Vertex-Function Poles and the Bootstrap Condition in Field Theory*

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We discuss the implications of maintaining finite mass renormalization when the wave-function renormalization constant Z_3 of an elementary particle is set equal to zero, making only the approximations of two-particle unitarity. We show that this defines a field-theory bootstrap for the elementary particle which is completely equivalent to the usual type of bootstrap based on the N/D method. As Z_3 goes to zero the vertex function and inverse propagator develop poles which move to μ^2, the elementary-particle mass, in the limit. For nonzero Z_3 this pole does not contribute to the scattering amplitude, but at $Z_3=0$ it cancels the elementary-particle pole in the single-particle reducible part, leaving the dynamical pole in the irreducible part. We suggest, further, that in this limit the bootstrapped state is a Regge pole.

I. INTRODUCTION

RECENT investigations in field theory and dispersion theory[1,2] have verified the result, first obtained in some model calculations,[3–6] that an elementary particle may be considered as composite when its wave-function renormalization constant Z_3 is set equal to zero. Specifically, it was shown in Ref. 1 that if two theories are defined by N/D equations with elastic unitarity which differ in their high-energy behavior according to Levinson's theorem,[7,8] then the $Z_3=0$ limit of the "elementary-particle" theory yielded a scattering amplitude identical to the "bound-state" theory. It has furthermore been suggested[9] that $Z_3=0$ should be made the criterion not only for compositeness but for a bootstrap theory where, loosely speaking, the forces responsible for binding a composite particle are a direct result of the existence of the particle itself. We examine this suggestion in the present paper and find that it is correct when the self-mass of the composite particle is finite.

Our method will be to use the results of renormalized field theory, making the approximation of elastic unitarity in the spectral representations of the propagator and vertex function. The recent results of Jin and MacDowell[10] on the properties of poles of the vertex and zeros of the propagator are quite important in this regard. In fact, we show that for small Z_3 the inverse propagator and hence the vertex function must develop a pole if $\delta\mu^2$ is to remain finite. These poles do not normally give rise to a pole in the scattering amplitude, a cancellation taking place between the irreducible single-particle part (IP) and reducible single-particle part (RP). However when $Z_3=0$ the elementary-particle pole itself is canceled by the other pole in the RP, this entire term vanishes and the pole in the IP is the pole in the scattering amplitude. That is, for $Z_3=0$ the "unphysical" vertex function pole becomes the bound state. Thus, along with the result that $Z_3=0$, $\delta\mu^2$ finite provide a satisfactory definition of a bootstrap, we also obtain a physical understanding of the vertex function pole as giving rise to the bootstrapped particle.

Jin and MacDowell[10] have suggested that the vertex function pole is associated with a Regge pole which, however, does not give rise to an elementary particle. Comparing this result with ours leads to the very suggestive idea that, in a field theory, particles with $Z_3=0$ and finite self-mass are bootstrapped states which lie on Regge trajectories. This result is in accord with the conclusions of Kaus and Zachariasen[9] in their comparison of the N/D bootstrap equations and field theory. They require $Z_1=0$ and $Z_1/Z_3=0$ as well as $Z_3=0$ and these conditions may be equivalent to $\delta\mu^2$ finite. Our work is thus complementary to theirs.[11]

Let us say a word about the finite self-mass condition which is crucial to our work. We know that this is not an empty condition because there exists a model theory, the Zachariasen model,[12] in which the A particle has finite self-mass even in the $Z_3=0$ limit.[13] In fact we shall show in the Appendix how our results are verified in this model. However, if the λB^4 coupling is set equal to zero then the self-mass of the A particle is infinite and our results no longer hold. Apparently some extra condition of this type is necessary and we specifically exclude theories of the type considered by Olesen[14] and Hagen.[15] In fact we shall show in Sec. IV that with our definition of a bootstrap, and if $\delta\mu^2$ is finite for nonzero

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[11] Our model does differ from that of Ref. 9, however, because we assume that the field theory has no dynamical pole for finite Z_3; cf. footnote 8 of Ref. 9.
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Z_3, then the requirement

$$\lim_{Z_3\to 0} Z_3\delta\mu^2=0$$

is necessary and our stronger assumption of finite $\delta\mu^2$ for $Z_3=0$ may be replaced by it.

