega_2/\omega\lambda) \\ &-(4/105)[\gamma_p(2\gamma_p{}^2-28\gamma_p+35)(\ln^2\gamma_p-\pi^2/6)-(4\gamma_p{}^3-48\gamma_p{}^2+87\gamma_p-46)\xi_p\eta_p(h(\xi_p)-\ln\gamma_p)] \\ &+[2/(105)^2][4(74\gamma_p{}^2+125\gamma_p-829)-(148\gamma_p{}^3-11627\gamma_p{}^2+33250\gamma_p-19320)\ln\gamma_p \\ &-(148\gamma_p{}^3-11331\gamma_p{}^2+21069\gamma_p-8002)\xi_p\eta_p]\}\,, \qquad \text{(VII.10)} \end{aligned}$$

where γ_p is defined in Eq. (VII.6) and

$$\xi_p = \ln(\epsilon_-/\epsilon_+)\,, \qquad \text{(VII.11)}$$

$$\eta_p = (\epsilon_+ - \epsilon_-)/\omega_1\,. \qquad \text{(VII.12)}$$

Collecting the terms, Eqs. (VII.8), (VII.9), and (VII.10), we find that the soft-photon radiative correction to pair production, Eq. (VII.2), analogously to Eq. (VI.5) may be written in the form

$$\delta_{\text{soft}}{}^P(\epsilon_+,\Delta\omega_1) = G_1(\epsilon_+/\omega_1) + G_2(\epsilon_+/\omega_1)\ln(\Delta\omega_2/\omega_1) + G_{\text{vac}}(\epsilon_+/\omega_1)(\ln R)^{-1}\,, \qquad \text{(VII.13)}$$

where

$$\begin{aligned} G_1(\epsilon_+/\omega_1) = -(\alpha/\pi a_{0p})\{&c_1 + c_2|\eta_p| + a_{5p}\eta_p D_p \\ &+(c_3+c_4\ln\gamma_p)\ln\gamma_p + a_{3p}\xi_p{}^2 + [a_{9p}+c_5\ln\gamma_p \\ &-a_{12p}h(\xi_p/2)+c_6 h(\xi_p)]\xi_p\eta_p\}\,, \qquad \text{(VII.14)} \end{aligned}$$

$$G_2(\epsilon_+/\omega_1) = (2\alpha/\pi a_{0p})[a_{2p}+a_{7p}\ln\gamma_p + a_{10p}\xi_p\eta_p]\,,$$

$$G_{\text{vac}}(\epsilon_+/\omega_1) = -(\alpha/\pi a_{0p})(123\gamma_p - 184)/81\,.$$

Here $D_p = L_2(\epsilon_+/\omega_1) - L_2(\epsilon_-/\omega_1)$. The coefficients a_{0p}, a_{3p}, a_{5p}, a_{9p}, a_{2p}, a_{7p}, and a_{10p} are obtained from the coefficients a_0, a_3, a_5, a_9, a_2, a_7, and a_{10} of Eq. (VI.7) by substituting $-\gamma_p$ for γ. The remaining coefficients

[13] I. Harris and L. M. Brown, Phys. Rev. 105, 1656 (1957).

are

$$c_1=-\frac{8}{315}\pi^2\gamma_p^3-\left(\frac{32}{105}-\frac{37}{45}\pi^2\right)\gamma_p^2-\left(\frac{59941}{(105)^2}+\pi^2\right)\gamma_p+4\frac{12229}{(105)^2}-\frac{2\pi^2}{3}-\frac{4\pi^2}{3\gamma_p},$$

$$c_2=\pi^2\left(\frac{8}{105}\gamma_p^3+\frac{32}{35}\gamma_p^2+\frac{174}{105}\gamma_p-\frac{92}{105}\right),$$

$$c_3=\frac{11333}{(105)^2}\gamma_p^2-\frac{144655}{2\times(105)^2}\gamma_p+4\frac{5565}{(105)^2}, \qquad \text{(VII.15)}$$

$$c_4=-\frac{8}{105}\gamma_p^3+\frac{109}{68}\gamma_p^2-\frac{31}{12}\gamma_p,$$

$$c_5=\frac{32}{35}\gamma_p^2-\frac{109}{70}\gamma_p+\frac{211}{105},$$

$$c_6=\frac{2}{5}\gamma_p^2-\frac{8}{15}\gamma_p-\frac{64}{15}.$$

The functions G_1, G_2, and $G_{\rm vac}$ are given in Table II. As for the case of bremsstrahlung, the contribution to the radiative correction due to vacuum polarization is always small. Neglecting $G_{\rm vac}(\ln R)^{-1}$ we obtain the curves for $\delta_{\rm soft}{}^P$ in Fig. 4. The form of the spectrum is independent of the atomic number of the target and of the initial photon energy ω_1, the only dependence on ω_1 is through the energy resolution $\Delta\omega_2/\omega_1$.

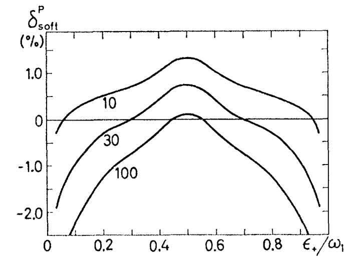

FIG. 4. Soft-photon radiative correction to the pair spectrum, $\delta_{\rm soft}{}^P=G_1+G_2\times\ln(\Delta\omega_2/\omega_1)$. Numbers affixed to the curves give the values of $\omega_1/\Delta\omega_2$.

