# Theory of Low-Energy $\pi\omega$ Scattering*

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(Received 30 July 1965; revised manuscript received 26 August 1965)

The $\pi\omega$ problem, as the simplest case of the pseudoscalar-meson and vector-meson system, is discussed from the standpoint of the $S$-matrix approach. A general procedure of constructing invariant amplitudes in spin and isospin space of the pseudoscalar-vector system is given, and for $\pi\omega$ scattering, a set of invariant amplitudes are conveniently chosen and their crossing properties are discussed. These amplitudes are expressed by one-dimensional representations which are derived from the Mandelstam representations by the Cini-Fubini technique. Partial-wave expansions as well as projections are done by the use of the Jacob-Wick helicity amplitudes. A prescription for calculating the driving forces from the exchange of particles is presented and applied to the exchange of the $\rho$ and $B$ mesons in states of the two possible quantum numbers $J^P=1^+$ and $2^-$. The procedure consists of a zero-width approximation to the transition amplitudes in states of given $J$ and $L$, crossing-symmetric relations, and the one-dimensional dispersion representations of the invariant amplitudes. The relationship between the invariant amplitudes and the helicity amplitudes greatly facilitates this procedure. The $t$-channel reaction is also analyzed. A method of solution of the partial-wave dispersion relations is discussed based on a recently developed formalism, and is extended further to avoid the difficulty associated with the zeros of the driving forces. A systematic program to understand the quantum numbers of the $B$ meson as a $\pi\omega$ resonance is also discussed. The qualitative nature of the forces due to the $B$ exchange in states of each possible quantum number is briefly sketched. A model calculation which favors a $2^-$ state of the $B$ meson is presented.

## I. INTRODUCTION

In recent years, a great deal of progress in the study of strong-interaction physics has been introduced by the dispersion-theoretic approach to elementary-particle physics.[1] The idea in this approach is to try to compute the physical scattering amplitudes from the knowledge of their singularities in the energy and momentum-transfer variables. However, the dispersion relations as applied to scattering have displayed only limited usefulness; we know no simple general rule for localizing all singularities of the amplitudes. A prescription proposed by Mandelstam[2] allows one to obtain a partial understanding about the momentum-transfer properties of the scattering amplitudes, when only two particles are present in both the initial and the final states of the scattering process. Also this prescription allows us to derive dispersion relations for the partial-wave amplitudes for which the unitarity relation takes a simple form. An approximate representation[3] for the amplitude of two-particle scattering has been deduced from the Mandelstam representation, by neglecting consistently inelastic processes in the unitarity condition.

Despite the lack of general methods for a complete calculational framework of the $S$ matrix, such as the Feynman rules in the case of perturbation theory, the dispersion-theoretic approach to the problems of strong interactions has provided a theoretical framework in which phenomenologically existing information can be analyzed and summarized, thus giving a better theoretical understanding of many problems. In particular, the problems of the pion-pion,[4] pion-nucleon,[5] and nucleon-nucleon[6] interactions have been extensively explored by many authors and the Mandelstam representation has provided a dynamical scheme for the discussion of scattering to obtain many quantitative features of the systems. In many cases, the dispersion-theoretic handling of the problems enabled us to predict a reasonable portion of the driving forces of the system via the Mandelstam prescription.

In the present article, we consider the pseudoscalar-meson–vector-meson scattering through the approach of dispersion theory. We would like to have a theoretical framework for understanding the existing experimental data, such as the $\pi\omega$ resonance[7] called $B$, and the two $\pi\rho$ resonances[8,9] $A_1$ and $A_2$. Owing to the different

* Work supported by the U. S. Atomic Energy Commission.

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[8] S. U. Chung, O. I. Dahl, L. M. Hardy, R. I. Hess, G. R. Kalbfleisch, J. Kirz, D. H. Miller, and G. A. Smith, Phys. Rev. Letters **12**, 621 (1964); G. Goldhaber, J. L. Brown, S. Goldhaber, J. A. Kaduk, B. C. Shen, and G. H. Trilling, *ibid.* **12**, 336 (1964).

[9] R. T. Deck, Phys. Rev. Letter **13**, 169 (1964) has suggested that a kinematic enhancement in the mass spectrum of the final $\pi\rho$ scattering could be responsible for the lower (the $A_1$) of the two peaks seen in the experiments of Ref. 8. This suggestion makes it difficult to interpret the $A_1$ peak as a resonance, because the $\pi\rho$ system associated with the kinematical peak does not occur in a definite angular momentum state. Experimental evidence in support of this suggestion has been presented by M. A. Abolins, D. D. Carmony, R. L. Lander, and N. Xuong in *Proceedings of the Second Topical Conference on Recently Discovered Resonant Particles* (Ohio University, Athens, Ohio, 1965) (to be published).

isotopic-spin quantum numbers of the mesons belonging to the pseudoscalar and vector-meson families,[10] we are unable to incorporate universal isotopic-spin projection operators for all systems of meson families. Thus we shall primarily consider $\pi\omega$ scattering as an explicit example, for a given value of the total angular momentum $J$. For then a complete characterization of any pseudoscalar-vector scattering for a given total $J$ will be easily accomplished by means of appropriate projection operators of the isotopic-spin states. Even without the isotopic-spin operators, the problem is considerably complicated by the presence of spin.

The pseudoscalar-vector system can be either in a spin-singlet or spin-triplet state. For each process, the Feynman amplitude can be expressed in terms of four invariant functions of the energy and the momentum-transfer variables.

In the next section, the usual scalar variables are defined and the singularities in the direct as well as crossed channels are described. We have outlined in Sec. III the procedure to introduce invariant amplitudes for the pseudoscalar-vector system in spin and isospin space. As for the $\pi\omega$ scattering, the isotopic-spin space appears trivial since the reaction goes through only the $I=1$ state. The crossing properties of the four invariant amplitudes of the $\pi\omega$ scattering are also discussed in Sec. IV. These four invariant amplitudes are expressed by the Mandelstam representation.[2] By using the Cini-Fubini technique,[3] the double dispersion relations are approximated to the one-dimensional representations which will be useful when the lower partial waves give dominant contributions to the absorptive parts of the invariant amplitudes. Partial-wave decompositions are considered in Sec. V. This is done by a modification of the Jacob-Wick formalism[11] of scattering. Singularities of the partial-wave amplitudes are sketched. Section VI contains calculations of the helicity amplitudes. The polarization vectors of the massive photon are discussed and expressed in terms of its helicities, so that one may calculate the helicity amplitudes. In Sec. VII, a general procedure to derive the driving forces is discussed. The procedure is verified by reproducing the Born terms of the vector and an axial-vector exchange as those from the Feynman graphs. The $2^-$ exchange[12] as a possible quantum-number state of the $B$ meson is calculated. Here a narrow-width approximation to the absorptive part of the transition amplitude in the physical region of the crossed channel and crossing relations of the invariant amplitudes are incorporated with the one-dimensional representation of the amplitudes to give the Born terms. The knowledge of the effective interactions at the vertices of the Feynman diagram is not necessary in our approach. Also we will see the correct kinematical factors to be removed in the partial-wave amplitudes in states of given $J^P$, as a consequence of making the invariant amplitudes free from kinematical singularities. In Sec. VIII we discuss the method of solution of the partial-wave dispersion relations based on a recently developed formalism.[13] This formalism gives a manifestly symmetric solution which does not depend on any subtraction parameters. The method is extended to avoid the difficulty associated with the zeros of the driving forces. Finally, Sec. IX deals with concluding remarks. A systematic approach to understand the $B$ meson in the viewpoint of a dynamical theory of the $\pi\omega$ interaction, which is under way, is described as an application of our theory, and a model calculation favoring a $2^-$ state of the $\pi\omega$ resonance is discussed.

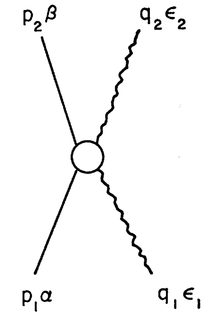


FIG. 1. Diagram representing the $\pi\omega$ scattering.

## II. KINEMATICS

Scattering amplitudes for the process $p_1+q_1\to p_2+q_2$ shown in Fig. 1 are considered as a function of the three scalar variables

$$s=-(p_1+q_1)^2=M^2+m^2+2k_s^2+2[(k_s^2+M^2)(k_s^2+m^2)]^{1/2}, \quad (1a)$$

$$t=-(p_1-p_2)^2=-2k_s^2(1-\cos\theta_s), \quad (1b)$$

$$u=-(p_1-q_2)^2=2(M^2+m^2)-s-t, \quad (1c)$$

where we have denoted the four-momenta of the pseudoscalar mesons by $p_1$ and $p_2$, the four-momenta of the vector mesons by $q_1$ and $q_2$, the vector-meson mass by $M$, the pseudoscalar meson mass by $m$, and the center-of-mass momentum and scattering angle by $k_s$ and $\theta_s$, respectively.

If the diagram in Fig. 1 is considered to represent the annihilation of two vector mesons $q_1$ and $-q_2$ into two pions $-p_1$ and $p_2$, then we can write

$$t=4(q^2+M^2)=4(p^2+m^2), \quad (2a)$$

$$s=-p^2-q^2+2pq\cos\theta_t, \quad (2b)$$

$$u=2(M^2+m^2)-s-t, \quad (2c)$$

[10] In order to construct a general theory, one should consider all possible combinations of the pseudoscalar mesons $(\pi,\eta,K,\bar{K})$ and the vector mesons $(\rho,\omega,\phi,K^*,\bar{K}^*)$.

[11] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

[12] The $B$ meson has $G$ parity$+1$ and isotopic spin $I=1$. Possible quantum numbers of spin and parity $J^P$ are $2^-$ and $1^+$. A $1^-$ interpretation of $B$ is not likely to be due to nonobservation of $K\bar{K}$. See A. H. Rosenfeld, A. Barbaro-Galtieri, W. Barkas, P. Bastien, J. Kirz, and M. Roos, Rev. Mod. Phys. **36**, 977 (1964).

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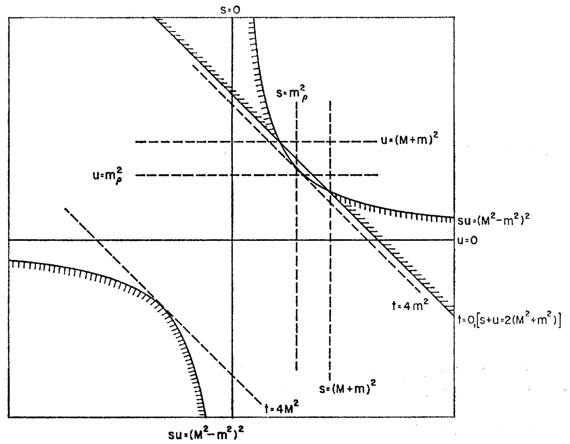


FIG. 2. Kinematics of the reactions (1), (2), and (3).

where $p$ and $q$ are the center-of-mass momentum of the pseudoscalar and vector meson, respectively, and $\theta_t$ is the scattering angle in this channel.

Figure 1 can also be interpreted as representing the scattering process $p_1+(-q_2)\rightarrow p_2+(-q_1)$. Then the role of $s$ and $u$ is interchanged;

$$u=M^2+m^2+2k_u{}^2+2[(k_u{}^2+M^2)(k_u{}^2+m^2)]^{1/2}, \quad \text{(3a)}$$

$$t=-2k_u{}^2(1-\cos\theta_u), \quad \text{(3b)}$$

$$s=2(M^2+m^2)-t-u. \quad \text{(3c)}$$

In (1), (2), and (3), there are only two independent variables because of the usual relations (1c), (2c), and (3c), and we have used a convention $a\cdot b=\mathbf{a}\cdot\mathbf{b}-a_0b_0$.