In Sec. II we collect several well-known formulas for the vertex function, propagator, and scattering amplitude which define the model. Section III is devoted to the $Z_3=0$ limit for finite $\delta\mu^2$, and in particular we study the motion of the various singularities of these functions. We discuss our definition of a bootstrap and some other conclusions in Sec. IV. The Zachariasen model is considered as an example in the Appendix.

II. FIELD THEORY

We consider a theory of three spinless bosons A, B, and C. We shall consider s wave AB elastic scattering and assume that particle C, of mass μ^2, occurs as a pole in the scattering amplitude, i.e., the Yukawa interaction $C\rightleftarrows A+B$ exists. The C-particle propagator satisfies the Lehmann[16] spectral representation where we assume that no subtractions are necessary,

$$\Delta(s)=\frac{1}{s-\mu^2}-\frac{1}{\pi}\int_{s_T}^{\infty}ds'\frac{\sigma(s')}{s'-s-i\epsilon}. \tag{1}$$

In the elastic unitarity approximation

$$\begin{aligned}\sigma(s)&=\rho(s)\,|\langle 0|C(0)|AB\rangle|^2\\&=g^2\rho(s)\,|\Delta(s)|^2|\Gamma(s)|^2,\end{aligned}\tag{2}$$

where $\rho(s)$ is the two-particle, $l=0$, phase-space factor given by

$$\rho(s)=(1/16\pi)k/\sqrt{s}\,, \tag{3}$$

and $\Gamma(s)$ is the ABC proper vertex function with C off the mass shell. The total center-of-mass energy squared of AB is s, and k is their momentum. Since $\sigma(s)$ is positive definite, $\Delta(s)$ may have a zero at $s=m^2$, where

$$\mu^2\leq m^2\leq s_T \tag{4}$$

is given by the solution of

$$0=\frac{1}{m^2-\mu^2}-\frac{g^2}{\pi}\int_{s_T}^{\infty}ds'\frac{\rho(s')|\Delta(s')|^2|\Gamma(s')|^2}{s'-m^2}. \tag{5}$$

The inverse propagator is

$$\Delta^{-1}(s)=(s-\mu^2)\left[1-C\frac{s-\mu^2}{s-m^2}+\frac{s-\mu^2}{\pi}g^2\times\int_{s_T}^{\infty}ds'\rho(s')\frac{|\Gamma(s')|^2}{(s'-\mu^2)(s'-s-i\epsilon)}\right], \tag{6}$$

where the quantity $-C(m^2-\mu^2)^2$ is the residue of the pole in $\Delta^{-1}(s)$ at $s=m^2$ and is given by

$$\begin{aligned}C&=-\frac{1}{(m^2-\mu^2)^2}\left[\frac{\partial\Delta}{\partial s}\bigg|_{s=m^2}\right]^{-1}\\&=\left[1+\frac{(m^2-\mu^2)^2}{\pi}g^2\times\int_{s_T}^{\infty}ds'\frac{\rho(s')|\Delta(s')|^2|\Gamma(s')|^2}{(s'-m^2)^2}\right]^{-1}.\end{aligned}\tag{7}$$

The asymptotic behavior of the propagator is used to define Z_3, the C-particle wave-function renormalization constant, by

$$Z_3^{-1}=\lim_{s\to\infty} s\Delta(s).$$

Using Eqs. (1) and (6) we derive two expressions for Z_3,

$$Z_3^{-1}=1+\frac{g^2}{\pi}\int_{s_T}^{\infty}ds'\rho(s')|\Delta(s')|^2|\Gamma(s')|^2, \tag{8a}$$

$$Z_3=1-\frac{g^2}{\pi}\int_{s_T}^{\infty}ds'\frac{\rho(s')|\Gamma(s')|^2}{(s'-\mu^2)^2}-C. \tag{8b}$$