VIII. THE REAL-HARD-PHOTON RADIATIVE CORRECTION TO PAIR PRODUCTION

We shall in this section calculate the contribution to pair-production effect from the process Eq. (VII.3) for the case that k_2 is a hard photon with energy larger than $\Delta\omega_2$. This contribution to the radiative correction, $\delta_{\rm hard}{}^P$, when added to $\delta_{\rm soft}{}^P$ gives the radiative correction to the spectrum δ^P when an extra photon k_2 of any energy is emitted during the pair-production process. This radiative correction is independent of $\Delta\omega_2$,

$$\delta^P(\epsilon_+)=\delta_{\rm soft}{}^P(\epsilon_+,\Delta\omega_2)+\delta_{\rm hard}{}^P(\epsilon_+,\Delta\omega_2). \qquad \text{(VIII.1)}$$

Again we use the Weizsäcker-Williams method. The cross section $d\sigma_{\rm real,\,hard}$ is obtained from the corresponding bremsstrahlung cross section $d\sigma_{\rm real,\,hard}{}^B$, with diagrams given in Fig. 1. The cross section $d\sigma_{\rm real,\,hard}{}^B$ is in turn obtained from the double Compton cross section which has been calculated by Mandl and Skyrme.[14] By a procedure exactly like that of Sec. III we obtain

$$d\sigma_{\rm real,\,hard}{}^P(\epsilon_+,\Delta\omega_2)=-(\alpha Zr_0)^2[\ln R/(2\pi)^3](d\epsilon_+/\omega_1)\int_{\Delta\omega_2}^{\omega_1-\epsilon_+}\left(\frac{\epsilon_-}{\epsilon_+{}^3}\right)\omega_2\,d\omega_2\int d\Omega' d\Omega_1'\,Xf_3{}^{-2}. \qquad \text{(VIII.2)}$$

The quantity X of Mandl and Skyrme is given by[14]

$$X=2(ab-c)[(a+b)(x+2)-(ab-c)-8]-2x(a^2+b^2)-8c+(4x/AB)[(A+B)(x+1)-(aA+bB)(2+z(1-x)/x)+x^2(1-z)+2z]-2\rho[ab+c(1-x)], \qquad \text{(VIII.3)}$$

with

$$a=\sum\kappa_i{}^{-1},\quad b=\sum(\kappa_i')^{-1},\quad c=\sum\kappa_i{}^{-1}(\kappa_i')^{-1},$$
$$x=\sum\kappa_i=\sum\kappa_i',\quad z=\sum\kappa_i\kappa_i',\quad A=\kappa_1\kappa_2\kappa_3, \qquad \text{(VIII.4)}$$
$$B=\kappa_1'\kappa_2'\kappa_3',\quad \rho=\sum(\kappa_i(\kappa_i')^{-1}+\kappa_i'\kappa_i{}^{-1}).$$

For the pair-production process with bremsstrahlung Eq. (VII.3), we find

$$\kappa_1=(\omega_1/\epsilon_+)(1-\cos\theta_1')^{-1},\quad \kappa_1'=(\omega_1/\epsilon_+)(f_1/f_3)\kappa_3,$$
$$\kappa_2=-(\omega_2/\epsilon_+)(1-\cos\theta_2')^{-1},\quad \kappa_2'=-(\omega_2/\epsilon_+)(f_2/f_3)\kappa_3, \qquad \text{(VIII.5)}$$
$$\kappa_3=f_3(\epsilon_+/\epsilon_-)(1-\cos\theta')^{-1}(1-\cos\theta_1')^{-1},\quad \kappa_3'=(\epsilon_-/\epsilon_+)\kappa_3,$$

with

$$f_1=\epsilon_+{}^{-1}\{-\omega_2(1-\cos\theta_1')+(\epsilon_++\omega_2)(1-\cos\theta_2')+(\omega_2/\epsilon_+)(\epsilon_++\omega_2)(1-\cos\theta_{12}')\},$$
$$f_2=\epsilon_+{}^{-1}\{\omega_1(1-\cos\theta_2')+(\epsilon_+-\omega_1)(1-\cos\theta_1')-(\omega_1/\epsilon_+)(\epsilon_+-\omega_1)(1-\cos\theta_{12}')\}, \qquad \text{(VIII.6)}$$
$$f_3=\epsilon_+{}^{-1}\{\omega_1(1-\cos\theta_2')-\omega_2(1-\cos\theta_1')+(\omega_1\omega_2/\epsilon_+)(1-\cos\theta_{12}')\}.$$

The angles θ_1' θ_2' and θ_{12}' are the angles in the rest system of the positron. θ_1' is the angle between the virtual photon and the incident photon k_1', θ_2' the angle between the virtual photon and the secondary photon k_2', and θ_{12}' the angle between k_1' and k_2'.

It is convenient to introduce the new variables

$$x=\omega_2/\omega_1,\quad y=\epsilon_+/\omega_1,\quad \Phi=\varphi/2\pi,$$
$$z_1=\tfrac{1}{2}(1-\cos\theta_1'),\quad z_2=\tfrac{1}{2}(1-\cos\theta_2').$$

[14] F. Mandl and T. H. R. Skyrme, Proc. Roy. Soc. (London) A215, 497 (1952).

TABLE II. The functions $G_1(\epsilon_+/\omega_1)$, $G_2(\epsilon_+/\omega_1)$ and $G_{\rm vac}(\epsilon_+/\omega_1)$ occurring in the soft-photon pair-production radiative correction Eq. (VII.13).

ϵ_+/ω_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$G_1\times 10^2$	2.70	2.22	2.16	2.30	2.53	2.30	2.16	2.22	2.70
$G_2\times 10^2$	1.07	0.750	0.632	0.553	0.526	0.553	0.632	0.750	1.07
$G_{\rm vac}\times 10^2$	0.174	0.171	0.168	0.166	0.165	0.166	0.168	0.171	0.174

The cross section Eq. (VIII.2) then becomes

$$d\sigma_{\text{real, hard}}{}^P(\epsilon_+,\Delta\omega_2)$$

$$=(2/\pi)(\alpha Zr_0)^2\ln R\frac{\omega_1{}^2d\epsilon_+}{\epsilon_+{}^3}\int_0^1 dz_1\int_0^1 dz_2\int_0^1 d\Phi \times\int_{\Delta\omega_2/\omega_1}^{1-y} dx\, x(x+y-1)X/f_3{}^2. \quad \text{(VIII.7)}$$

The fourfold integration in Eq. (VIII.7) is performed by the Monte Carlo method, that is, we evaluate the integrand a large number of times for random combinations of the variables x, Φ, z_1, and z_2. The probable error in the result, the root-mean-square deviation, is also obtained.