The boundaries of the physical region for the $s$ channel are given by the limiting values of $\cos\theta_s=\pm1$. We obtain a curve

$$su=(M^2-m^2)^2 \quad \text{(4a)}$$

and a line

$$s+u=2(M^2+m^2) \quad \text{(4b)}$$

as shown in Fig. 2. The physical region for the $u$ channel is symmetric to that of the $s$ channel about the line $s=u$. If there occurs a single-particle exchange of mass $\bar{m}$, say smaller than the threshold value $M+m$, then the pole appears at $s=\bar{m}^2$ and at $u=\bar{m}^2$. The point where $s$ has the least permissible value for $k_s$ to be real has the coordinates $(s,t,u)=((M+m)^2,0,(M-m)^2)$ while the corresponding point in the $u$ channel has $(s,t,u)=((M-m)^2,0,(M+m)^2)$.

The physical region for the $t$ channel is bounded by a branch of (4a) and the point with the least permissible value of $t$ for $q$ to be real is $(s,t,u)=(-(M^2-m^2), 4M^2, -(M^2-m^2))$. But there are also threshold singularities for intermediate states of lower energy involving pions, namely, $t=4m^2$, or $16m^2$, and so on.

## III. CROSSING SYMMETRY

Let us first consider the process $p_1+q_1\rightarrow p_2+q_2$. If this is to represent the $\pi\omega$ scattering, the reaction may take place only in the isotopic spin state 1. If this is to represent the $\pi\rho$ scattering, then the reaction will take place in any of the three isotopic-spin states $I=0$, 1, and 2. For this process, the projection operators for isotopic spin states are

$$\mathcal{P}_0=3^{-1}(\mathbf{I}_1\cdot\mathbf{I}_2+1)(\mathbf{I}_1\cdot\mathbf{I}_2-1), \quad (5)$$

$$\mathcal{P}_1=-2^{-1}(\mathbf{I}_1\cdot\mathbf{I}_2+2)(\mathbf{I}_1\cdot\mathbf{I}_2-1), \quad (6)$$

$$\mathcal{P}_2=6^{-1}(\mathbf{I}_1\cdot\mathbf{I}_2+1)(\mathbf{I}_1\cdot\mathbf{I}_2+2), \quad (7)$$

where $\mathbf{I}_1$, and $\mathbf{I}_2$ are the usual isotopic-spin operators for pion and $\rho$ meson, whose elements are given by

$$\langle\alpha|I^j|\beta\rangle=-i\epsilon_{j\alpha\beta}. \quad (8)$$

As for the $\pi K^*$ scattering, the projection operators of the $I=\frac{3}{2}$ and $I=\frac{1}{2}$ states are given by

$$\mathcal{P}_{3/2}=3^{-1}(2+\mathbf{I}\cdot\boldsymbol{\tau}), \quad (9)$$

$$\mathcal{P}_{1/2}=3^{-1}(1-\mathbf{I}\cdot\boldsymbol{\tau}), \quad (10)$$

where $\mathbf{I}$ and $\boldsymbol{\tau}$ are the usual isotopic operators for $\pi$ and $K^*$, while as for the $KK^*$ scattering, we have

$$\mathcal{P}_0=4^{-1}(1-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2) \quad (11)$$

and

$$\mathcal{P}_1=4^{-1}(3+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2). \quad (12)$$

Since the spin of the pseudoscalar-vector systems is $0+1=1$, we observe that for a given value of the total angular momentum $J$, the orbital angular momentum $L$ can be $J$ or $J\pm1$. For $L=J$, one amplitude is sufficient, as parity conservation forbids transitions to $L=J\pm1$, while for $L=J\pm1$, three amplitudes are required to describe the transitions $J+1\leftrightarrow J+1$, $J-1\leftrightarrow J-1$, and $J-1\leftrightarrow J+1$, respectively. Because of time-reversal invariance, the transition amplitudes for $J-1\rightarrow J+1$ and $J+1\leftrightarrow J-1$ are equal. Thus, there must be four invariant amplitudes describing the scattering, and four independent combinations of the spin are necessary.

The usual procedure to include the spins and charges of the particles is to define the usual Feynman amplitudes as an operator in spin and isospin space and to situate the operator between the appropriate spin and isospin state vectors.

We shall write the Feynman amplitude $F$ for process $p_1+q_1\rightarrow p_2+q_2$ in the form

$$F(s,t,u)=\sum_I\sum_{i=1}^{4}F_i{}^I(s,t,u)O_i\mathcal{P}_I, \quad (13)$$

where $F_i{}^I(s,t,u)$'s are functions of the invariant variable $s$, $t$, and $u$ and the $O_i$'s are the four independent combinations of the spin nature of the particles involved. We can treat $F$ as a matrix in isospin space. However, we shall discuss this procedure for the $\pi\omega$ scattering for which the reaction will take place only in the $I=1$ state. The Feynman amplitude $F$ and the scattering amplitude

$f$ are given by

$$F_{\alpha\beta}=F\delta_{\alpha\beta}=\mathfrak{M}_{\mu\nu}\epsilon_{2\mu}\epsilon_{1\nu}\delta_{\alpha\beta} \tag{14}$$

and

$$F=8\pi s^{1/2}f, \tag{15}$$

where $\epsilon_1$ and $\epsilon_2$ are the initial and final vector-meson polarization four-vectors, normalized to unity, and $\alpha$ and $\beta$ are the isospin indices of pions.

Let us write

$$\begin{aligned}\mathfrak{M}_{\mu\nu}=&F_1\delta_{\mu\nu}+F_2P_\mu P_\nu\\&+F_3R_\mu R_\nu+2^{-1}F_4(P_\mu R_\nu+P_\nu R_\mu)+F_5q_{2\mu}q_{1\nu}\\&+2^{-1}F_6(q_{2\mu}P_\nu+q_{1\nu}P_\mu)+2^{-1}F_7(q_{2\mu}R_\nu+q_{1\nu}R_\mu),\end{aligned} \tag{16}$$

where $F_1$ through $F_7$ are scalar functions of $s$, $t$, and $u$, and

$$P=p_1+q_1, \tag{17a}$$

$$R=p_1-q_2, \tag{17b}$$

then we notice the $P^2=-s$ and $R^2=-u$. Only four of the seven functions $F_1$ through $F_7$ actually count, since we will be working with the subsidiary condition

$$\epsilon_{1\mu}q_\mu=\epsilon_{2\mu}q_\mu=0 \tag{18}$$

and $F(s,t,u)$ becomes

$$\begin{aligned}F(s,t,u)=&(\epsilon_1\cdot\epsilon_2)F_1(s,t,u)\\&+(\epsilon_1\cdot P)(\epsilon_2\cdot P)F_2(s,t,u)+(\epsilon_1\cdot R)(\epsilon_2\cdot R)F_3(s,t,u)\\&+2^{-1}[(\epsilon_1\cdot P)(\epsilon_2\cdot R)+(\epsilon_1\cdot R)(\epsilon_2\cdot P)]F_4(s,t,u).\end{aligned} \tag{19}$$

Thus we have identified the $O_i$'s.[14]

As for the process $p_1+(-q_2)\to p_2+(-q_1)$, the Feynman amplitude will have a representation analogous to (19) with $(q_1,\epsilon_1)$ and $(-q_2,\epsilon_2)$ interchanged. This process also takes place only in the $I=1$ state, and furthermore

$$F(s,t,u)=F(u,t,s). \tag{20}$$

On the other hand, the invariant operators $O_i$ must also be crossed, thus giving the crossing properties of the scalar amplitudes

$$F_i(u,t,s)=\sum_{j=1}^{4}\chi_{ij}F_j(s,t,u)\quad(i=1,2,3,4) \tag{21}$$

with

$$(\chi_{ij})=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}. \tag{22}$$

This relation is useful in obtaining the $\pi\omega$ interaction generated by the exchange of resonant or single-particle intermediate states in the $u$ channel.

In the $t$ channel where two pions annihilate into two $\omega$ mesons, the only allowed isotopic state is $I=0$. Moreover, the orbital angular momentum in this channel is uniquely related to the total angular momentum $J$. Bose statistics restricts values of $J=L$ only to even values; thus the $t$ channel may not contribute when the odd-parity or the $I=1$ particle exchanges are considered in all the channels. One can also easily verify for this process that[15]

$$F_{\alpha\beta}=3^{-1/2}F(t,s,u)\delta_{\alpha\beta}. \tag{23}$$

The crossing relation under $s\leftrightarrow t$ with $u$ fixed can be worked out in an analogous manner as in the $\pi N$ scattering. We will relate the $F_j$ of the $\pi\omega$ scattering to the production amplitudes $f_{\lambda_1\lambda_2;00}$, where $\lambda_1$ and $\lambda_2$ denote the helicity of the two $\omega$ mesons, when we discuss the partial-wave amplitudes.

## IV. AN APPROXIMATE REPRESENTATION OF THE INVARIANT FUNCTIONS

According to Mandelstam's postulate,[2] each of the four invariant amplitudes $F_i$ is an analytic function of the energy-momentum variables except where $s$, $t$, and $u$ equal the thresholds for energy conserving intermediate states:

$$\begin{aligned}F_j(s,t,u)=P_j(s,t,u)+&\frac{1}{\pi^2}\int_{(M+m)^2}^{\infty}dx\int_{(M+m)^2}^{\infty}dy\frac{\rho_{su}{}^j(x,y)}{(x-s)(y-u)}\\&+\frac{1}{\pi^2}\int_{(M+m)^2}^{\infty}dx\int_{4m^2}^{\infty}dy\frac{\rho_{st}{}^j(x,y)}{(x-s)(y-t)}+\frac{1}{\pi^2}\int_{(M+m)^2}^{\infty}dx\int_{4m^2}^{\infty}dy\frac{\rho_{ut}{}^j(x,y)}{(x-u)(y-t)},\end{aligned} \tag{24}$$

where the term $P_j(s,t,u)$ denotes the single-particle exchange terms such as the $\rho$-exchange terms in the $s$ and $u$ channels of the $\pi\omega$ scattering, and the anomalous threshold cuts are neglected. We shall overlook the possibility of subtractions. The spectral functions $\rho^j(x,y)$ are not independent, but it follows from (21)

[14] This choice is equivalent to that of M. L. Mehta, Phys. Rev. **134**, B1377 (1964). Construction of invariant scattering amplitudes for arbitrary spins are discussed from group-theoretical considerations by A. O. Barut, I. Muzinich, and D. N. Williams, Phys. Rev. **130**, 442 (1963).

[15] Following the method of W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960), we write the isotopic-spin decomposition as $\langle\bar{\omega}\omega|F|\beta\alpha\rangle=F(t,s,u)\langle\bar{\omega}\omega|g^0|\beta\alpha\rangle$, where $\alpha$, $\beta=1$, 2, 3 and $g^0=|I=0,\ I_z=0(\omega)\rangle\langle(\pi)I=0,\ I_z=0|$ is the projection operator for the $I=0$ state. Thus one can easily get $\langle\bar{\omega}\omega|g^0|\beta\alpha\rangle=3^{-\frac{1}{2}}\delta_{\alpha\beta}$.

and (22) that

$$\rho_{su}{}^j(x,y)=\rho_{us}{}^j(y,x) \quad \text{for} \quad j=1,4, \tag{25a}$$

$$\rho_{st}{}^j(x,y)=\rho_{ut}{}^j(y,x) \quad \text{for} \quad j=1,4, \tag{25b}$$

$$\rho_{su}{}^2(x,y)=\rho_{us}{}^3(y,x), \tag{25c}$$

$$\rho_{st}{}^2(x,y)=\rho_{ut}{}^3(y,x), \tag{25d}$$

and

$$\rho_{ut}{}^2(x,y)=\rho_{st}{}^3(y,x). \tag{25e}$$

We can also derive from (24) one-dimensional dispersion relations with one of the $s$, $t$, and $u$ variables fixed. It is convenient in our case to write a fixed-$t$ dispersion relation, which then still exhibits the $s$-$u$ symmetry.

$$F_j(s,t,u)=P_j(s,t,u)+\frac{1}{\pi}\int_{(M+m)^2}^{\infty}dx\frac{A_s{}^j(x,t)}{x-s}+\frac{1}{\pi}\int_{(M+m)^2}^{\infty}dy\frac{A_u{}^j(y,t)}{y-u}. \tag{26}$$