Following Lehmann[16] we may also derive two expressions for $\delta\mu^2\equiv\mu^2-\mu_0^2$, the C-particle mass renormalization,

$$\delta\mu^2=-Z_3\frac{g^2}{\pi}\int_{s_T}^{\infty}ds'\rho(s')(s'-\mu^2)|\Gamma(s')|^2|\Delta(s')|^2, \tag{9a}$$

$$\delta\mu^2=-Z_3^{-1}\frac{g^2}{\pi}\int_{s_T}^{\infty}ds'\rho(s')\frac{|\Gamma(s')|^2}{s'-\mu^2}-\frac{C}{Z_3}(m^2-\mu^2). \tag{9b}$$

Jin and MacDowell[10] have shown that the s-wave AB scattering amplitude

$$T(s)=e^{i\delta(s)}\sin\delta(s)/\rho(s)$$

has the decomposition[17]

$$T(s)=g^2\Gamma(s)\Delta(s)\Gamma(s)+U(s)\,, \tag{10}$$

where the first term is the RP and the second is the IP referred to in the Introduction. This decomposition has the feature that the second term satisfies unitarity by itself and the entire left-hand cut of $T(s)$ is contained in $U(s)$. Thus $U(s)$ is the dynamical part of $R(s)$. Furthermore, the imaginary part of $\Gamma(s)$ across the physical cut is directly related to $U(s)$,[1,10,18] and if there are no anomalous thresholds

$$\Gamma(s)=1+R\frac{(s-\mu^2)}{(s-m^2)}-\frac{s-\mu^2}{\pi}\int_{s_T}^{\infty}ds'\frac{\rho(s')U^*(s')\Gamma(s')}{(s'-\mu^2)(s'-s-i\epsilon)}. \tag{11}$$

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[16] H. Lehmann, Nuovo Cimento 11, 342 (1954).

[17] This decomposition was discussed by S. D. Drell and F. Zachariasen, Phys. Rev. 105, 1407 (1957).

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We have included the possibility of a pole in $\Gamma(s)$ at the position of the zero of the propagator according to the results of Ref. 10.

Since $\Gamma(s)$ has the phase of $U(s)$ along the physical cut we may write[19]

$$U(s)=\mathfrak{N}(s)\Gamma(s)\,, \tag{12}$$

where $\mathfrak{N}(s)$ has the left-hand cut of $U(s)$ and Eq. (11) becomes

$$\Gamma(s)=1+\frac{R(s-\mu^2)}{s-m^2} -\frac{s-\mu^2}{\pi}\int_{s_T}^{\infty}ds'\rho(s')\frac{\mathfrak{N}(s')\,|\,\Gamma(s')\,|^2}{(s'-\mu^2)(s'-s-i\epsilon)}. \tag{13a}$$

Using Eq. (13a) we write a dispersion relation for $\Gamma^{-1}(s)$,

$$\Gamma^{-1}(s)=1+\frac{s-\mu^2}{\pi}\int_{s_T}^{\infty}ds'\frac{\rho(s')\mathfrak{N}(s')}{(s'-s-i\epsilon)(s'-\mu^2)}. \tag{13b}$$

The information that $\Gamma^{-1}(s)$ has a zero at $s=m^2$ is expressed by

$$0=1+\frac{m^2-\mu^2}{\pi}\int_{s_T}^{\infty}ds'\rho(s')\frac{\mathfrak{N}(s')}{(s'-m^2)(s'-\mu^2)}, \tag{14}$$

while the constant R is given by

$$R^{-1}=(m^2-\mu^2)\frac{d\Gamma^{-1}(s)}{ds}\bigg|_{s=m^2} =(m^2-\mu^2)\frac{1}{\pi}\int_{s_T}^{\infty}ds'\frac{\rho(s')\mathfrak{N}(s')}{(s'-m^2)^2}. \tag{15}$$

We are, of course, making the assumption that the asymptotic behavior of $\mathfrak{N}(s)$ is such that Eq. (13) is valid.