For small values of x the integrand behaves as $1/x$. In order to separate out the term $\ln(\Delta\omega_2/\omega_1)$ we write the integrand in the form

$$x(x+y-1)X/f_3{}^2 =x(x+y-1)X/f_3{}^2-[x(x+y-1)X/f_3{}^2]_{x\to 0} +[x(x+y-1)X/f_3{}^2]_{x\to 0} \quad \text{(VIII.8)}$$

where, since $\Delta\omega_2/\omega_1\ll 1$, we may put $\Delta\omega_2=0$ in the integral Eq. (VIII.7) for the combination of the first two terms of the integrand Eq. (VIII.8). The last term of Eq. (VIII.8), when integrated over x, gives $\ln[(\omega_1-\epsilon_+)/\Delta\omega_2]$ multiplied with the known function $d\sigma_0{}^PG_2(\epsilon_+/\omega_1)$, where $G_2(\epsilon_2/\omega_1)$ is given in Eq. (VII.14).

We may therefore write

$$d\sigma_{\text{real, hard}}{}^P(\epsilon_+,\Delta\omega_2)=d\sigma_0{}^P\{-(\alpha\gamma_p/\pi a_{0p})\Delta I +G_2\ln[(\omega_1-\epsilon_+)/\Delta\omega_2]\}, \quad \text{(VIII.9)}$$

where

$$\Delta I=y^{-3}\int_0^1 dz_1\int_0^1 dz_2\int_0^1 d\Phi\int_0^{1-y} dx[(x(x+y-1)X/f_3{}^2) -2(1-y)y^2x^{-1}\{4z_1{}^2-4z_1+2-y^{-1}(1-y)^{-1}\} \times\{y^2+(2z_1(1-y)/f_2)^2-2(1-2y(1-y)z_1)/f_2\}].$$

The integral ΔI was performed on the NBS IBM computer. The results are given in Table III. The statistical error is always smaller than 5%.

IX. THE TOTAL RADIATIVE CORRECTION TO THE PAIR SPECTRUM

When the result Eq. (VIII.9) is combined with $\delta_{\rm soft}{}^P$, Eq. (VII.13), we obtain the radiative correction δ^P, Eq. (VIII.1):

$$\delta^P(\epsilon_+)=G_1+G_2\ln[1-(\epsilon_+/\omega_1)] +G_{\rm vac}(\ln R)^{-1}-(\alpha\gamma_p/\pi a_{0p})\Delta I.$$

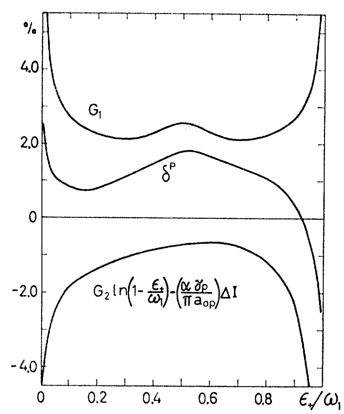

FIG. 5. Total radiative correction to the pair spectrum, $\delta^P(\epsilon_+/\omega_1)$, Eq. (IX.1). The virtual-photon contribution G_1, and real-photon contribution $G_2\ln[1-(\epsilon_+/\omega_1)]-(\alpha\gamma_p/\pi a_{0p})\Delta I$, are also shown.

Again we neglect the vacuum polarization contribution and δ^P is then a function of ϵ_+/ω_1 only, independent of the initial photon energy and of the target material:

$$\delta^P(\epsilon_+/\omega_1)=G_1(\epsilon_+/\omega_1)+G_2(\epsilon_+/\omega_1)\ln[1-(\epsilon_+/\omega_1)] -(\alpha\gamma_p/\pi a_{0p})\Delta I(\epsilon_+/\omega_1). \quad \text{(IX.1)}$$

This radiative correction is shown in Fig. 5. The statistical errors in δ^P are less than 5% and are due to the errors in ΔI. It should be noted that the pair spectrum is not symmetric about $\epsilon_+=\frac{1}{2}\omega_1$ when the radiative correction is included. The shift towards lower positron or electron energies is due to the energy loss by the emission of the secondary quantum k_2.

TABLE III. The integral $\Delta I(\epsilon_+/\omega_1)$ occurring in the real, hard pair-production radiative correction Eq. (VIII.9).

ϵ_+/ω_1	0.01	0.02	0.03	0.04	0.05	0.06	0.1	0.15	0.2	0.3
ΔI	−34.5	−29.3	−26.3	−24.2	−19.0	−17.2	−13.4	−10.1	−7.9	−5.0
ϵ_+/ω_1	0.4	0.5	0.6	0.7	0.8	0.85	0.9	0.95	0.97	1.0
ΔI	−3.35	−2.1	−1.0	−0.10	0.60	0.82	0.60	0.50	0.27	0.0

TABLE IV. The radiative correction to the total pair cross section Δ^P for incomplete screening, supplementing Eq. (X.4) (no screening) and Eq. (X.3) (complete screening).

	$Z=13$			$Z=29$		
ω_1	200	400	800	150	300	600
$\Delta^P(\%)$	1.03	0.98	0.94	1.04	0.99	0.94
		$Z=50$			$Z=82$	
ω_1	125	250	500	100	200	400
$\Delta^P(\%)$	1.04	0.99	0.94	1.06	1.00	0.94

When the present result is combined with the high-energy Coulomb-corrected pair specrtum,[15] we obtain

$$d\sigma^P(\epsilon_+)=\alpha Z^2r_0^2(d\epsilon_+/\omega_1^3)\{(\epsilon_+^2+\epsilon_-^2)[\Phi_1-\tfrac{4}{3}\ln Z-4f(Z)] +\tfrac{2}{3}\epsilon_+\epsilon_-[\Phi_2-\tfrac{4}{3}\ln Z-4f(z)]\}(1+\delta^P), \quad \text{(IX.2)}$$

which is the formula for the pair spectrum with a relative error which is given by the larger of the two numbers $(\alpha Z)^2 \ln\omega_1/\omega_1$ and 0.0005. The error of the order $(\alpha Z)^2 \ln\omega_1/\omega_1$ is due to the inaccuracy of the Coulomb correction for lower energies[16] and the number 0.0005 is the error in δ^P, Eq. (IX.1), due to the uncertainty in ΔI.

The functions Φ_1 and Φ_2 are given[11] and tabulated[17] by Bethe and Heitler. The quantity $f(z)$ is given by Davies, Bethe, and Maximon[15]. In Eq. (IX.2) we have neglected the Coulomb-correction effects on δ^P which is justified for the accuracy given. We have further assumed that δ^P is independent of screening also for the case of partial screening, since this is true both for no screening and for complete screening, as we have seen.