Then Eq. (24) shows

$$A_s{}^j(x,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}dt'\frac{\rho_{st}{}^j(x,t')}{t'-t}+\frac{1}{\pi}\int_{(M+m)^2}^{\infty}du'\frac{\rho_{su}{}^j(x,u')}{u'+x+t-2(M^2+m^2)} \tag{27}$$

and

$$A_u{}^j(y,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}dt'\frac{\rho_{ut}{}^j(y,t')}{t'-t}+\frac{1}{\pi}\int_{(M+m)^2}^{\infty}ds'\frac{\rho_{su}{}^j(s',y)}{s'+y+t-2(M^2+m^2)}. \tag{28}$$

From (25), one finds

$$A_s{}^j(x,t)=A_u{}^j(x,t) \quad \text{for} \quad j=1,4, \tag{29a}$$

$$A_s{}^2(x,t)=A_u{}^3(x,t), \tag{29b}$$

$$A_s{}^3(x,t)=A_u{}^2(x,t). \tag{29c}$$

The functions $A_i{}^j(s,t)$ ($i=s$, $u$; $j=1, 2, 3, 4$) are to be calculated by considering the absorptive part of the amplitude for the $\pi\omega$ reactions. However, we shall adopt a modest attitude of treating the low-energy region and make here an approximation to (24), using the technique introduced by Cini and Fubini.[3] This approximation is believed to give the essential structure of the invariant amplitude in the region of low-energy and low-momentum transfer. From unitarity, the $A_i{}^j(x,t)$ can be expressed as the sum of contribution from lower mass intermediate states involved in the elastic $\pi\omega$ scattering and an inelastic contribution where additional mesons are produced in the intermediate state:

$$A_i{}^j(s,t,u)=A_{il}{}^j(x,t)+A_{ih}{}^j(x,t), \tag{30}$$

where the contribution from the higher mass intermediate states $A_{ih}{}^j(x,t)$ vanishes in the low-energy region, say, for $x<(M+2m)^2$. Then we can write (26) as

$$F_j(s,t,u)=P_j(s,t,u)+\frac{1}{\pi}\int_{(M+m)^2}^{\infty}dx\frac{A_{sl}{}^j(x,t)}{x-s}+\frac{1}{\pi}\int_{(M+m)^2}^{\infty}dx\frac{A_{ul}{}^j(x,t)}{x-u}+\frac{1}{\pi}\int_{(M+2m)^2}^{\infty}dx\frac{A_{sh}{}^j(x,t)}{x-s}+\frac{1}{\pi}\int_{(M+2m)^2}^{\infty}dx\frac{A_{uh}{}^j(x,t)}{x-u}. \tag{31}$$

In the first two integrals of (31), it is easy to understand from Fig. 3 that the lowest singularity in $t$ starts at $16m^2$, while for the last two integrals in (31), the nearest cut in $t$ begins at $4m^2$ with the cuts in $s$ and $u$ at $(M+2m)^2$. We shall expand the third integral in power series of $s$ and $u$, keeping only the first few terms. Then

$$\frac{1}{\pi}\int_{(M+2m)^2}^{\infty}dx\frac{A_{sh}{}^j(x,t)}{x-s}=\frac{1}{\pi}\int_{4m^2}^{\infty}dt'\frac{a_s{}^j(t',s,u)}{t'-t}+\frac{1}{\pi^2}\int_{(M+2m)^2}^{\infty}dx\int_{(M+m)^2}^{\infty}du'\frac{\rho_{suh}{}^j(x,u')}{(u'+x+t-2M^2-2m^2)(x-s)}, \tag{32}$$

where

$$a_s{}^j(t',s,u)=\frac{1}{\pi}\int_{(M+2m)^2}^{\infty}dx\frac{\rho_{sth}{}^j(x,t')}{x-s}. \tag{33}$$

Treating the fourth integral in (31) in the same manner, we get

$$\frac{1}{\pi}\int_{(M+2m)^2}^{\infty}dx\frac{A_{uh}{}^j(x,t)}{x-u}=\frac{1}{\pi}\int_{4m^2}^{\infty}dt'\frac{a_u{}^j(t',s,u)}{t'-t}+\frac{1}{\pi^2}\int_{(M+2m)^2}^{\infty}dx\int_{(M+m)^2}^{\infty}ds'\frac{\rho_{suh}{}^j(s',x)}{(s'+x+t-2M^2-2m^2)(x-u)} \tag{34}$$

with

$$a_u{}^j(t',s,u)=\frac{1}{\pi}\int_{(M+2m)^2}^{\infty} dx\frac{\rho_{uth}{}^j(x,t')}{x-u}. \quad (35)$$

The second terms in (32) and (34) should have only a weak dependence on all three variables since the cuts are all distant and they will be expressed in a phenomenological manner; furthermore they will be assumed to be absorbed in $P_j(s,t,u)$ which contain adjustable parameters in general.

Thus we obtain an approximate representation for the invariant amplitudes $F_j(s,t,u)$:

$$F_j(s,t,u)=P_j(s,t,u)+\frac{1}{\pi}\int_{(M+m)^2}^{\infty} dx\frac{A_{sl}{}^j(x,t)}{x-s} +\frac{1}{\pi}\int_{(M+m)^2}^{\infty} dx\frac{A_{ul}{}^j(x,t)}{x-u}+\frac{1}{\pi}\int_{4m^2}^{\infty} dx\frac{a^j(x,s,u)}{x-t}, \quad (36)$$

where

$$a^j(x,s,u)=a_s{}^j(x,s,u)+a_u{}^j(x,s,u) \quad (37)$$

is a sum of a real polynomial of low degree in $s$ and a polynomial in $u$, and $A_{il}{}^j(s,t)$ is a real polynomial of low degree in $t$.

The use of unitarity to calculate $A_{il}{}^j(x,t)$ and $a^j(t,s,u)$ in (36) is simple, since they are given by the imaginary parts of the low partial waves of the reaction in question and the related crossed processes. The integrals of (36) will get dominant contribution from the imaginary parts of partial waves in which a resonance is present in the corresponding channel.

In the $t$ channel, the nearest singularity is at $t=4m^2$ as shown in Fig. 3. This two-pion state must occur in the $I=0$ state, and have even values of the orbital angular momentum. Thus the $s$ and $d$ waves will be important in our approximation. A possibility of the $I=0$ $d$-wave resonance of two-pion called $f^0$ has been confirmed,[16] and even the $I=0$ $s$-wave resonance of a two-pion state has been reported.[17] Furthermore there seem to be **two** possible candidates in this state and the mixing of these two scalar resonant particles has been argued,[18] while in some calculations,[19] and in some semi-empirical analyses,[20] a strong final-state interaction is favorable in the $I=0$ $s$-wave state instead of a resonance.

If we do not assume any $I=0$ two-pion $s$-wave resonance, then the nearest singularity in $t$ occurs further

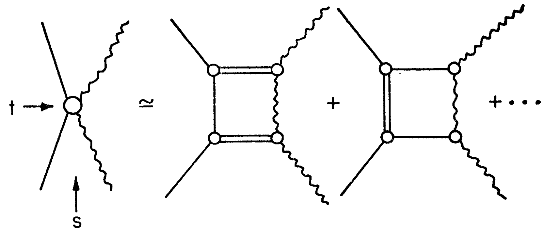


FIG. 3. Determination of the lowest singularity in $s$ and $t$ of the $\pi\omega$ scattering.

away in the last two integrals of (31). The pole coming from $f^0$ exchange in $t$ channel locates much further. Thus effectively the lowest singularity in $t$ will start further at $t=16m^2$ and in $s$ and $u$ at $(M+3m)^2$ for the last two integrals in (31). The last integral of (36) will have thus a weak dependence on $s$, $t$, and $u$ as the cuts will effectively be distant. In particular, if only the $\rho$ and $B$ exchanges are considered in all the channels, then the third integral in (36) will not appear.

## V. PARTIAL-WAVE DECOMPOSITIONS

It is convenient to express the $S$ matrix in terms of transition amplitudes in states of given quantum numbers of the total angular momentum $J$ and parity. We will treat reactions of $\pi\omega$, employing the formalism developed by Jacob and Wick.[11] This formalism has an intrinsic simplicity of expansion in terms of amplitudes for transitions between states of given helicities.

We define the $T$ matrix by the relation

$$T_{fi}=(S_{fi}-\delta_{fi})/(2ik_f{}^{1/2}k_i{}^{1/2}), \quad (38)$$

where $k_f$ and $k_i$ refer to the center-of-mass momentum in the final and initial states of the $\pi\omega$ scattering and in the case of elastic scattering $k_f=k_i=k$. The scattering amplitudes of Jacob and Wick can be expressed for given helicities $\lambda_i$ and $\lambda_f$ of the initial and final $\omega$ mesons, as follows:

$$\langle\lambda_f|f|\lambda_i\rangle=k_f{}^{1/2}k_i{}^{-1/2}\sum_J(2J+1) \times\langle\lambda_f|T^J|\lambda_i\rangle d_{\lambda_i\lambda_f}{}^J(\theta)e^{i(\lambda_i-\lambda_f)\phi}, \quad (39)$$

where $d_{\lambda_i\lambda_f}{}^J(\theta)$ is the reduced rotation matrix and $\langle\lambda_f|T^J|\lambda_i\rangle$ means $\langle JM;\lambda_f|T|JM;\lambda_i\rangle$. Without loss of generality, we shall set the azimuthal angle of the final momentum of $\omega$ equal to zero. For the helicity amplitudes, the invariance under space inversion implies

$$\langle\lambda_f|T^J|\lambda_i\rangle=\langle-\lambda_f|T^J|-\lambda_i\rangle \quad (40)$$

and time-reversal invariance gives

$$\langle\lambda_f|T^J|\lambda_i\rangle=\langle\lambda_i|T^J|\lambda_f\rangle. \quad (41)$$

Using (40), (41) and the relation

$$d_{\lambda_i\lambda_f}{}^J(\theta)=d_{-\lambda_i-\lambda_f}{}^J(\theta)=(-)^{\lambda_i-\lambda_f}d_{\lambda_f\lambda_i}{}^J(\theta) \quad (42)$$

one can again reduce the possible nine helicity amplitudes to four independent ones, say, $f_{++}$, $f_{-+}$, $f_{0+}$, and

[16] W. Selove, V. Hagopian, H. Brody, A. Baker, and E. Leboy, Phys. Rev. Letters **9**, 272 (1962); Y. Y. Lee, B. P. Roe, D. Sinclair, and J. C. Vander Velde, *ibid.* **12**, 342 (1964).

[17] M. Feldman, W. Frati, J. Halpern, A. Kanofsky, M. Nussbaum, S. Richert, P. Yamin, A. Choudry, S. Devons, and J. Grun Haus, Phys. Rev. Letters **14**, 869 (1965); W. Selove, V. Hagopian, J. Alitti, J. P. Baton, and M. Neveu-Rene, Bull. Am. Phys. Soc. **11**, CB7 (1965). Not all the experiments seem unanimous for the existence of a scalar meson. For example, see W. D. Walker, in *Proceedings of the Second Topical Conference on Resonant Particles* (Ohio University, Athens, Ohio, 1965) (to be published).

[18] L. M. Brown, Phys. Rev. Letters **14**, 836 (1965).

[19] K. Kang, Phys. Rev. **139**, B126 (1965).

[20] C. Kacser, P. Singer, and T. N. Truong, Phys. Rev. **137**, B1605 (1965).

$f_{00}$. Here the helicity of the $\omega$ meson is labled by $+$, $-$, and 0.