Jin and MacDowell[10] point out that the residues R and C are related. In fact, from Eq. (6) we see that the pole in $\Delta^{-1}(s)$ arises from the distortion of the integration contour around the pole of $\Gamma(s)$ as it emerges from the second sheet at threshold. Using Eq. (20) of Ref. 10, we have

$$\mathfrak{N}(m^2)(m^2-\mu^2)=g^2R/C. \tag{16}$$

Of course if there are no poles in $\Gamma(s)$ and $\Delta^{-1}(s)$ we put $C=R=0$ and ignore the equations determining them and m^2.

III. THE $Z_3=0$ LIMIT

In this section we study the $Z_3=0$ limit of the theory. We make the assumption that $\delta\mu^2$ is finite, even in this limit, and although we shall not continually qualify our remarks, all statements made in the remainder of this paper are subject to this condition.

We now show that the propagator must develop a zero[20] for some critical, maximum Z_3 and, in particular, this zero is present for $Z_3=0$. The residue C, as given by Eq. (7), is greater than zero, and the pole position m^2 is greater than μ^2; thus there can be no cancellation of infinite terms in Eq. (9b), and finite $\delta\mu^2$ implies

$$\lim_{Z_3\to 0}\int_{s_T}^{\infty}ds'\rho(s')\frac{|\,\Gamma(s')\,|^2}{s'-\mu^2}=0, \tag{17}$$

and, if $C\neq 0$,

$$\lim_{Z_3\to 0}\,[C(m^2-\mu^2)]\sim Z_3\to 0. \tag{18}$$

(Of course both C and m^2 are functions of Z_3.) But the integrand in Eq. (17) is positive definite, so that

$$\lim_{Z_3\to 0}|\,\Gamma(s)\,|^2=0. \tag{19}$$

Taking the $Z_3=0$ limit of Eq. (8b), and using Eq. (19), we have

$$0=1-\lim_{Z_3\to 0} C. \tag{20}$$

So $C=1$ at $Z_3=0$, which proves the contention that $C\neq 0$ and Eq. (18) becomes

$$\lim_{Z_3\to 0}(m^2-\mu^2)\sim Z_3\to 0. \tag{21}$$

Thus at $Z_3=0$, the propagator has a zero which has moved down to the elementary-particle position at μ^2, and since its position is a continuous function of Z_3 it must move from the second sheet onto the first sheet at s_t for some critical nonzero Z_3. It should be noted that Eqs. (20) and (21) are consistent with the expression (7) for C.

Now consider Eq. (14) which is the condition that $\Gamma(s)$ have a pole at $s=m^2$. In the $Z_3=0$ limit we have

$$\lim_{Z_3\to 0}\frac{m^2-\mu^2}{\pi}\int_{s_T}^{\infty}ds'\frac{\rho(s')\mathfrak{N}(s')}{(s'-\mu^2)^2}=-1, \tag{22}$$

where we have used (21) to replace m^2 by μ^2 where it occurs explicitly in the integrand. Now Eq. (15) yields

$$\lim_{Z_3\to 0} R=-1. \tag{23}$$

Thus we have the picture that as Z_3 is decreased a pole in $\Gamma(s)$ and a zero in $\Delta(s)$ appear to keep $\delta\mu^2$ finite. This singularity moves to the left as Z_3 is further decreased and approaches μ^2, the position of the elementary particle as Z_3 approaches zero. At the same time

[19] The relation of $\Gamma(s)$ to $U(s)$ is the same as that of $F(s)$, the form factor defined by $F(s)(s-\mu^2)^{-1}=\Gamma(s)\Delta(s)$, to $T(s)$.