X. THE RADIATIVE CORRECTION TO THE TOTAL PAIR CROSS SECTION

The radiative correction to the total pair cross section Δ^P is obtained by integrating $d\sigma_{\text{soft}}^P$, Eq. (VII.2), and $d\sigma_{\text{real, hard}}^P$, Eq. (VIII.9), over the positron energy ϵ_+

$$\Delta^P=(1/\sigma_0^P)\int_1^{\omega_1-1} d\epsilon_+\left\{\left(\frac{d\sigma_{\text{soft}}^P}{d\epsilon_+}\right)+\left(\frac{d\sigma_{\text{real, hard}}^P}{d\epsilon_+}\right)\right\}$$

$$=(1/\sigma_0^P)\int_1^{\omega_1-1} d\epsilon_+\left(\frac{d\sigma_0^P}{d\epsilon_+}\right) \times\left\{G_1+G_2\ln\left(1-\frac{\epsilon_+}{\omega_1}\right)-\frac{\alpha\gamma_p}{\pi a_{0p}}\Delta I\right\}, \quad \text{(X.1)}$$

according to Eq. (IX.1). Here σ_0^P is the total pair cross section without radiative corrections.

The first two integrals in Eq. (X.1) were calculated analytically and the last intergral numerically. We use the Weizsäcker-Williams approximation for $d\sigma_0^P/d\epsilon_+$ obtained from $d\sigma_0^B/d\omega_1$, Eq. (II.11):

$$d\sigma_0^P/d\epsilon_+=4\alpha Z^2r_0^2 \ln R\omega_1^{-3}\{\epsilon_+^2+\epsilon_-^2+\tfrac{2}{3}\epsilon_+\epsilon_-\}. \quad \text{(X.2)}$$

The result is for complete screening,

$$\Delta^P=(0.93\pm0.05)\% \quad \text{(complete screening)}, \quad \text{(X.3)}$$

where the uncertainty is due to the Monte Carlo method as discussed previously. The vacuum-polarization contribution $\Delta_{\text{vac}}^P=0.032(1-0.06\ln Z)^{-1}\%$, is very small.

For the case of no screening we find, correspondingly,

$$\Delta^P=0.93\frac{\ln 2\omega_1-1.58}{\ln 2\omega_1-2.08}\% \text{(no screening)}, \quad \text{(X.4)}$$

and $\Delta_{\text{vac}}^P=0.17(\ln 2\omega_1-2.08)^{-1}\%$.

The deviation of (X.4) from (X.3) is small; indeed its largest value occurs at the lowest photon energy for which the present theory is valid, $\omega_1=30$ (15 MeV). Equation (X.4) gives $\Delta^P=1.12\%$ which is only slightly above the complete screening value 0.93%.

For the case of incomplete screening, sufficiently accurate values for Δ^P are obtained using the Weizsäcker-Williams approximation, Eq. (X.2), for $d\sigma_0^P/d\epsilon_+$, with $\ln R=-\tfrac{1}{2}\ln[(\omega_1/2\epsilon_+\epsilon_-)^2+(Z^{1/3}/183)^2]$. The results are given in Table IV for some energies and elements. These values together with Eqs. (X.3) and (X.4) are sufficient for calculation of Δ^P for all elements and energies above 15 MeV.

The relative error in Δ^P is in all cases of the same order as that given in Eq. (X.3), viz., of the order 5%.

The total pair cross section is then given by

$$\sigma^P=\alpha Z^2r_0^2\left\{\int_1^{\omega_1-1}\frac{d\epsilon_+}{\omega_1^3}[(\epsilon_1^2+\epsilon_2^2)\Phi_1+\tfrac{2}{3}\epsilon_1\epsilon_2\Phi_2] -(28/9)[\tfrac{1}{3}\ln Z+f(z)]\right\}(1+\Delta^P). \quad \text{(X.5)}$$

The integral involving Φ_1 and Φ_2 has to be computed numerically.[18]

We have also computed separately the contribution to the total-cross-section radiative correction due to the soft photons:

$$\Delta_{\text{soft}}^P=(1/\sigma_0^P)\int_1^{\omega_1-1} d\epsilon_+\frac{d\sigma_{\text{soft}}^P}{d\epsilon_+}.$$

The calculation is described in Appendix V. The result is, for the case of complete screening,

$$\Delta_{\text{soft}}^P=A+B\ln(\Delta\omega_2/\omega_1), \quad \text{(X.6)}$$

[15] H. Davies, H. A. Bethe, and L. C. Maximon, Phys. Rev. **93**, 788 (1954). See also H. Olsen and L. C. Maximon, Phys. Rev. **114**, 887 (1959), Eqs. (10.9) and (8.7).

[16] H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954). See also G. White Grodstein, Natl. Bur. Std. (U. S.) Circ. 583 (1957).

[17] H. A. Bethe and W. Heitler, Proc. Roy. Soc. (London) **A146**, 83 (1934).

[18] The most accurate evaluation of the screening effect using Hartree-Fock potentials is given by A. Sørenssen, Nuovo Cimento **38**, 745 (1965).

TABLE V. Comparison between experimental (Ref. 5) and theoretical values of the radiative correction to the total pair cross section for 1-BeV photons. (The theoretical value of Δ^P is 0.93% for all elements.)