The parity operation $P$ on the helicity state $|JM;\lambda\rangle$ produces the effect

$$P|JM;\lambda\rangle=\eta(-)^{J-S}|JM;-\lambda\rangle, \quad (43)$$

where $S$ and $\eta$ are spin and intrinsic parity of the $\omega$ meson, and the helicity and intrinsic parity of the pion are omitted in forming the helicity state $|JM;\lambda\rangle$ of the $\pi\omega$ system. Explicitly $\eta(-)^{J-S}$ in the present situation is $(-)^J$. Now we define eigenstates $|JM;\lambda\rangle_\pm$ of parity such that

$$P|JM;\lambda\rangle_\pm=\pm(-)^J|JM;\lambda\rangle_\pm. \quad (44)$$

This is satisfied by

$$|JM;\lambda\rangle_\pm=2^{-1/2}(|JM;\lambda\rangle\pm|JM;-\lambda\rangle). \quad (45)$$

When the parity is conserved, $T$ matrix will have the nonvanishing elements only between the same parity states:

$$_\pm\langle JM;\lambda_f|T|JM;\lambda_i\rangle_\pm =\langle\lambda_f|T^J|\lambda_i\rangle\pm\langle-\lambda_f|T^J|\lambda_i\rangle. \quad (46)$$

Let us define parity-conserving scattering amplitude by the rule[21]

$$\begin{aligned}\langle\lambda_f|f^\pm|\lambda_i\rangle &=[2^{1/2}\cos(\theta/2)]^{-|\lambda_i+\lambda_f|}[2^{1/2}\sin(\theta/2)]^{-|\lambda_i-\lambda_f|}\\ &\times\langle\lambda_f|f|\lambda_i\rangle\pm(-)^{\lambda_i+\lambda_m}[2^{1/2}\sin(\theta/2)]^{-|\lambda_i+\lambda_f|}\\ &\times[2^{1/2}\cos(\theta/2)]^{-|\lambda_i-\lambda_f|}\langle-\lambda_f|f|\lambda_i\rangle, \quad (47)\end{aligned}$$

where

$$\lambda_m=\max(|\lambda_i|,|\lambda_f|).$$

Finally, one can express (47) in terms of parity-conserving $T$-matrix elements:

$$\begin{aligned}\langle\lambda_f|f^\pm|\lambda_i\rangle=&\textstyle\sum_J(2J+1)\\ &\times[e_{\lambda_i\lambda_f}{}^{J+}(\theta)\,{}_\pm\langle JM;\lambda_f|T|JM;\lambda_i\rangle_\pm\\ &+e_{\lambda_i\lambda_f}{}^{J-}(\theta)\,{}_\mp\langle JM;\lambda_f|T|JM;\lambda_i\rangle_\mp], \quad (48)\end{aligned}$$

where

$$\begin{aligned}e_{\lambda_i\lambda_f}{}^{J\pm}(\theta)=&2^{-1}\{[2^{1/2}\cos(\theta/2)]^{-|\lambda_i+\lambda_f|}\\ &\times[2^{1/2}\sin(\theta/2)]^{-|\lambda_i-\lambda_f|}d_{\lambda_i\lambda_f}{}^J(\theta)\\ &\pm(-)^{\lambda_i+\lambda_m}[2^{1/2}\sin(\theta/2)]^{-|\lambda_i+\lambda_f|}\\ &\times[2^{1/2}\cos(\theta/2)]^{-|\lambda_i-\lambda_f|}d_{\lambda_i-\lambda_f}{}^J(\theta)\}. \quad (49)\end{aligned}$$

For completeness, we give the partial-wave projections;

$$\begin{aligned}&_\pm\langle JM;\lambda_f|T|JM;\lambda_i\rangle_\pm\\ &=2^{-1}\int_{-1}^{1}d(\cos\theta)[C_{\lambda_i\lambda_f}{}^{J+}(\theta)\langle\lambda_f|f^\pm|\lambda_i\rangle\\ &\qquad+C_{\lambda_i\lambda_f}{}^{J-}(\theta)\langle\lambda_f|f^\mp|\lambda_i\rangle], \quad (50)\end{aligned}$$

[21] We follow the definition of M. Gell-Mann, M. L.Goldberger, F. E. Low, E. Marx, and A. Zachsiasen, Phys. Rev. **133**, B145 (1964).

where

$$\begin{aligned}C_{\lambda_i\lambda_f}{}^{J\pm}(\theta)=&2^{-1}\{[2^{1/2}\cos(\theta/2)]^{|\lambda_i+\lambda_f|}\\ &\times[2^{1/2}\sin(\theta/2)]^{|\lambda_i-\lambda_f|}d_{\lambda_i\lambda_f}{}^J(\theta)\\ &\pm(-)^{\lambda_i+\lambda_m-1}[2^{1/2}\sin(\theta/2)]^{|\lambda_i+\lambda_f|}\\ &\times[2^{1/2}\cos(\theta/2)]^{|\lambda_i-\lambda_f|}d_{\lambda_i-\lambda_f}{}^J(\theta)\}. \quad (51)\end{aligned}$$

We shall use notations $T_{\lambda_f\lambda_i}{}^{J\pm}={}_\pm\langle JM;\lambda_f|T|JM;\lambda_i\rangle_\pm$ and $f_{\lambda_i\lambda_f}{}^\pm=\langle\lambda_f|f^\pm|\lambda_i\rangle$. From (47), one finds

$$f_{++}{}^-=(1+\cos\theta)^{-1}f_{++}-(1-\cos\theta)^{-1}f_{-+}, \quad (52)$$

$$f_{++}{}^+=(1+\cos\theta)^{-1}f_{++}+(1-\cos\theta)^{-1}f_{-+}, \quad (53)$$

$$f_{0+}{}^+=2[\sin\theta]^{-1}f_{0+}, \quad (54)$$

and

$$f_{00}{}^+=2f_{00}, \quad (55)$$

while from (50), the partial-wave amplitudes in states of given parity and total spin are given:

$$\begin{aligned}T_{++}{}^{J-}=&2^{-1}\int_{-1}^{1}d(\cos\theta)\{(2J+1)^{-1}\\ &\times[(J+1)P_{J-1}(\cos\theta)+JP_{J+1}(\cos\theta)]f_{++}{}^-\\ &+P_J(\cos\theta)f_{++}{}^+\}, \quad (56)\end{aligned}$$

$$\begin{aligned}T_{++}{}^{J+}=&2^{-1}\int_{-1}^{1}d(\cos\theta)\{(2J+1)^{-1}\\ &\times[(J+1)P_{J-1}(\cos\theta)+JP_{J+1}(\cos\theta)]f_{++}{}^-\\ &+P_J(\cos\theta)f_{++}{}^-\}, \quad (57)\end{aligned}$$

$$\begin{aligned}T_{0+}{}^{J+}=&2^{-1}(2J+1)^{-1}[J(J+1)]^{1/2}\int_{-1}^{1}d(\cos\theta)\\ &\times[P_{J+1}(\cos\theta)-P_{J-1}(\cos\theta)]f_{0+}{}^+W^{-1}M, \quad (58)\end{aligned}$$

$$T_{00}{}^{J+}=2^{-1}\int_{-1}^{1}d(\cos\theta)P_J(\cos\theta)f_{00}{}^+[MW^{-1}]^2. \quad (59)$$

Here $T_{0+}{}^{J+}$ and $T_{00}{}^{J+}$ are modified by the factors $M/W$ and $(M/W)^2$ to remove the kinematical singularities in $f_{0+}{}^+$ and $f_{00}{}^+$ as will be seen later. This will necessarily modify the partial-wave expansions (48) so that $W/M$ and $(W/M)^2$ should be multiplied to $f_{0+}{}^+$ and $f_{00}{}^+$, respectively. $W$ is given by $(M^2+k^2)^{1/2}$ in the center-of-mass system. The expression for the transition amplitudes in states of given orbital angular momentum $L$ are readily obtained by using the relation[22]

$$\begin{aligned}|JMLS\rangle=&[(2L+1)/(2J+1)]^{1/2}\\ &\times\textstyle\sum_\lambda C(LSJ;0\lambda)C(S_\omega S_\pi S;\lambda 0)|JM;\lambda\rangle, \quad (60)\end{aligned}$$

where the summation is over the helicity states of $\omega$ and $S_\omega$ and $S_\pi$ are spins of $\omega$ and pion, respectively. Here we have explicitly put the helicity of the pion to zero. We shall denote the transition amplitude $\langle JMLS|T|JML'S\rangle$ by $T^J(L\leftrightarrow L')$. Let us first consider the singlet state $L=J$. (60) gives

$$T^J(J\leftrightarrow J)=T_{++}{}^J-T_{-+}{}^J; \quad (61)$$

[22] See the Appendix B of Ref. 11.

thus from (46)

$$T^J(J\leftrightarrow J)=T_{++}{}^{J-}. \tag{62}$$

As for the triplet states, where $L=J\pm1$, we get as follows:

$$T^J(J-1\leftrightarrow J-1)=(2J+1)^{-1}\times\{(J/2)T_{00}{}^{J+}+(J+1)T_{++}{}^{J+}+[2J(J+1)]^{1/2}T_{0+}{}^{J+}\}, \tag{63}$$

$$T^J(J-1\leftrightarrow J+1)=(2J+1)^{-1}[J(J+1)]^{1/2}\times\{-2^{-1}T_{00}{}^{J+}+T_{++}{}^{J+}-[2J(J+1)]^{-1/2}T_{0+}{}^{J+}\}, \tag{64}$$

$$T^J(J+1\leftrightarrow J+1)=(2J+1)^{-1}\times\{2^{-1}(J+1)T_{00}{}^{J+}+JT_{++}{}^{J+}-[2J(J+1)]^{1/2}T_{0+}{}^{J+}\}. \tag{65}$$

Partial-wave projections in the $t$ channel can also be carried out by constructing[23]

$$|JM;00\rangle_+=2^{1/2}|JM;00\rangle \tag{66a}$$

and

$$|JM;\lambda_1\lambda_2\rangle_+=2^{-1}\{|JM;\lambda_1\lambda_2\rangle+|JM;\lambda_2\lambda_1\rangle+|JM;-\lambda_1-\lambda_2\rangle+|JM;-\lambda_2-\lambda_1\rangle\}. \tag{66b}$$

The $T$ matrix becomes

$$_+\langle JM;\lambda_1\lambda_2|T|JM;00\rangle_+=2\times2^{1/2}\langle\lambda_1\lambda_2|T^J|00\rangle \tag{67a}$$

or

$$T_{\lambda_1\lambda_2;00}{}^{J+}=2\times2^{1/2}T_{\lambda_1\lambda_2;00}{}^J, \tag{67b}$$

where $\lambda_1$ and $\lambda_2$ are the helicities of the two $\omega$ mesons in the final state. Again we define the scattering amplitudes by the rule of (47) in which $\lambda_i$ and $\lambda_f$ are replaced by 0 and $\lambda_1-\lambda_2$, respectively. The scattering amplitude in terms of $T$ matrix is then given by

$$f_{\lambda_1\lambda_2;00}{}^+(t,\theta_t)=q^{1/2}p^{-1/2}2^{-1}\times(\sin\theta_t)^{-|\lambda_1-\lambda_2|}\sum_J(2J+1)[d_{0\lambda_1-\lambda_2}{}^J+(-)^{|\lambda_1-\lambda_2|}d_{0\lambda_2-\lambda_1}{}^J(\theta_t)]T_{\lambda_1\lambda_2;00}{}^{J+}, \tag{68}$$

where

$$T_{\lambda_1\lambda_2;00}{}^{J+}=-i2^{-1}(pq)^{-1/2}S_{\lambda_1\lambda_2;00}{}^{J+}. \tag{69}$$

Finally we get the partial-wave amplitudes

$$T_{\lambda_1\lambda_2;00}{}^{J+}=4^{-1}p^{1/2}q^{-1/2}\int_{-1}^{1}d(\cos\theta_t)(\sin\theta_t)^{|\lambda_1-\lambda_2|}\times[d_{0\lambda_1-\lambda_2}{}^J(\theta_t)+(-)^{|\lambda_1-\lambda_2|}d_{0\lambda_2-\lambda_1}{}^J(\theta_t)]f_{\lambda_1\lambda_2;00}{}^+. \tag{70}$$

One can relate $T_{\lambda_1\lambda_2;00}{}^{J+}$ to the invariant function $F_j$ by calculating $(8\pi t^{1/2})^{-1}F(t,s,u)$ in the barycentric system of the $t$ channel for the corresponding helicity states of the two $\omega$ mesons. In doing so, one also obtains the relations between the invariant amplitudes and the amplitudes for production of two $\omega$ mesons with helicity $\lambda_1$ and $\lambda_2$. For example,

$$f_{++;00}{}^+(t,\cos\theta_t)=-(16\times3^{1/2}\pi t^{1/2})^{-1}\times p^2\sin^2\theta_t[F_2+F_3+F_4]. \tag{71}$$

Singularities of the partial-wave amplitudes $T_{\lambda_i\lambda_f}{}^{J\pm}$ defined by Eqs. (56)–(59) will occur when the curves $s+u=2(M^2+m^2)$ and $su=(M^2-m^2)^2$ meet singularities of $F_j(s,t,u)$. Thus a pole in the $u$ channel at $u=\bar{m}^2$ will give a branch cut in the $s$ plane from $s=(M^2-m^2)^2/\bar{m}^2$ to $s=2(M^2+m^2)-\bar{m}^2$. The normal threshold in $s$ gives a branch cut $s=(M+m)^2$ to $s=\infty$ while the normal threshold in $u$ giving a branch cut $s=(M-m)^2$ to $s=-\infty$. The normal threshold in $t$ at $t=4m^2$ gives a branch cut $s=-(M^2-m^2)$ to $s=-\infty$ and a cut along the circumference of a circle $|s|=M^2-m^2$. Finally there will be a pole at $s=\bar{m}^2$ corresponding to a bound state. However, we can avoid consideration of such cuts on the complex domains, by evaluating exchange terms directly.