[20] A similar remark was made by J. S. Dowker, Nuovo Cimento **36**, 304 (1965).

the residue constants C and R approach one and minus one, respectively.

It is apparent in the simultaneous limit $Z_3=0$, $s=\mu^2$ that $\Gamma(s)$ and $\Delta(s)$ are poorly defined. For example, depending on how we take the above limits, $\Delta(\mu^2)$ is either zero or has a pole while $\Gamma(\mu^2)$ is either one or has a pole. This behavior is not too surprising since neither $\Gamma(s)$ nor $\Delta(s)$ are observable functions and to make further progress we turn to expression (10) for the scattering amplitude.

We have previously noted that the RP, $\Gamma(s)\Delta(s)\Gamma(s)$, has poles at $s=\mu^2$ and $s=m^2$ while the IP, $U(s)$, has a pole at $s=m^2$. We write a dispersion relation for each of these functions using the fact that $\Gamma(s)\Delta(s)\Gamma(s)$ has only a right-hand cut while $U(s)=\mathfrak{N}(s)\Gamma(s)$ satisfies two-particle unitarity on the right-hand cut,

$$g^2\Gamma(s)\Delta(s)\Gamma(s)=\frac{g^2}{s-\mu^2}-\frac{g^2R/C}{s-m^2}+\frac{g^2}{\pi}\int_{s_T}^{\infty}ds'\frac{\operatorname{Im}(\Gamma(s')\Delta(s')\Gamma(s'))}{s'-s-i\epsilon},\quad(24)$$

$$\mathfrak{N}(s)\Gamma(s)=\frac{\mathfrak{N}(m^2)R(m^2-\mu^2)}{s-m^2}+\frac{1}{\pi}\int_{s_T}^{\infty}ds'\frac{\rho(s')|\mathfrak{N}(s')|^2|\Gamma(s')|^2}{s'-s-i\epsilon}+\int_L ds'\frac{\operatorname{Im}(\mathfrak{N}(s'))\Gamma(s')}{s'-s-i\epsilon}.\quad(25)$$

(The question of possible subtractions need not concern us here.) Equation (16) ensures that the poles at $s=m^2$ cancel and so do not appear in the scattering amplitude as has been noted by many people.[10,21,22] However, in the $Z_3=0$ limit, Eqs. (20), (21), and (23) show that the two poles in Eq. (24) cancel each other while Eq. (25) becomes

$$\mathfrak{N}(s)\Gamma(s)=g^2/(s-\mu^2)+\cdots.\quad(26)$$

Thus, in this limit the term $\Gamma(s)\Delta(s)\Gamma(s)$ no longer has a pole corresponding to the C particle at all, while $\Gamma(s)$ contributes the particle pole at $s=\mu^2$ to $T(s)$ with the correct residue g^2.

In fact we can go further and show that the entire RP vanishes in the limit $Z_3=0$. This is because $F(s)$, the form factor,[19] is given by

$$F(s)=1/D(s),$$

where

$$T(s)=N(s)/D(s),$$

and is well behaved and nonvanishing in the $Z_3=0$ limit since the scattering amplitude is nonvanishing [essentially because $F(s)$ is an observable whereas $\Gamma(s)$ is not]. But

$$\Gamma(s)\Delta(s)\Gamma(s)=\Gamma(s)(s-\mu^2)^{-1}F(s)$$

and thus, according to Eq. (19)

$$\lim_{Z_3\to 0}(\Gamma(s)\Delta(s)\Gamma(s))=0.$$

We emphasize that this behavior is possible and consistent with Eq. (24) only because the pole at $s=\mu^2$ is cancelled and vanishes at $Z_3=0$.