Z Element	13 Al	22 Ti	29 Cu	42 Mo	50 Sn	73 Ta	82 Pb	92 U
$\Delta_{\rm expt}{}^P(\%)$	1.4 ± 1.3	2.7 ± 1.4	-0.8 ± 1.4	2.5 ± 1.3	-0.1 ± 0.8	-0.7 ± 1.9	-0.6 ± 1.0	-1.8 ± 1.1
$\Delta_{\rm theor}{}^P(\%)$				0.93				

with

$$A=-\frac{9\alpha}{14\pi}\left(\frac{512}{35}L_3(2)-\frac{1452}{35}\zeta(3)-\frac{316}{225}\pi^2+\frac{6283}{210}-\frac{128}{35}\pi^2\ln2\right),$$

$$B=-\frac{9\alpha}{14\pi}\left(6-\frac{128}{105}\pi^2\right).$$

Introducing the values for the Riemann zeta function ζ, $\zeta(3)=1.202$ and for the Euler trilogarithm L_3, $L_3(2)=2.762$, we obtain[19]

$$\Delta_{\rm soft}{}^P=[2.75+0.90\ln(\Delta\omega_2/\omega_1)]10^{-2}\quad\text{(complete screening)}.\quad\text{(X.7)}$$

Because of the complexity of the calculations involved in computing $\Delta_{\rm soft}{}^P$ we have checked the result, Eq. (X.6), by an independent method of calculation. We have integrated the differential radiative-correction pair cross sections $d\sigma_{\rm vir}{}^P(\mathbf{p}_+)$ and $d\sigma_{\rm real,\,soft}{}^P(\mathbf{p}_+,\Delta\omega_2)$, corresponding to the bremsstrahlung radiative-correction cross sections $d\sigma_{\rm vir}{}^B(\mathbf{k}_1)$ of Eq. (III.1) and $\sigma_{\rm real,\,soft}{}^B(\mathbf{k}_1,\Delta\omega_2)$ of Eq. (V.3), directly over angles and energies without first obtaining the energy spectrum. As this calculation is performed in a way substantially different from the way it is done in the text, one obtains an independent check on the soft-photon radiative correction Eq. (X.6) and thereby a check on the results of Secs. VI and VII.

XI. RELATIONS TO EXPERIMENTS

The comparison with experiments is made difficult because the theoretical pair cross section without radiative corrections is in many cases not known to the accuracy required, viz., to a fraction of a percent. The uncertainty in the theoretical cross section arises from the fact that the error in the Coulomb-corrected cross section is of the order $(\alpha Z)^2\ln\omega_1/\omega_1$ as stated below Eq. (IX.2). Another uncertainty, namely, that the screening correction based on the Thomas-Fermi model is not sufficiently accurate, seems now to be removed to some extent through the recent calculation of the screening effect based on Hartree-Fock potentials.[18] Because of the uncertainty in the theoretical triplet cross section we shall leave out in the present comparison elements lighter than aluminum.

When comparing with the available experiments[5] we have selected experiments where ω_1 and Z meet the requirement that the uncertainty in the cross section $(\alpha Z)^2\ln\omega_1/\omega_1$ should be less than 0.005 giving an uncertainty in the theoretical cross section which is less than 0.5%.[20]

For the highest photon energy, 13.5 BeV, for which the gamma absorption coefficient has been measured, the experimental value[21] of Δ^P is $(1.17\pm2.0)\%$ for copper and $(1.85\pm2.4)\%$ for lead. This is consistent with the theoretical value of Δ^P which is 0.93% for all elements.

For 1-BeV photons Table V shows that only for the case of uranium is there a serious disagreement between experimental and theoretical value of Δ^P. The average experimental radiative correction based on all elements in Table V is

$$\Delta_{\rm expt}{}^P=(0.33\pm2.0)\%,$$

which again is consistent with $\Delta_{\rm theor}{}^P=0.93\%$. The large (in the present context) experimental uncertainties prevent detailed comparison with theory.

Experimental[5,22–29] and theoretical radiative corrections for other energies are given in Fig. 6. Again the number of cases where serious disagreements occur are few. For all cases in Fig. 6 the pair production process is the dominant contribution to the total gamma-absorption cross section. The largest contribution from the Compton effect occurs for 60 MeV for aluminum, but the contribution is small, only 15% of the total cross

[19] This partial result has been given before, K. Mork and H. Olsen, Nuovo Cimento 18, 395 (1960).

[20] This requirement means that for Sn only experiments with energies above 50 MeV, for Pb above 225 MeV, and for U above 300 MeV can be used in the comparison.

[21] Since Sørenssen's results (Ref. 18) were not available at that time, the theoretical cross sections in Ref. 5 are based on Thomas-Fermi screening. We have for the case of 13.5- and 1-BeV photons corrected the theoretical cross sections in accord with Sørenssen's results.

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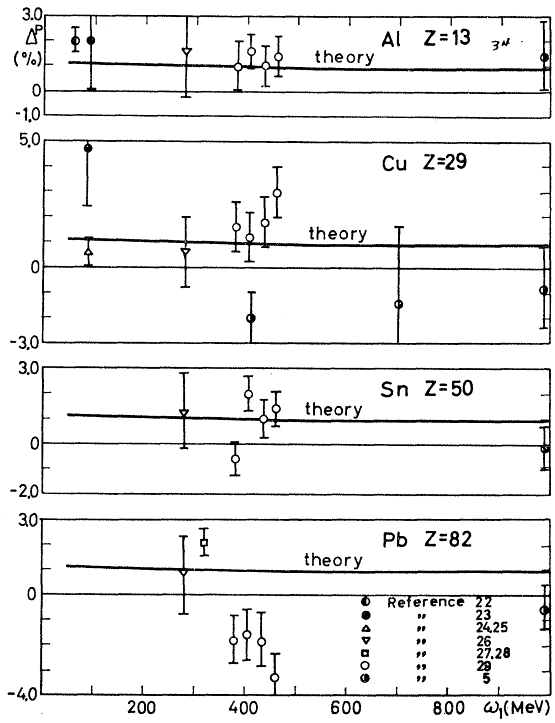

FIG. 6. Comparison with experiments. For the selection of experiments see Ref. 20. For the high-energy experiments of Malamud (Ref. 5) we have corrected the theoretical cross section according to Ref. 18. Other experimental points are taken from Malamud's paper (Ref. 5).

section. Thus for all cases in Fig. 6 the radiative correction to the total cross section is practically equal to the radiative correction to pair production.

It should be noted that qualitatively the increase in the radiative correction towards the lower energies of Fig. 6 is in accord with theory (Table IV). The experimental increase is, however, considerably larger than that predicted by theory, but again the agreement is fair within the experimental limits of uncertainty.

Finally as another relation of the present theory to experiments, the application to the photonuclear measurements might be mentioned. In these measurements the gamma absorption coefficient is measured as a function of energy and the photonuclear contribution is obtained by subtracting from the measured gamma-absorption coefficient the theoretical electromagnetic gamma-absorption coefficient. Since the photonuclear contribution is always considerably smaller than the electromagnetic contribution, the radiative correction to the electromagnetic processes is of the order of 1% when compared to the electromagnetic cross section but much larger when compared to the photonuclear cross sections. Thus to obtain a reliable interpretation, the radiative correction to pair production is needed. Also, the radiative correction to the Compton effect is necessary in order to obtain the necessary accuracy for the gamma-absorption coefficient. The latter will be given in a later paper.