## VI. CALCULATION OF THE HELICITY AMPLITUDES

Before we consider the $\pi\omega$ interaction generated by the exchange of resonant and single-particle intermediate states in the $t$ and $u$ channel, we shall first calculate the helicity amplitudes in terms of the invariant amplitudes.

The scattering amplitude may be calculated for the given helicity states of the initial and final $\omega$. One can easily obtain the helicity amplitudes in terms of the invariant amplitudes from Eq. (19):

$$F_{++}=2^{-1}(1+\cos\theta)[F_1-k^2(1-\cos\theta)F_3], \tag{72}$$

$$F_{-+}=2^{-1}(1-\cos\theta)[F_1+k^2(1+\cos\theta)F_3], \tag{73}$$

$$F_{0+}=-2^{-1/2}M^{-1}\sin\theta[WF_1+k^2(W\cos\theta+E)F_3+2^{-1}k^2(W+E)F_4], \tag{74}$$

$$F_{00}=M^{-2}[(W^2\cos\theta-k^2)F_1+sk^2F_2+k^2(W\cos\theta+E)^2F_3+s^{1/2}k^2(W\cos\theta+E)F_4], \tag{75}$$

where $k$ is the barycentric momentum, $W=(M^2+k^2)^{1/2}$, $E=(m^2+k^2)^{1/2}$, $s=(W+E)^2$, and $F_{\lambda_f\lambda_i}=8\pi s^{1/2}f_{\lambda_f\lambda_i}$. The parity-conserving amplitudes defined by (52)–(55) are given by

$$F_{++}{}^-=-k^2F_3, \tag{76}$$

$$F_{++}{}^+=F_1+k^2\cos\theta F_3, \tag{77}$$

$$F_{0+}{}^+=-2^{1/2}(W/M)[F_1+k^2(\cos\theta+E/W)F_3+2^{-1}W^{-1}s^{1/2}k^2F_4], \tag{78}$$

$$F_{00}{}^+=2(W/M)^2[(\cos\theta-k^2/W^2)F_1+W^{-2}k^2sF_2+W^{-2}k^2(W\cos\theta+E)^2F_3+k^2s^{1/2}(W\cos\theta+E)F_4]. \tag{79}$$

---

[23] When both helicities $\lambda_1$ and $\lambda_2$ equal to zero and $\eta_1\eta_2(-)^{S_1+S_2}=1$, Eq. (45) gives $|JM;00\rangle_+=2^{1/2}|JM;00\rangle$ and $|JM;00\rangle_-=0$. See footnote 8 of Ref. 21. As for the $\omega\bar{\omega}$ state, we have to use a correctly symmetrized wave function, since $\bar{\omega}$ is treated the same as $\omega$. Starting with an eigenstate of the exchange operator $P_{12}$, $2^{-1/2}\{|JM;\lambda_1\lambda_2\rangle+|JM;\lambda_2\lambda_1\rangle\}$, one obtains the parity eigenstate (66b). Notice that only even values of $J$ are allowed for the $t$-channel process.

Thus by identifying the invariant functions, one can readily calculate the parity-conserving amplitudes and the transition amplitudes for given angular momentum $L$ from (62)–(65) of Sec. V.

Before closing this section, let us write down the $\rho$ exchange and axial-vector meson exchange ($A$ exchange) terms in the $s$ and $u$ channels (Fig. 4). The $\rho\pi\omega$ and $A\pi\omega$ vertices shown in Fig. 5 are described in terms of the effective interactions,

$$(g_{\rho\pi\omega}/m_\rho)\delta_{\alpha i}\epsilon_{\mu\nu\lambda\sigma}\epsilon_\mu(\omega)k_\nu\epsilon_\lambda(\rho)q_\sigma \tag{80}$$

and

$$g_{A\pi\omega}m_A\delta_{\alpha i}\epsilon_\mu(\omega)\epsilon_\mu(A). \tag{81}$$

In (81), the $d$-wave interaction $\epsilon_\mu(A)k_\mu\epsilon_\nu(\omega)p_\nu$ is neglected. One can easily verify the contributions of the $\rho$ and $A$ exchanges in the $s$ channel to the invariant amplitudes as follows:

The $\rho$ exchange:

$$F_1(s,t,u)=sk^2\cos\theta[\gamma_\rho/(m_\rho^2-s)], \tag{82}$$

$$F_2(s,t,u)=-(E^2-k^2\cos\theta)[\gamma_\rho/(m_\rho^2-s)], \tag{83}$$

$$F_3(s,t,u)=-s[\gamma_\rho/(m_\rho^2-s)], \tag{84}$$

$$F_4(s,t,u)=2Es^{1/2}[\gamma_\rho/(m_\rho^2-s)]. \tag{85}$$

The $A$ exchange:

$$F_1(s,t,u)=\gamma_A/(m_A^2-s), \tag{86}$$

$$F_2(s,t,u)=m_A^{-2}\gamma_A/(m_A^2-s), \tag{87}$$

$$F_3(s,t,u)=F_4(s,t,u)=0. \tag{88}$$

Also the exchanges in the $u$ channel give:

The $\rho$ exchange:

$$F_1(u,t,s)=[k^4\sin^2\theta+k^2(E^2+W^2)\cos\theta+2k^2EW]\times[\gamma_\rho/(m_\rho^2-u)], \tag{89}$$

$$F_2(u,t,s)=-u[\gamma_\rho/(m_\rho^2-u)], \tag{90}$$

$$F_3(u,t,s)=-(E^2-k^2\cos\theta)[\gamma_\rho/(m_\rho^2-u)], \tag{91}$$

$$F_4(u,t,s)=-2[k^2(1+\cos\theta)-E(E-W)]\times[\gamma_\rho/(m_\rho^2-u)]. \tag{92}$$

The $A$ exchange:

$$F_1(u,t,s)=\gamma_A/(m_A^2-u), \tag{93a}$$

$$F_3(u,t,s)=m_A^{-2}\gamma_A/(m_A^2-u), \tag{93b}$$

$$F_2(u,t,s)=F_4(u,t,s)=0. \tag{93c}$$

Here, $\gamma_\rho=(g_{\rho\pi\omega}/m_\rho)^2$ and $\gamma_A=(g_{A\pi\omega}m_A)^2$.

Indeed, the invariant amplitudes satisfy the crossing relation (21) of Sec. II, if we notice the following relations under $s\leftrightarrow u$, $t\leftrightarrow t$:

$$sk_s^2\cos\theta_s\to uk_u^2\cos\theta_u =k_s^4\sin^2\theta_s+(E^2+W^2)k_s^2\cos\theta_s+2k_s^2EW, \tag{94}$$

$$k_s^2\cos\theta_s-E^2\to k_u^2\cos\theta_u-k_u^2-m^2 =k_s^2\cos\theta_s-E^2, \tag{95}$$

$$s^{1/2}E=2^{-1}(s+m^2-M^2)\to[u(k_u^2+m^2)]^{1/2} =E(E-W)-k_s^2(1+\cos\theta_s), \tag{96}$$

where $E=(k_s^2+m^2)^{1/2}$ and $W=(k_s^2+M^2)^{1/2}$. Those relations are trivial consequence of the two invariants

$$2k_s^2(1-\cos\theta_s)=s+u-2(M^2+m^2) =2k_u^2(1-\cos\theta_u), \tag{97}$$

$$2sk_s^2(1+\cos\theta_s)=(M^2-m^2)^2-us =2uk_u^2(1+\cos\theta_u). \tag{98}$$

Finally, the exchange of a scalar meson ($s$ exchange) in the $t$ channel gives

$$F_1(s,t,u)=2\times3^{-1/2}\gamma_s/(t-m_s^2),\quad F_2(s,t,u)=F_3(s,t,u)=F_4(s,t,u)=0, \tag{99}$$

where $\gamma_s=\gamma_{s\pi\pi}\gamma_{s\omega\omega}$. Here we have also neglected the $d$-wave interaction $(\epsilon_1\cdot k_2)(\epsilon_2\cdot k_1)$. Thus the parity-conserving amplitudes for given helicity states are readily obtainable from (76)–(79), and then their partial-wave projections will immediately follow.

## VII. BORN TERMS

While we were calculating the helicity amplitudes in the last section, the method of localizing the poles of the $S$ matrix was also mentioned on the basis of the Feynman graphs of perturbation theory. It should be noticed that in dispersion theory a single-pole term includes all effects from propagator and vertex renormalization and all kinds of stable particles whether elementary or composite give rise to poles according to the same rules.

In this section, we shall outline a general procedure to derive the driving forces. The advantage of this general discussion is that we can obtain the Born terms of particle exchanges in any spin-parity state. This formalism will become very useful in the $\pi\omega$ scattering when we want to use the experimental information such as the exchange of the $B$ meson for which we do not know yet the definite $J^P$ state.[12] The $B$ meson should be included in the discussion of the $\pi\omega$ scattering along with other possible exchanges, particularly when we want to understand the $\pi\omega$ resonance and its spin and parity state. One should naturally consider all possible assignment of the $J^P$ state and observe the characterization of the driving forces in each of the possible states.

Our procedure will be carried out essentially through two steps: First we shall make use of a narrow-width

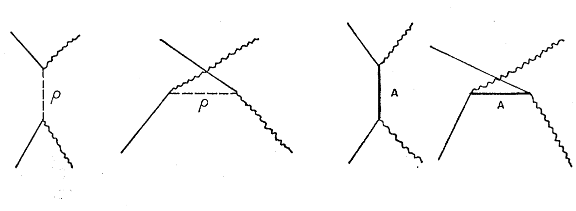


FIG. 4. Diagrams representing the exchange of the $\rho$ and $A$ (a $1^+$ state of the $B$) in the $s$ and $u$ channels.

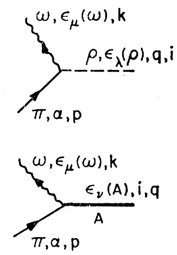


FIG. 5. The vertex graphs for the $\pi\omega\rho$ and $\pi\omega A$ vertices.

resonance approximation in particular transition amplitudes, and secondly, from them we shall calculate the contribution of the crossed $\pi\omega$ cut by using crossing symmetry The procedure will be justified by reproducing the $\rho$ and $A$ exchanges (82)–(88) of Sec. VI.