The reason why $\Delta(s)$ develops a zero for small Z_3 is now clear. For Z_3 nonzero, the pole at $s=m^2$ of the RP is harmless and is just canceled by the pole of $U(s)$. At $Z_3=0$, however, the RP vanishes. This is possible only if the pole at $s=\mu^2$ due to the intermediate C particle is canceled (its residue is g^2 which is nonzero) and the extra pole at $s=m^2$ effects this cancellation leaving us with

$$T(s)=U(s)=\mathfrak{N}(s)\Gamma(s).\quad(27)$$

The pole in $\Gamma(s)$ contributes the C-particle pole to $T(s)$ and the C particle is now a composite.

IV. DISCUSSION

We now see why the C particle should be termed composite when its wave-function renormalization constant is zero. Although $T(s)$ is perfectly well behaved in this limit and has a pole at $s=\mu^2$, the interpretation of this pole is different. For nonzero Z_3 the pole was due to an elementary C particle in an intermediate state, reflected by the fact that the pole arose from the RP, $\Gamma(s)\Delta(s)\Gamma(s)$, i.e., from the C-particle propagator. At $Z_3=0$, however, this propagator no longer contributes to $T(s)$ and the pole arises from $U(s)$, the dynamical term.

The above provides the connection between a field-theory bootstrap and the more familiar calculations based on the N/D method.[23] In the latter, one assumes the existence of an elementary C particle and calculates $D(s)$ by assuming that the exchange of C provides the left-hand cut of $T(s)$. The requirement that $T(s)$ have a pole at $s=\mu^2$ with residue g^2 then provides the bootstrap. In field theory, however, one would expect the assumed existence of C to provide a pole in $T(s)$ via the RP, and it would seem to be inconsistent to ignore this term. Thus, it would appear that if one actually carried through a consistent N/D bootstrap in field theory one would obtain the C-particle pole twice, once dynamically and once kinematically. In Ref. 1, however, it was shown that the N/D compositeness conditions are equivalent to $Z_3=0$ for C, and thus with the additional requirement of finite mass renormalization our result shows that the RP vanishes, its pole disappearing, and thus the N/D bootstrap is completely equivalent to the field-theory calculation.

[21] C. Goeble and B. Sakita, Phys. Rev. Letters **11**, 293 (1963).
[22] S. D. Drell, A. C. Finn, and A. C. Hearn, Phys. Rev. **136**, B1439 (1964).
[23] F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961).

The pole of the dynamical term $U(s)$ is just the pole which appears as the zero of the D function in the N/D method.

Jin and MacDowell[10] have given an interpretation of the pole in $\Gamma(s)$ at $s=m^2$ as a Regge pole. They argue that if the C particle is not a Regge pole and if a Regge trajectory exists, with the quantum numbers of C and the proper signature, which passes through $l=0$ at $s=m^2$, then the pole in $\Gamma(s)$ develops to cancel the Regge pole in $U(s)$ so that it does not contribute to $T(s)$. But we have shown that for $Z_3=0$, the pole in $U(s)$ *is* the pole of $T(s)$. Thus, if Jin and MacDowell are correct we conclude that the conditions $Z_3=0$, $\delta\mu^2$ finite not only define a field-theory bootstrap but are the conditions which make C a Regge pole.

We may reverse the arguments given in the above by assuming that a bootstrap solution exists. So we assume that a pole develops in the RP which, in the limit of the bootstrap, has moved down to μ^2. Equation (7) shows that if this occurs then in this limit

$$C=1\,, \tag{28a}$$

while from Eqs. (14) and (15)

$$R=-1\,. \tag{28b}$$

Therefore, making only the assumption that a pole in the RP moves down to μ^2, it automatically follows that the residues take on the correct values to effect all the cancellations necessary for the bootstrap. Furthermore, from Eq. (8b) and the conditions

$$0\leq Z_3\leq Z\,,$$

we see that

$$Z_3=0 \tag{28c}$$

since the right-hand side of Eq. (8b) is negative, semidefinite for $C=1$, and therefore