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APPENDIX I: THE INTEGRALS I_n^B AND I_n^P

We list here the integrals I_n^B occurring in Eq. (III.11) for bremsstrahlung and the integrals I_n^P occurring in Eq. (VII.8) for pair production. With the definitions

$$\gamma=\omega_1^2/(\epsilon_1\epsilon_2)\,,\qquad \gamma_p=\omega_1^2/(\epsilon_+\epsilon_-)\,,$$
$$\xi=\ln(\epsilon_2/\epsilon_1)\,,\qquad \xi_p=\ln(\epsilon_-/\epsilon_+)\,,$$
$$\eta=(\epsilon_1+\epsilon_2)/\omega_1\,,\qquad \eta_p=(\epsilon_+-\epsilon_-)/\omega_1\,,$$
$$D=L_2(\epsilon_1/\omega_1)-L_2(-\epsilon_2/\omega_1)\,,$$
$$D_p=L_2(\epsilon_+/\omega_1)-L_2(\epsilon_-/\omega_1)\,,$$
$$h(x)=x^{-1}\int_0^x(\coth u)u\,du\,,$$

we obtain

$$I_1{}^B=7\gamma+(22/3)+\gamma(\gamma+\tfrac{5}{2})\ln\gamma+\gamma(\gamma+\tfrac{1}{2})\xi\eta\,,$$

$$I_2{}^B=\tfrac{1}{3}[-\gamma-4+2(3\gamma-2)\xi\eta+(\gamma^2-\gamma-(\gamma-6)\eta)\pi^2/2$$
$$-(3\gamma^2+19\gamma+20-32/\gamma)\xi^2/4+\tfrac{3}{4}\gamma(\gamma-1)\ln^2\gamma+\tfrac{1}{2}(\gamma-6)\xi\eta\ln\gamma-(3\gamma^2+2\gamma-6)\eta D]\,,$$

$$I_3{}^B=-3\gamma+(2/9)-\gamma(2\gamma+1)\eta D+\tfrac{1}{2}\gamma(\gamma+3)(\ln^2\gamma-\xi^2+\tfrac{2}{3}\pi^2)-(\gamma^2+3\gamma+\tfrac{2}{3})\ln\gamma+(\gamma^2+5\gamma+\tfrac{2}{3})\xi\eta\,,$$

$$I_4{}^B=2\gamma[\tfrac{1}{3}+(1+\tfrac{1}{6}\gamma)\ln\gamma-\tfrac{1}{6}(\gamma+4)\xi\eta]\,,$$

$$I_5{}^B=-(\tfrac{3}{2}\gamma+1-4/(3\gamma))\xi^2-4\gamma/45+2/15+\gamma(131\gamma/90+1)\ln\gamma+(131\gamma^2/90+49\gamma/45-4/15)\xi\eta\,,$$

$$I_6{}^B=(1/15)[2(3\gamma^2+2\gamma-32)\xi\eta h(\xi/2)+\tfrac{1}{6}\gamma(3\gamma-5)(3\xi^2+2\pi^2)$$
$$+(\gamma/30)(231\gamma-400)\ln\gamma+(1/30)(231\gamma^2-232\gamma-544)\xi\eta-7\gamma/5+59/15]\,,$$

$$I_7{}^B=(8/105)[(2\gamma^3+24\gamma^2+87\gamma/2+23)\xi\eta(2h(\xi/2)-h(\xi))+\gamma(2\gamma^2+28\gamma+35)\tfrac{1}{4}\pi^2+(1/210)(599\gamma^2+10701\gamma^2/2+6702\gamma+1586)\xi\eta+(1/210)(599\gamma^2+11417\gamma/2+6125)\gamma\ln\gamma+(1/105)(179\gamma^2-230\gamma-934)],$$

$$I_8{}^B=-(105)^{-2}[105(16\gamma^3+128\gamma^2-58\gamma+144+816/\gamma)\xi^2+105(16\gamma^3+96\gamma^2+202\gamma-132)\xi\eta\ln\gamma+5384\gamma^2+20102\gamma+18614+(2692\gamma^2+58113\gamma^2+57683\gamma-22578)\xi\eta+(2692\gamma^2+60137\gamma^2+36435\gamma+9030)\ln\gamma],$$

$$I_9{}^B=(16/105)[2\gamma^2+25\gamma+23+\tfrac{1}{2}(2\gamma^2+28\gamma+35)\gamma\ln\gamma+\tfrac{1}{2}(2\gamma^3+24\gamma^2+87\gamma/2+23)\xi\eta].$$

The corresponding quantities for pair production $I_n{}^P$ may be obtained from $I_n{}^B$ in the following way: $I_1{}^P$, $I_3{}^P$, $I_4{}^P$, $I_8{}^P$, and $I_9{}^P$ are obtained from $I_1{}^B$, $I_3{}^B$, $I_4{}^B$, $I_8{}^B$, and $I_9{}^B$, respectively, by replacing γ by $-\gamma_p$, ξ by ξ_p, η by η_p, $\ln\gamma$ by $\ln\gamma_p$, and D by D_p. $I_2{}^P$ is obtained from $I_2{}^B$ by replacing γ by $-\gamma_p$, ξ by ξ_p, η by η_p, $\ln\gamma$ by $\ln\gamma_p$, D by D_p, and $[\gamma^2-\gamma-(\gamma-6)\eta]\pi^2/2$ by $(\gamma_p{}^2+12\gamma_p-16/\gamma_p-10)\pi^2/2$. $I_5{}^P$, $I_6{}^P$, and $I_7{}^P$ are obtained from $I_5{}^B$, $I_6{}^B$, and $I_7{}^B$, respectively, by replacing γ by $-\gamma_p$, η by η_p, $\ln\gamma$ by $\ln\gamma_p$, ξ (when occurring linearly) by ξ_p (but ξ^2 by $\xi_p{}^2-\pi^2$), $h(\xi/2)$ by $h(\xi_p)-h(\xi_p/2)$, and $[2h(\xi/2)-h(\xi)]$ by $-[2h(\xi_p/2)-h(\xi_p)]+\pi^2/(2\xi_p|\eta_p|)$.