Let us first invert the relations (76)–(79) of the last section and get

$$F_1(s,t,u)=F_{++}{}^{+}+\cos\theta\, F_{++}{}^{-}, \tag{100}$$

$$\begin{aligned}F_2(s,t,u)=(sk^2)^{-1}&\\ \times[2^{-1}M^2F_{00}{}^{+}&+2^{1/2}M(W\cos\theta+E)F_{0+}{}^{+}\\ &+(W^2\cos\theta+2WE+k^2)F_{++}{}^{+}\\ &+(k^2\cos\theta-E^2)F_{++}{}^{-}],\end{aligned} \tag{101}$$

$$F_3(s,t,u)=-k^{-2}F_{++}{}^{-}, \tag{102}$$

$$\begin{aligned}F_4(s,t,u)=(s^{1/2}k^2)^{-1}&\\ \times[-2^{1/2}MF_{0+}{}^{+}&-2WF_{++}{}^{+}+2EF_{++}{}^{-}],\end{aligned} \tag{103}$$

where

$$\begin{aligned}F_{++}{}^{\pm}=\sum_J(2J+1)[J(J+1)]^{-1}[P_J'(\cos\theta)T_{++}{}^{J\pm}&\\ +P_J''(\cos\theta)(\cos\theta T_{++}{}^{J\pm}-T_{++}{}^{J\mp})],&\end{aligned} \tag{104}$$

$$\begin{aligned}F_{0+}{}^{+}=-(W/M)\sum_J(2J+1)&\\ \times[J(J+1)]^{-1/2}P_J'(\cos\theta)T_{0+}{}^{J+},&\end{aligned} \tag{105}$$

$$F_{00}{}^{+}=(W/M)^2\sum_J(2J+1)P_J(\cos\theta)T_{00}{}^{J+}. \tag{106}$$

Here $P_J'(\cos\theta)$ means $dP_J(\cos\theta)/d(\cos\theta)$ and again we mention that $F_{\lambda_f\lambda_i}{}^{\pm}=8\pi s^{1/2}f_{\lambda_f\lambda_i}{}^{\pm}$.

By substituting now a narrow-width approximation for the transition amplitude of given $J$ and $L$ state, we can evaluate contributions to the partial-wave amplitudes with given helicity from (62)–(65) of Sec. V. One then obtains the parity-conserving amplitudes from (104)–(106) which in turn are used to calculate contributions to the invariant amplitudes by the relations (100)–(103). Finally from crossing symmetry, we get the Born terms.

We shall now verify our procedure by reproducing the $\rho$-exchange terms. The $\rho$ meson comes into the $\pi\omega$ scattering as its bound state. It has $J=L=1$. Thus we put

$$\begin{aligned}&T^{J=1}(1\leftrightarrow 1)=D_\rho,\\ &T^{J=1}(0\leftrightarrow 0)=T^{J=1}(0\leftrightarrow 2)=T^{J=1}(2\leftrightarrow 2)=0,\end{aligned} \tag{107}$$

where $D_\rho$ is essentially a pole term with correct threshold factor and a kinematical factor which will remove kinematical singularities in the invariant amplitudes. From the relations (62)–(65) of Sec. V, (107) becomes

$$T_{++}{}^{1-}=D_\rho,\quad T_{++}{}^{1+}=T_{0+}{}^{1+}=T_{00}{}^{1+}=0 \tag{108}$$

and we get from (104)–(106) that

$$F_{++}{}^{-}=(3/2)D_\rho,\quad F_{++}{}^{+}=F_{0+}{}^{+}=F_{00}{}^{+}=0. \tag{109}$$

By substituting (109) from Eqs. (100)–(103), the contribution to the invariant amplitudes becomes

$$\begin{aligned}&F_1(s,t,u)=\tfrac{3}{2}\cos\theta\, D_\rho,\\ &F_2(s,t,u)=\tfrac{3}{2}(sk^2)^{-1}(k^2\cos\theta-E^2)D_\rho,\\ &F_3(s,t,u)=-\tfrac{3}{2}k^{-2}D_\rho,\\ &F_4(s,t,u)=3(s^{1/2}k^2)^{-1}ED_\rho.\end{aligned} \tag{110}$$

Thus if we take

$$D_\rho=\tfrac{2}{3}sk^2[\gamma_\rho/(m_\rho{}^2-s)], \tag{111}$$

then (110) becomes the $\rho$-exchange terms (82)–(85) of Sec. VI and the form of (111) has not only a correct threshold behavior but also makes the invariant amplitudes $F_i$ free from kinematical singularities. It should be noticed that our procedure necessitates in a natural way the use of the $sk^2$ in order to impose the threshold condition on the partial-wave amplitude $T^{J=1}(1\leftrightarrow 1)$ which has no kinematical singularities.[24] Using the relations (94)–(96) under $s\leftrightarrow u$ crossing, (110) with (111) gives the contributions due to the $\rho$ exchange in the $u$ channel, which are (89)–(92) of Sec. VI. Thus the Born term or the driving force due to the $\rho$ exchange is obtained. For completeness, we give the contributions of the $\rho$-exchange pole in the $u$ channel to the parity-conserving amplitudes from the relations (76)–(79) of the last section.

$$F_{++}{}^{-}=k^2(E^2-k^2\cos\theta)[\gamma_\rho/(m_\rho{}^2-u)], \tag{112a}$$

$$F_{++}{}^{+}=k^2(k^2+2WE+W^2\cos\theta)[\gamma_\rho/(m_\rho{}^2-u)], \tag{112b}$$

$$F_{0+}{}^{+}=-2^{1/2}Mk^2(E+W\cos\theta)[\gamma_\rho/(m_\rho{}^2-u)], \tag{112c}$$

$$F_{00}{}^{+}=-2M^2k^2\sin^2\theta[\gamma_\rho/(m_\rho{}^2-u)]. \tag{112d}$$

From (112) and the relations (56)–(59) of Sec. V, the partial-wave projections and thus the transition amplitudes are readily available.

Let us next look at the exchange of the $B$ meson. In Sec. VI, we discussed the contributions to the invariant amplitudes due to an axial vector exchange assuming only $s$-wave interactions. We shall first reproduce them from the procedure developed in this section and then derive the Born terms due to the particle exchange with $J^P=2^-$ assuming $p$-wave interactions. *If we assume that the B meson has $J^P=1^+$*, then the interaction can be either $s$ or $d$ wave. Neglecting $d$-wave interactions, we then put

$$\begin{aligned}&T^{J=1}(0\leftrightarrow 0)\equiv D_A,\\ &T^{J=1}(1\leftrightarrow 1)=T^{J=1}(0\leftrightarrow 2)=T^{J=1}(2\leftrightarrow 2)=0.\end{aligned} \tag{113}$$

[24] This statement supports the choice of the phase-space factor for the $1^-$ amplitude of the $\pi\omega$ channel made by J. Fulco, G. Shaw, and D. Wong, Phys. Rev. **137**, B1242 (1965).

Equation (113) gives

$$T_{++}{}^{1+}=T_{0+}{}^{1+}=T_{00}{}^{1+}=(2/3)D_A\,,\quad T_{++}{}^{1-}=0\,,\quad (114)$$

which results in

$$F_{++}{}^{+}=D_A\,,\quad F_{0+}{}^{+}=-2^{1/2}(W/M)D_A\,,$$
$$F_{00}{}^{+}=2\cos\theta(W/M)^2D_A\,,\quad F_{++}{}^{-}=0\,.\quad (115)$$

Therefore we obtain

$$\begin{aligned}\operatorname{Im}F_1(s,t,u)&=\operatorname{Im}D_A\,,\\ \operatorname{Im}F_2(s,t,u)&=\operatorname{Im}(D_A/s)\,,\\ \operatorname{Im}F_3(s,t,u)&=\operatorname{Im}F_4(s,t,u)=0\,.\end{aligned}\quad (116)$$

Here we take

$$D_A=m_A{}^{-2}s[\gamma_A/(m_A{}^2-s)]\quad (117)$$

and make an approximation

$$\operatorname{Im}D_A=\pi\gamma_A m_A{}^{-2}s\delta(s-m_A{}^2)\,.\quad (118)$$

Again we mention that the amplitude $T^{J=1}(0\leftrightarrow 0)/s$ will have no kinematical singularities. It is easy to observe that (86)–(88) of VI is reproduced when (116) with (118) is used to evaluate the first integrals in Eq. (36) of Sec. VI;

$$F_j(s,t,u)=\frac{1}{\pi}\int_{(M+m)^2}^{\infty}dx\frac{A_{sl}{}^j(x,t)}{x-s}\quad (j=1,\,2,\,3,\,4)\,,\quad (119)$$

where $A_{sl}{}^j(x,t)\simeq\operatorname{Im}F_j(x,t,u)$. From crossing symmetry, it follows that

$$\begin{aligned}\operatorname{Im}F_1(u,t,s)&=\pi\gamma_A m_A{}^{-2}u\delta(u-m_A{}^2)\,,\\ \operatorname{Im}F_3(u,t,s)&=\pi\gamma_A m_A{}^{-2}\delta(u-m_A{}^2)\,,\\ \operatorname{Im}F_2(u,t,s)&=\operatorname{Im}F_4(u,t,s)=0\,.\end{aligned}\quad (120)$$

These absorptive functions give the Born terms by evaluating the second integrals in Eq. (36) of Sec. IV;

$$F_j(u,t,s)=\frac{1}{\pi}\int_{(M+m)^2}^{\infty}dx\frac{A_{ul}{}^j(x,t)}{x-u}\quad (j=1,\,2,\,3\;\,4)\,,\quad (121)$$

where

$$A_{ul}{}^j(x,t)\simeq\operatorname{Im}F_j(x,t,s)\,,$$

which have a cut along the crossed $\pi\omega$ cut and are analytic everywhere else in the $s$ plane. Equations (120) and (121) give the contribution from the crossed channel to the invariant amplitudes, thus reproducing (93) of Sec. VI. Again, we give the contribution of the axial vector exchange in the $u$ channel to the parity conserving amplitudes:

$$\begin{aligned}F_{++}{}^{-}&=-m_A{}^{-2}k^2[\gamma_A/(m_A{}^2-u)]\,,\\ F_{++}{}^{+}&=(1+m_A{}^{-2}k^2\cos\theta)[\gamma_A/(m_A{}^2-u)]\,,\\ F_{0+}{}^{+}&=-2^{1/2}M^{-1}[W+m_A{}^{-2}k^2(W\cos\theta+E)]\\ &\qquad\times[\gamma_A/(m_A{}^2-u)]\,,\\ F_{00}{}^{+}&=2M^{-2}[W^2\cos\theta-k^2+m_A{}^{-2}k^2(W\cos\theta+E)^2]\\ &\qquad\times[\gamma_A/(m_A{}^2-u)]\,.\end{aligned}\quad (122)$$

From (122), the partial-wave amplitudes and the transition amplitudes are easily obtainable.

So far we have considered a possible assignment of $J^P=1^+$ for the $B$-meson exchanges. There is still another possible $J^P$ state which has not yet been excluded from the experimental information, that is, $J^P=2^-$. We have noticed that this $J^P$ state can give rise to three transitions. However, we shall assume again the lowest partial-wave interactions only in calculating the Born terms. *If the B meson has* $J^P=2^-$, then neglecting $f$-wave interactions we put

$$\begin{aligned}&T^{J=2}(1\leftrightarrow 1)\equiv D_T\,,\\ &T^{J=2}(2\leftrightarrow 2)=T^{J=2}(1\leftrightarrow 3)=T^{J=2}(3\leftrightarrow 3)=0\,,\end{aligned}\quad (123)$$

where $D_T$ is a propagator with correct threshold and kinematical factor to make the invariant amplitudes free from kinematical singularities. Equation (123) gives

$$T_{++}{}^{2-}=0\,,$$
$$(5/3)T_{++}{}^{2+}=(5/2\sqrt{3})T_{0+}{}^{2+}=(5/4)T_{00}{}^{2+}=D_T\quad (124)$$

and therefore

$$\begin{aligned}F_{++}{}^{-}&=-\tfrac{3}{2}D_T\,,\\ F_{++}{}^{+}&=3\cos\theta D_T\,,\\ F_{0+}{}^{+}&=-3\sqrt{2}(W/M)\cos\theta D_T\,,\\ F_{00}{}^{+}&=2(W/M)^2(3\cos^2\theta-1)D_T\,.\end{aligned}\quad (125)$$

We obtain

$$\begin{aligned}\operatorname{Im}F_1(s,t,u)&=\tfrac{3}{2}\cos\theta\operatorname{Im}D_T\,,\\ \operatorname{Im}F_2(s,t,u)&=\tfrac{3}{2}(sk^2)^{-1}\\ &\qquad\times(k^2\cos\theta+E^2-2W^2/3)\operatorname{Im}D_T\,,\\ \operatorname{Im}F_3(s,t,u)&=\tfrac{3}{2}k^{-2}\operatorname{Im}D_T\,,\\ \operatorname{Im}F_4(s,t,u)&=-3(s^{1/2}k^2)^{-1}E\operatorname{Im}D_T\,.\end{aligned}\quad (126)$$