$$|\Gamma(s)|^2\rightarrow 0\,. \tag{28d}$$

Thus the condition that the RP vanish, which we take as the definition of a bootstrap, is ensured by assuming that a pole in $\Gamma(s)$ moves to μ^2 and this yields all the conditions (28). Our original assumptions of $\delta\mu^2$ finite, $Z_3=0$ are thus sufficient for a bootstrap, but we cannot show that the first one is strictly necessary. If, however, $\delta\mu^2$ is finite for finite Z_3, then Eq. (9b), together with (28) yields

$$\lim_{Z_3\to 0} Z_3\delta\mu^2=0\,,$$

which *is* a necessary and sufficient condition on the behavior of $\delta\mu^2$ for the existence of a bootstrap solution in field theory.

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APPENDIX: THE ZACHARIASEN MODEL

In this Appendix[24] we illustrate the conclusions of Sec. III in a version of the Zachariasen model[12] which includes three- and four-particle interactions. The scattering amplitude in this model is pure s wave, has no left-hand cut, and satisfies elastic unitarity. The model is easily solved using the method developed by Whippman and Gerstein,[6] where it is shown that the absence of a left-hand cut in $T(s)$ requires $U(s)$ to be proportional to $\Gamma(s)$ so that

$$T(s)=g^2\Gamma(s)\Delta(s)\Gamma(s)+\alpha\Gamma(s)\,. \tag{A1}$$

The constant α is given by

$$\alpha=\lambda/Z_3\,.$$

Formulas applicable to this model are now obtained by specializing the appropriate equation of the text to

$$\mathfrak{N}(s)=\alpha=\lambda/Z_3\,.$$

From Eq. (13b) we obtain

$$\Gamma^{-1}(s)=1+\frac{s-\mu^2}{\pi}\frac{\lambda}{Z_3}\int_{s_T}^{\infty}ds'\frac{\rho(s')}{(s'-\mu^2)(s'-s-i\epsilon)}\,. \tag{A2}$$

The integral in Eq. (A2) can be evaluated explicitly, and it can be shown that as Z_3 becomes small (provided $\lambda\neq 0$) the function $\Gamma^{-1}(s)$ develops a pole at $s=m^2$ given by the solution of

$$(m^2-\mu^2)=-\frac{Z_3}{\lambda}\left[\frac{1}{\pi}\int_{s_T}^{\infty}ds'\frac{\rho(s')}{(s'-\mu^2)(s'-m^2)}\right]^{-1}. \tag{A3}$$

This zero first appears at the threshold, $s=s_T$, from the second sheet for some critical value of Z_3 and then travels down to $s=\mu^2$. We do not consider the case $\lambda=0$ since, as we shall see, it leads to $\delta\mu^2=\infty$ and so does not define a theory which satisfies our assumptions. To obtain the detailed behavior of m^2 as it approaches μ^2 we note that the integral in Eq. (A3) is finite for any $m^2<s_T$ so we have

$$\lim_{Z_3\to 0}\frac{m^2-\mu^2}{Z_3}=-\frac{1}{\lambda}\left[\int_{s_T}^{\infty}ds'\frac{\rho(s')}{(s'-\mu^2)^2}\right]^{-1}. \tag{A4}$$

From Eq. (A2) we verify the result

$$\lim_{Z_3\to 0}\Gamma(s)=0\,. \tag{A5}$$

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[24] The results of this section are based on part of a Ph.D. thesis submitted by N. G. Deshpande to the University of Pennsylvania, 1965 (unpublished).