APPENDIX II: VACUUM POLARIZATION

According to Eq. (IV.3) of the text, the vacuum polarization contribution to the radiative correction is given by

$$d\sigma_{\mathrm{vac}}{}^B(\mathbf{k})=-2d\sigma_0{}^B(\mathbf{k})q^2\Pi_f(q^2) \qquad \text{(AII.1)}$$

where $d\sigma_0{}^B(\mathbf{k})$ is the differential bremsstrahlung cross section and $\Pi_f(q^2)$ is given in Eq. (IV.2). In order to integrate $d\sigma_{\mathrm{vac}}{}^B(\mathbf{k})$ over angles we use a method due to Bethe.[11] Bethe integrates the bremsstrahlung cross section using three variables x, y, and q^2. It is convenient for us to use the same method since Π_f is a function of q only. At high energies we find, according to Bethe,[11] that the cross section integrated over x and y is given by

$$d\sigma_{\mathrm{vac}}{}^B(\omega_1)=-2\alpha Z^2r_0{}^2\frac{d\omega_1}{\epsilon_1{}^2\omega_1}\int_\delta^\infty\frac{d(q^2)}{q^2}\Pi_f(q^2)\times\{(\epsilon_1{}^2+\epsilon_2{}^2)q^2 2\zeta\xi/(1+\zeta)^{1/2}-8\epsilon_1\epsilon_2(1-2\zeta\xi/(1+\zeta)^{1/2})\},$$

where we have used the fact that the contribution to the integral is negligible for q of the order δ, due to the rapid vanishing of $q^2\Pi_f(q^2)$ for small values of q. Here $\xi=\frac{1}{2}\ln[((1+\zeta)^{1/2}+1)/((1+\zeta)^{1/2}-1)]$, where $\zeta=4/q^2$. When ξ is introduced as a new variable we get

$$d\sigma_{\mathrm{vac}}{}^B(\omega_1)=-2(\alpha Zr_0)^2\frac{4}{3\pi}\int_0^\infty\frac{d\xi}{\sinh^2\xi}\times\{(\epsilon_1{}^2+\epsilon_2{}^2)\xi-\epsilon_1\epsilon_2(\coth\xi-\xi(\sinh^2\xi)^{-1})\}\times\{5/3-(\sinh^2\xi)^{-1}-(2-(\sinh^2\xi)^{-1})\xi\coth\xi\}. \qquad \text{(AII.2)}$$

When the integrals are performed the result given in Eq. (IV.4) is obtained.

APPENDIX III: THE REAL-SOFT-PHOTON CROSS SECTION

A. General

In the expression for the real-soft-photon cross section, Eq. (V.3), there appears the integral $I(\mathbf{p}_1,\mathbf{p}_2,\Delta\omega_2)$ which may be written, according to Eq. (V.4),

$$I(\mathbf{p}_1,\mathbf{p}_2,\Delta\omega_2)=\frac{1}{4\pi}\int_{\omega<\Delta\omega_2}\frac{d^3k}{\omega}\times\left\{\frac{1}{(p_1k)^2}+\frac{1}{(p_2k)^2}+\frac{2p_1p_2}{(p_1k)(p_2k)}\right\}. \qquad \text{(AIII.1)}$$

Since this integral is often encountered in calculations involving soft photons we shall first give a general expression for I in terms of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\Delta\omega_2$. We introduce the variable

$$\zeta=\omega/|\mathbf{k}|=[1+(\lambda/|\mathbf{k}|)^2]^{1/2},$$

λ being the "photon mass," and the integral becomes

$$I=\frac{1}{4\pi}\int_{\zeta_0}^\infty\frac{d\Omega d\zeta}{\zeta^2-1}\left\{\frac{1}{\epsilon_1{}^2(\zeta-\boldsymbol{\beta}_1\cdot\mathbf{e})^2}+\frac{1}{\epsilon_2{}^2(\zeta-\boldsymbol{\beta}_2\cdot\mathbf{e})^2}+\frac{2p_1p_2}{\epsilon_1\epsilon_2(\zeta-\boldsymbol{\beta}_1\cdot\mathbf{e})(\zeta-\boldsymbol{\beta}_2\cdot\mathbf{e})}\right\},$$

where

$$\boldsymbol{\beta}_1=\mathbf{p}_1/\epsilon_1,\quad \boldsymbol{\beta}_2=\mathbf{p}_2/\epsilon_2,\quad \mathbf{e}=\mathbf{k}/|\mathbf{k}|,$$

and

$$\zeta_0=[1+(\lambda/\Delta\omega_2)^2]^{1/2}.$$

We use the Feynman parametric method[30] to rewrite the last term in the integral in the following way:

$$\frac{1}{(\zeta-\boldsymbol{\beta}_1\cdot\mathbf{e})(\zeta-\boldsymbol{\beta}_2\cdot\mathbf{e})}=\int_0^1\frac{dx}{(\zeta-\boldsymbol{\beta}'\cdot\mathbf{e})^2},$$

with

$$\boldsymbol{\beta}'=(\boldsymbol{\beta}_1-\boldsymbol{\beta}_2)x+\boldsymbol{\beta}_2.$$

The integrations over angles and ζ are elementary and we obtain

$$I=(2+\mathcal{J}_1)\ln(2\Delta\omega_2/\lambda)-(1/2\beta_1)\ln[(1+\beta_1)/(1-\beta_1)]-(1/2\beta_2)\ln[(1+\beta_2)/(1-\beta_2)]-\mathcal{J}_2.$$