Notice that (126) is very similar to those of the $\rho$ exchanges (110) but $\operatorname{Im}F_2(s,t,u)$ which behaves like $s^{-2}k^{-2}\operatorname{Im}D_T$. Taking a narrow-width-resonance approximation with an appropriate kinematic factor for $\operatorname{Im}D_T$ in the form of

$$\operatorname{Im}D_T=\tfrac{2}{3}\pi\gamma_T s^2k^2\delta(s-m_T{}^2)\,,\quad (127)$$

we obtain the absorptive amplitudes

$$\begin{aligned}\operatorname{Im}F_1(s,t,u)&=\pi\gamma_T s\{4^{-1}[s-(M-m)^2][s-(M+m)^2]+s[M^2+m^2-2^{-1}(s+u)]\}\delta(s-m_T{}^2)\,,\\ \operatorname{Im}F_2(s,t,u)&=\pi\gamma_T\{3^{-1}[s-(M-m)^2][s-(M+m)^2]+s[M^2/3+2m^2-2^{-1}(s+u)]\}\delta(s-m_T{}^2)\,,\\ \operatorname{Im}F_3(s,t,u)&=\pi\gamma_T s^2\delta(s-m_T{}^2)\,,\\ \operatorname{Im}F_4(s,t,u)&=-\pi\gamma_T s(s+m^2-M^2)\delta(s-m_T{}^2)\,.\end{aligned}\quad (128)$$

From crossing symmetry, $\mathrm{Im}F_j(u,t,s)$ is readily available and thus we get the Born terms from (121):

$$
\begin{aligned}
F_1(u,t,s)&=m_T^2\{4^{-1}[m_T^2-(M-m)^2][m_T^2-(M+m)^2]+m_T^2[M^2+m^2-2^{-1}(s+m_T^2)]\}[\gamma_T/(m_T^2-u)],\\
F_2(u,t,s)&=m_T^4[\gamma_T/(m_T^2-u)],\\
F_3(u,t,s)&=\{3^{-1}[m_T^2-(M-m)^2][m_T^2-(M+m)^2]+m_T^2[M^2/3+2m^2-2^{-1}(s+m_T^2)]\}[\gamma_T/(m_T^2-u)],\\
F_4(u,t,s)&=-m_T^2(m_T^2+m^2-M^2)[\gamma_T/(m_T^2-u)].
\end{aligned}
\tag{129}
$$

The parity-conserving amplitudes are then given by

$$
F_{++}{}^{-}=-k^2\{3^{-1}[m_T^2-(M-m)^2][m_T^2-(M+m)^2]+m_T^2[M^2/3+2m^2-2^{-1}(s+m_T^2)]\}[\gamma_T/(m_T^2-u)],\tag{130a}
$$

$$
\begin{aligned}
F_{++}{}^{+}=\{4^{-1}m_T^2[m_T^2-(M-m)^2][m_T^2-(M+m)^2]&\\
+m_T^4[M^2+m^2-2^{-1}(s+m_T^2)]+k^2\cos\theta[3^{-1}[m_T^2-(M-m)^2][m_T^2-(M+m)^2]&\\
+m_T^2[3^{-1}M^2+2m^2-2^{-1}(s+m_T^2)]]\}[\gamma_T/(m_T^2-u)],&
\end{aligned}
\tag{130b}
$$

$$
\begin{aligned}
F_{0+}{}^{+}=-2^{1/2}(W/M)\{4^{-1}m_T^2[m_T^2-(M-m)^2][m_T^2-(M+m)^2]+m_T^4[M^2+m^2-2^{-1}(s+m_T^2)]&\\
+k^2(\cos\theta+E/W)[3^{-1}[m_T^2-(M-m)^2][m_T^2-(M+m)^2]+m_T^2[3^{-1}M^2+2m^2-2^{-1}(s+m_T^2)]]&\\
-2^{-1}s^{1/2}k^2m_T^2W^{-1}(m_T^2+m^2-M^2)\}[\gamma_T/(m_T^2-u)],&
\end{aligned}
\tag{130c}
$$

$$
\begin{aligned}
F_{00}{}^{+}=2(W/M)^2\{(\cos\theta-k^2W^{-2})[4^{-1}m_T^2[m_T^2-(M-m)^2][m_T^2-(M+m)^2]+m_T^4[M^2+m^2-2^{-1}(s+m_T^2)]]&\\
+sk^2m_T^4W^{-2}+k^2(\cos\theta+EW^{-1})^2[3^{-1}[m_T^2-(M-m)^2][m_T^2-(M+m)^2]&\\
+m_T^2[3^{-1}M^2+2m^2-2^{-1}(s+m_T^2)]]-s^{1/2}k^2(W\cos\theta+E)m_T^2(m_T^2+m^2-M^2)\}[\gamma_T/(m_T^2-u)],&
\end{aligned}
\tag{130d}
$$

from which one can have partial-wave projections.

Partial-wave projections to the states of $J^P=1^{\pm}$, $2^-$ from these exchange forces are explicitly given in the Appendix.

## VIII. PARTIAL-WAVE DISPERSION RELATIONS

Having the Born terms of the transition amplitudes determined, we come now to the method of solution of the partial-wave dispersion relations. Dispersion relations are often used as a tool to impose the unitarity condition on the partial-wave amplitudes in the physical region when the driving forces are known. In the $\pi\omega$ scattering, one can assume that the driving forces are given by the Born terms due to the $\rho$ and $B$ exchanges. The elastic unitarity condition on the partial-wave amplitudes in this problem might be permitted up to a considerably high energy because of the empirical absence of significant connection to the $\pi\varphi$ channels. Once the Born terms are given, one can solve the partial-wave dispersion relations by various methods.

The well-known $N/D$ method[25] has been applied to many calculations with various approximations. In some cases, the integral equations have been reduced to computationally simple forms. Also a cutoff parameter has often been introduced for all dispersion integrals to avoid the difficulty of the well-known divergent behavior associated with forces arising from the exchange of spin-one or higher spin particles. Indeed the Born terms evaluated from the $\rho$ and $B$ exchanges (for both possible $J^P$ states of $B$) of the last section behave like $s\ln s$ for large $s$. Terminating the dispersion integrals at a finite energy will yield a one-parameter solution for all partial-wave amplitudes, provided unsubtraction relations are used for both $N$ and $D$ equations.

In a recent paper[13] we have discussed another method of solution (which we will call the $L/F$ method) of the partial-wave dispersion relations. The $L/F$ method gives a solution of the partial-wave amplitude $A$ as

$$
\begin{aligned}
A^{-1}(s)&=[LF^{-1}]^{-1}\\
&=L^{-1}(s)-L^{-1}(s)\Bigg[\frac{1}{\pi}\int_{C_R}ds'\frac{L(s')\rho(s')L(s')}{s'-s}\\
&\qquad-\frac{1}{\pi}\int_{C_L}ds'\frac{\mathrm{Re}[L(s')-A(s')]\,\mathrm{Im}F(s')}{s'-s}\Bigg]L^{-1}(s),
\end{aligned}
\tag{131}
$$

where $L(s)$ is the left hand-cut contribution, and $C_R$ and $C_L$ denote the right- and left-hand cut, respectively. This solution is free from any subtraction parameter and is manifestly symmetric if one notices

$$
\mathrm{Re}[L(s)-A(s)]\,\mathrm{Im}F(s)=R(s)\,\mathrm{Im}(A^{-1}(s))R(s),\tag{132}
$$

where for $s$ on $C_L$

$$
R(s)=\mathrm{Re}A(s)-\mathrm{Re}L(s)\tag{133}
$$

and

$$
\mathrm{Im}A^{-1}(s)=-[R(s)+L^*(s)]^{-1}\times(\mathrm{Im}L(s))[R(s)+L(s)]^{-1}.\tag{134}
$$

The kinematic factor $\rho(s)$ is given by the unitarity condition

$$
\mathrm{Im}A^{-1}(s)=-\rho(s),\quad\text{for } s \text{ on } C_R,\tag{135}
$$

[25] See the first reference of the footnote 4; additional references on the $N/D$ method can be found in G. Shaw, Phys. Rev. Letters **12**, 345 (1964).

which has one element for transitions in states with $J=L$, while it is a $2\times 2$ diagonal matrix, for the scattering amplitudes in the triplet states with $J\neq L$. One can think of various approximations as was discussed in Ref. 13.

Still another possible approximation[26] is to use a zero-width form consistently to evaluate the second integrals of (131), that is, to approximate $R(s)$ of (133) by the same pole term as that used to obtain the Born terms $L(s)$; namely $R^{J=1}(1\leftrightarrow 1)=D_\rho(s)$ as defined by (111) of Sec. VII, and

$$R^{J=1,P=+1}=\begin{pmatrix} D_A(s) & 0 \\ 0 & 0 \end{pmatrix} \quad\text{or}\quad R^{J=2,P=-1}=\begin{pmatrix} D_T(s) & 0 \\ 0 & 0 \end{pmatrix},$$

depending on the assumed quantum numbers of the $B$ meson. Again $D_A(s)$ and $D_T(s)$ are defined in the last section. In the sense of an effective-range calculation, however, we may neglect the integral over $C_L$ in (131) and try to obtain a one-parameter solution depending on a cutoff introduced for the dispersion integral over $C_R$ in (131). The solution given by (131) forces the partial-wave amplitude $A(s)$ to have zeros at the same position where $L(s)$ vanishes. There is likely to be zeros for $L(s)$ which is obtained by the exchange of spin-nonzero particles. In such cases, we propose to use instead a new $L(s)$ which is divided out by its zeros and evaluate a new $F(s)$ by a similar procedure as that of Ref. 13. In particular, if $L(s)$ has a zero on real axis of $s$ at $s=s_1$, then we use $L_1(s)=(s-s_1)^{-1}L(s)$ and write $A(s)=L_1(s)F(s)^{-1}$, where

$$F(s)=-\frac{1}{\pi}\int_{C_R} ds'\frac{\rho(s')L_1(s')}{s'-s}+\frac{1}{\pi}\int_{C_L} ds'\frac{\mathrm{Im}F(s')}{s'-s}. \tag{136}$$

Here a zero of $L(s)$ means a zero of $\det L(s)$ in general. After putting the identity relation

$$\mathrm{Im}L_1(s)=[\mathrm{Im}A(s)][\mathrm{Re}F(s)]+[\mathrm{Re}A(s)][\mathrm{Im}F(s)] \tag{137}$$

into the definition

$$L_1(s)=\frac{1}{\pi}\int_{C_L} ds'\frac{\mathrm{Im}L_1(s')}{s'-s} \tag{138}$$

and interchanging the order of integration in correct manner, one gets the solution

$$A^{-1}(s)=L_1^{-1}(s)\left[\frac{L_1(s)-L_1(s_1)}{s-s_1} -\frac{1}{\pi}\int_{C_R} ds'\frac{L_1(s')\rho(s')L_1(s')}{s'-s} +\frac{1}{\pi}\int_{C_L} ds'\frac{R(s')(\mathrm{Im}A^{-1}(s'))R(s')}{(s'-s_1)^2(s'-s)}\right]L_1^{-1}(s) \tag{139}$$

which is again symmetric and does not force $A(s)$ to have the same zeros as those of $L(s)$. It should be stressed that in arriving at the solution (139) it is necessarily assumed that $A(s)$ is to be nonsingular. We would also like to remark that one would get still the solution (139) even if a once-subtraction dispersion relation was assumed for $F(s)$ instead of (136). This is not surprising since the solution should not depend on the subtraction parameter as we have discussed in Ref. 13. This method can be easily generalized for the case of finitely many zeros of $L(s)$.

We feel that we have thus far prepared to understand the $\pi\omega$ resonances which lie in the elastic region of the $\pi\omega$ scattering, and the procedure we have developed will shed light in extending our discussion to the other pseudoscalar-vector systems. In the next section, we will outline the programs concerning the application of our theory, and present the results of a model calculation of the $\pi\omega$ resonance.