The dispersion relation Eq. (13a) for $\Gamma(s)$ becomes

$$\Gamma(s)=1+\frac{R(s-\mu^2)}{s-m^2}-\frac{s-\mu^2}{\pi}\frac{\lambda}{Z_3}\int_{s_T}^{\infty}ds'\frac{\rho(s')|\Gamma(s')|^2}{(s'-\mu^2)(s'-s-i\epsilon)}, \quad (A6)$$

where

$$R^{-1}=\frac{m^2-\mu^2}{\pi}\frac{\lambda}{Z_3}\int_{s_{\frac{1}{2}}}^{\infty}ds'\frac{\rho(s')}{(s'-m^2)^2}. \quad (A7)$$

Using Eq. (A3) we have

$$R=-\int_{s_T}^{\infty}ds'\frac{\rho(s')}{(s'-\mu^2)(s'-m^2)}\Big/\int_{s_T}^{\infty}ds'\frac{\rho(s')}{(s'-m^2)^2}, \quad (A8)$$

and so, when $Z_3=0$, since $m^2=\mu^2$

$$R=-1. \quad (A9)$$

The analog of Eq. (16) for the Zachariasen model is

$$(\lambda/Z_3)(m^2-\mu^2)=g^2R/C, \quad (A10)$$

which guarantees that the pole in $\Gamma(s)$ at m^2 does not appear in $T(s)$ for $Z_3\neq 0$. Using Eqs. (8b), (A2), (A6), and (A10) we obtain by the method of Ref. 6

$$Z_3=1-\frac{g^2}{\pi}\int_{s_T}^{\infty}ds'\frac{\rho(s')}{(s'-\mu^2)^2}. \quad (A11)$$

Mass renormalization is obtained by taking the $s\to\infty$ limit of Eq. (A6), taking into account Eq. (A5) and using Eqs. (9b) and (A10),

$$\delta\mu^2=g^2/\lambda. \quad (A12)$$

Thus we see that $\lambda\neq 0$ is necessary for finite $\delta\mu^2$ and since Eq. (A12) is independent of Z_3, $\delta\mu^2$ remains finite in the $Z_3=0$ limit. The behavior of C when $Z_3=0$ is obtained from Eq. (A10) using Eqs. (A4), (A9), and (A11),

$$\lim_{Z_3\to 0} C=\lim_{Z_3\to 0} g^2R/[(\lambda/Z_3)(m^2-\mu^2)]=1. \quad (A13)$$

[It should be noted that we cannot use this method to obtain C in the full field theory considered in the text since we do not know, in general, the $Z_3=0$ behavior of $\mathfrak{N}(s)$ *a priori.*]

Thus we have verified explicitly in the Zachariasen model that when $Z_3=0$ and $\delta\mu^2$ is finite we have

$$C=1,\quad R=-1,\quad m^2=\mu^2,\quad \Gamma(s)=0, \quad (A14)$$

and we have shown in detail how these functions approach their limits.

Now let us consider the scattering amplitude $T(s)$. We note that the form factor $F(s)=(s-\mu^2)\Gamma(s)\Delta(s)$ is given by[12]

$$F(s)=\left[1+\frac{(s-\mu^2)}{\pi}\int_{s_T}^{\infty}ds'\times\frac{\rho(s')}{(s'-\mu^2)(s'-s-i\epsilon)}\left(\lambda+\frac{g^2}{s'-\mu^2}\right)\right]^{-1}, \quad (A15)$$

and is perfectly finite at $Z_3=0$. So from Eqs. (A1) and (A5) the first term in $T(s)$ vanishes and we have

$$T(s)=\lim_{Z_3\to 0}(\lambda/Z_3)\Gamma(s), \quad (A16)$$

which is finite. This term has a pole at $s=\mu^2$ with residue given by

$$\text{residue}=\lim_{Z_3\to 0} R(m^2-\mu^2)\lambda/Z_3 = g^2,$$

where

$$g^2=\left[\frac{1}{\pi}\int_{s_T}^{\infty}ds'\frac{\rho(s')}{(s'-\mu^2)^2}\right]^{-1},$$

as is required by $Z_3=0$. Thus the pole in $(\lambda/Z_3)\Gamma(s)$ materializes at $s=\mu^2$ as a pole of the scattering amplitude in the $Z_3=0$ limit and has the correct residue required by this limit, just as in the full field theory the cancellation of the poles in $\Gamma(s)\Delta(s)\Gamma(s)$ allows this term to vanish.