[30] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

The integral $\mathcal{I}_1$ is easily found to give

$$\mathcal{I}_1=\frac{2p_1p_2}{\epsilon_1\epsilon_2}\int_0^1\frac{dx}{1-\beta'^2}=4y\coth y,$$

where y is defined by

$$\sinh^2 y=\tfrac{1}{4}(p_1-p_2)^2. \qquad \text{(AIII.2)}$$

The other integral,

$$\mathcal{I}_2=\frac{p_1p_1}{\epsilon_1\epsilon_2}\int_0^1\frac{dx}{\beta'(1-\beta'^2)}\ln\frac{1+\beta'}{1-\beta'},$$

is more complicated. Introducing β' as a new variable, we find that $\mathcal{I}_2$ may be separated in two terms:

$$\mathcal{I}_2=\frac{p_1p_2}{\epsilon_1\epsilon_2\beta_-(1-a^2)^{1/2}}\times\{F(\beta_1)+\theta(\beta_2{}^2-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)F(\beta_2)\}, \qquad \text{(AIII.3)}$$

where $\theta(x)=\pm 1$ for $x\gtrless 0$ and

$$F(\beta)=(1-a^2)^{1/2}\int_a^\beta\frac{d\beta'}{(\beta'^2-a^2)^{1/2}}\frac{1}{1-\beta'^2}\ln\frac{1+\beta'}{1-\beta'},$$

with

$$a=\{(\beta_1{}^2\beta_2{}^2-(\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)^2)/\beta_-{}^2\}^{1/2} \qquad \text{(AIII.4)}$$

and

$$\boldsymbol{\beta}_-=\boldsymbol{\beta}_1-\boldsymbol{\beta}_2.$$

In order to convert $F(\beta)$ into known functions we take as new variable

$$\eta=(b(\beta'-a)/(\beta'+a))^{1/2},$$

with

$$b=(1-a)/(1+a). \qquad \text{(AIII.5)}$$

We then get

$$F(\beta)=\int_0^c d\eta\left(\frac{1}{1-\eta^2}+\frac{b}{b^2-\eta^2}\right)\ln\frac{b(1-\eta^2)}{(b^2-\eta^2)},$$

where

$$c(\beta)=(b(\beta-a)/(\beta+a))^{1/2}, \qquad \text{(AIII.6)}$$

with the result

$$F(\beta)=\tfrac{1}{2}\ln\frac{1+c}{1-c}\ln\frac{4b^2(1-c^2)^3}{(b^2-c^2)^2(1-b)^4}+\tfrac{1}{2}\ln\frac{b+c}{b-c}\ln\frac{(1-b)^4}{4(b^2-c^2)}$$
$$+\ln\frac{b+c}{2b}\ln\frac{b-c}{2b}-\ln\frac{1+c}{2}\ln\frac{1-c}{2}-2L_2\left(\frac{1+c}{2}\right)$$
$$+2L_2\left(\frac{b+c}{2b}\right)+2L_2\left(\frac{b+c}{1+c}\right)-2L_2\left(\frac{b-c}{1-c}\right), \qquad \text{(AIII.7)}$$

where the arguments of all L_2 functions are all positive and smaller than one since $0<c<b<1$, where b and c are given in Eqs. (AIII.5) and (AIII.6).

We have then

$$\begin{aligned}I(\mathbf{p}_1,\mathbf{p}_2,\Delta\omega_2)&=2(1-2y\coth 2y)\ln(2\Delta\omega_2/\lambda)\\&\quad-(1/2\beta_1)\ln[(1+\beta_1)/(1-\beta_1)]\\&\quad-(1/2\beta_2)\ln[(1+\beta_2)/(1-\beta_2)]\\&\quad+\tfrac{1}{2}\coth 2y[F(\beta_1)+\theta(\beta_2{}^2-\boldsymbol{\beta}_1\cdot\boldsymbol{\beta}_2)F(\beta_2)]. \qquad \text{(AIII.8)}\end{aligned}$$

B. $\varepsilon_1\gg 1$, $\varepsilon_2\gg 1$, $-p_1p_2\sim 1$

For the case of high energies and small angles needed in Sec. V of the text, Eq. (AIII.8) simplifies considerably. It should be noted that the expression for I obtained is valid for any process in which a high-energy particle is deflected through a small angle.

On terms of the variables

$$\eta=(\epsilon_1+\epsilon_2)/\omega_1 \quad \zeta=\tanh y$$

we find from (AIII.4)

$$1-a^2=\frac{\zeta^2(\eta^2-1)}{\epsilon_1\epsilon_2(1-\zeta^2)(\zeta^2\eta^2-1)},$$

so that $a\sim 1$, and consequently from Eqs. (AIII.5) and (AIII.6) it follows that $b\sim 1/\epsilon_1{}^2$, $c\sim 1/\epsilon_1{}^2$, and $F(\beta)$ simplifies to

$$F(\beta)=-\tfrac{1}{2}\ln\frac{b+c}{b-c}\ln 4(b^2-c^2)+L_2\left(\frac{b+c}{2b}\right)-L_2\left(\frac{b-c}{2b}\right),$$

with

$$b=\frac{\zeta^2(\eta^2-1)}{4\epsilon_1\epsilon_2(1-\zeta^2)(\zeta^2\eta^2-1)},$$

and

$$c(\beta_{1,2})=\frac{\zeta(\eta\mp 1)(\zeta^2\eta\pm 1)}{4\epsilon_1\epsilon_2(1-\zeta^2)(\zeta^2\eta^2-1)},$$

where the upper signs are valid for β_1 and the lower for β_2.

We then finally get

$$\begin{aligned}I(\mathbf{p}_1,\mathbf{p}_2,\Delta\omega_2)&=2(1-2y\coth 2y)\ln\frac{\Delta\omega_1}{\lambda(\epsilon_1\epsilon_2)^{1/2}}\\&\quad+\tfrac{1}{2}\coth 2y\left\{2y\ln\frac{(1-\zeta^2)(\zeta^2\eta^2-1)}{4\zeta^2(\eta^2-1)}+\ln\frac{\eta+1}{\eta-1}\ln\frac{\zeta\eta+1}{\zeta\eta-1}\right.\\&\quad+L_2\left[\frac{(1+\zeta)(\zeta\eta+1)}{2\zeta(\eta+1)}\right]-L_2\left[\frac{(1-\zeta)(\zeta\eta-1)}{2\zeta(\eta+1)}\right]\\&\quad\left.+L_2\left[\frac{(1+\zeta)(\zeta\eta-1)}{2\zeta(\eta-1)}\right]-L_2\left[\frac{(1-\zeta)(\zeta\eta+1)}{2\zeta(\eta-1)}\right]\right\} \qquad \text{(AIII.9)}\end{aligned}$$

which is used in Eq. (V.5).