## IX. DISCUSSION

We have discussed the construction of the invariant amplitudes in the case of the $\pi\omega$ scattering which is the simplest example of the pseudoscalar meson and vector meson scattering problem. These invariant amplitudes are expressed by an approximate one-dimensional representation. By the use of the Jacob-Wick helicity amplitudes, the partial-wave expansions as well as the partial-wave projections of the scattering amplitudes in a definite parity state are obtained. A prescription for obtaining the driving forces is discussed. The procedure is checked to reproduce the same vector and axial-vector exchange terms as those of the Feynman diagrams and is further applied to calculate the $2^-$ exchange terms of the $B$ meson. Also we have discussed the method of solution of the partial-wave dispersion relations. Although we have considered mainly $\pi\omega$ scattering, the theory can be extended to the other pseudoscalar and vector system where more than one isotopic-spin state is present.

To this end, we want to make some remarks about the $\pi\omega$ resonance. Since the discovery of the $B$ particle,[7] several authors[27–31] have discussed its possible quantum numbers, using a dynamical scheme in analogy with the pion-nucleon scattering. A static-model calculation[28] for the $B$ meson in the $\pi\omega$ elastic scattering as well as a relativistic calculation[30] suggested a $2^-$ state, while another relativistic calculation[29] favored a $1^+$ resonance. Recently the existence[31] of a reciprocal bootstrap mechanism[32] between the $\rho$ and a $2^-$ state of the $B$ meson has been discussed in a static-model calculation, in analogy with the reciprocal bootstrap between $N$ and $N^*$

[26] This approximation is suggested by J. Franklin, Phys. Rev. **139**, B912 (1965).

[27] W. Frazer, S. Patil, and L. Watson, Phys. Rev. Letters **11**, 231 (1963).

[28] R. F. Peierls, Phys. Rev. Letters **12**, 50 and 119 (E) (1964).

[29] E. Abers, Phys. Rev. Letters **12**, 55 (1964).

[30] T. K. Kuo, Phys. Rev. Letters **12**, 465 (1964).

[31] J. Franklin, Phys. Rev. **137**, B994 (1965).

[32] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).

in the $\pi N$ scattering.[33] It should be noticed that the relativistic calculations[29,30] assumed the dominant force from the $\rho$ exchange only and the force due to the exchange of the $B$ meson itself was absent. We feel that the $B$-meson exchange is very important even in a qualitative discussion of the $\pi\omega$ scattering. Furthermore, if one wants to accept the analogy with pion-nucleon scattering, the $B$ exchange must be brought into the calculation; namely, the exchange of the $B$ meson in the $\pi\omega$ scattering may give rise to a sufficiently strong attractive force to produce the $\rho$ meson as a $\pi\omega$ bound state. The static-model calculation, on the other hand, is limited to providing relations between the $\rho\pi\omega$ coupling constant and the width of the $B$ resonance, as the calculation of Ref. 31 indicates. This is due to the fact that in the static theory, the masses of the $\rho$ meson and the $B$ meson are each separately controlled by a cutoff parameter. Thus a fully relativistic calculation including the $B$-meson exchange itself will be necessary to discuss the quantum numbers of the $B$ meson. In order to treat the $\rho$-meson pole and the $B$ resonance on an equal footing in the calculation of $\pi\omega$ scattering amplitudes, the $t$-channel contribution may need to be considered, recalling our experience with the $\rho$ exchange in $t$ channel of $\pi N$ scattering. We have also sketched the possibility of taking into account the exchange of a two-pion resonance in the $s$-wave state of the $t$ channel. However, the experimental information on the existence of a scalar resonance of the two-pion state is not yet very conclusive.[34] If this is a nonresonant state, then one can expect the effect to be absorbed in the other parameter, such as the cutoff, of the scattering amplitude, in the sense of the Cini-Fubini approximation.

As a preliminary step to understanding the quantum numbers of the $B$ meson, we have made a simple-minded low-energy approximation to the driving forces and solved an approximate relation for the partial-wave amplitude in the Appendix. The scattering amplitudes in the states of $1^+$ and $2^-$ are effectively represented by the transition amplitudes in the $s$ and $p$ states, respectively, so that we may have to solve a one-channel relation only. Furthermore, the driving forces are used by the threshold approximation, and the contributions from the two-pion states of the $t$ channel are neglected in this estimation.

An axial vector exchange of the $B$ meson gives a very weak repulsion to the $J^P=1^-$ state, while a $2^-$ exchange of the $B$ meson shows an attractive force to the $1^-$ state. The $\rho$ exchange gives an attraction to the $1^+$ state but much smaller than a $1^+$ exchange of the $B$ meson. The $2^-$ state gets the most attractive force from the $\rho$ exchange. Thus the existence of the reciprocal bootstrap mechanism between the $\rho$ meson and a $1^+$ state of the $B$ meson is very unlikely, and $2^-$ appear to be favorable quantum numbers for the $B$ meson.

From the approximate solution of the function $(s^{3/2}k^3/8\pi)\cot\delta^{2^-}$ obtained in the Appendix, we have set the phase shift to be zero at the experimentally observed energy of the $B$ resonance. Then by varying $(\gamma_\rho/4\pi)$ between the values 0.35 and 0.50, corresponding to the observed width of the $\omega$ meson,[35,36] we have tried to compute the width of the $B$ meson. It was seen that for $(\gamma_\rho/4\pi)=0.4869\sim0.4970$, the quantity $\gamma_T$ turned out to be in reasonable agreement with experiments ($\approx 122$ MeV). In particular, for $(\gamma_\rho/4\pi)=0.49692$, we have $\gamma_T=3.0789\times10^{-2}$ which gives the width of the $B$-resonance $\Gamma=128$ MeV from the relation $\Gamma=(\gamma_T/12\pi)s_Bk_B{}^3$. We have also noticed that $(\gamma_\rho/4\pi)<0.52$ in order to have a positive $\gamma_T$.

Although a numerical result of solving the matrix equations for the amplitudes in the state of $2^-$ with a correct form of the driving forces may be taken more seriously, it is nevertheless of some interest to perform a qualitative computation such as the one in the Appendix that can be solved algebraically. It should be noticed that we have no arbitrary parameters in the calculation and that we have ended up with a resonance whose width is in good agreement with experiments.

## ACKNOWLEDGMENTS

The author wishes to record his gratitude to the members of the Physics Department at Brown University, in particular, to Professor D. Feldman, Professor H. Fried, and Dr. M. M. Islam for numerous conversations and for their encouragement.

## APPENDIX

### I. Partial-Wave Projections of the Exchange Forces

In Sec. VII, we developed a prescription to obtain the Born terms from the one-dimensional representation of the invariant amplitudes. This procedure was applied to the $\rho$ and $B$ exchanges. In this Appendix, we will just state the partial-wave projections of those exchange forces to the partial-wave states of $J^P=1^\mp$ and $2^-$.

#### *A. $1^-$ State*

The $\rho$ exchange in the $u$ channel yields

$$\begin{aligned}\tilde{T}_\rho{}^{J=1}(1\leftrightarrow1)=\gamma_\rho(4sk^2)^{-1}\{[E^2-(4EW+k^2)d_\rho\\+(E^2+2W^2)d_\rho{}^2+k^2d_\rho{}^3]Q_0(d_\rho)+(4EW+k^2)\\-(E^2+2W^2)d_\rho-k^2d_\rho{}^2-k^2/3\}\,,\quad\text{(A1)}\end{aligned}$$

[33] A relativistic calculation on the reciprocal bootstrap between $N$ and $N^*$ has been carried out by J. S. Ball and D. Y. Wong, Phys. Rev. **133**, B179 (1964).

[34] See Ref. 17.

[35] The authors of Refs. 24 and 29 used the value 0.35 in the units of $\hbar=c=m=1$, while the authors of Refs. 28 and 30 used 0.45 after the analysis of Refs. 36 and 27.

[36] M. Gell-Mann, D. Sharp, and W. D. Wagner, Phys. Rev. Letters **8**, 261 (1962).

where

$$d_i=(4k^2)^{-1}[s-2(M^2+m^2-m_i^2)-(M^2-m^2)^2/s] \quad (i=\rho, A, T), \tag{A2}$$

$$Q_0(x)=2^{-1}\ln[(x+1)/(x-1)], \tag{A3}$$

and

$$\tilde{T}_i^{J=1}(1\leftrightarrow 1)=T_i^{J=1}(1\leftrightarrow 1)/(sk^2) \quad (i=\rho, A, T). \tag{A4}$$

We have mentioned in the text the removal of the kinematic factor $sk^2$ from this transition amplitude.

An axial vector exchange of the $B$ meson in the $u$ channel gives

$$\tilde{T}_A^{J=1}(1\leftrightarrow 1) =\gamma_A(2sk^2)^{-1}\{[(2m_A^2)^{-1}(d_A^2-1)-k^{-2}d_A]Q_0(d_A) +k^{-2}-(2m_A^2)^{-1}d_A\}, \tag{A5}$$

while a $2^-$ exchange of the $B$ meson contributes

$$\tilde{T}_T^{J=1}(1\leftrightarrow 1) =\gamma_T(4sk^4)^{-1}\{[k^2A_1(d_T^2-1)-2A_2d_T]Q_0(d_T) +2A_2-k^2A_1d_T\}, \tag{A6}$$

where

$$A_1=3^{-1}[m_T^2-(M-m)^2][m_T^2-(M+m)^2] +m_T^2[M^2/3+2m^2-2^{-1}(s+m_T^2)], \tag{A7}$$

$$A_2=4^{-1}m_T^2[m_T^2-(M-m)^2][m_T^2-(M+m)^2] +m_T^4[M^2+m^2-2^{-1}(s+m_T^2)]. \tag{A8}$$

### B. $1^+$ State

The transition amplitude in the $1^+$ state can be written as

$$T_i^{J=1}=\begin{pmatrix} T_{11}^i & T_{12}^i \\ T_{12}^i & T_{22}^i \end{pmatrix} \quad (i=\rho, A), \tag{A9}$$

where

$$T_{11}^i=T_i^{J=1}(0\leftrightarrow 0), \tag{A10}$$

$$T_{12}^i=T_{21}^i=T_i^{J=1}(0\leftrightarrow 2), \tag{A11}$$

$$T_{22}^i=T_i^{J=1}(2\leftrightarrow 2). \tag{A12}$$

The $\rho$ exchange gives

$$T_{11}^\rho=3^{-1}\gamma_\rho\{-[2^{-1}(k^2+2WE)+M^2EW^{-1}]I_0(d_\rho)+[2^{-1}W^{-2}M^4-M^2-2^{-1}(W^2+2E^2)]I_1(d_\rho) +[M^2EW^{-1}-2^{-1}(2WE-k^2)]I_2(d_\rho)+[M^2-2M^4W^{-2}-2^{-1}W^2]I_3(d_\rho)\}, \tag{A13}$$

$$T_{12}^\rho=(2^{1/2}/3)\gamma_\rho\{[-4^{-1}(k^2+2EW)+4^{-1}M^2EW^{-1}]I_0(d_\rho)-[4^{-1}k^2+2^{-1}E^2+2^{-1}M^4W^{-2}]I_1(d_\rho) -[4^{-1}(2WE-k^2)+4^{-1}M^2EW^{-1}]I_2(d_\rho)+[2^{-1}M^4W^{-2}-4^{-1}W^2-4^{-1}M^2]I_3(d_\rho)\}, \tag{A14}$$

$$T_{22}^\rho=3^{-1}\gamma_\rho\{[-4^{-1}(k^2+2WE)+M^2EW^{-1}]I_0(d_\rho)+[M^4W^{-2}-4^{-1}(W^2+2E^2)+M^2]I_1(d_\rho) -[4^{-1}(2EW-k^2)+M^2EW^{-1}]I_2(d_\rho)-[M^2W^{-2}+4^{-1}W^2+M^2]I_3(d_\rho)\}, \tag{A15}$$

while a $1^+$ exchange of the $B$ meson gives

$$T_{11}^A=3^{-1}\gamma_A\{-[3(2k^2)^{-1}+E(Wm_A^2)^{-1}]I_0(d_A)+2^{-1}[W^{-2}-m_A^{-2}-E^2(Wm_A)^{-2}]I_1(d_A)\}, \tag{A16}$$

$$T_{12}^A=(2^{1/2}/3)\gamma_A\{E(4Wm_A^2)^{-1}I_0(d_A)+[2^{-1}m_A^{-2}-2^{-1}W^{-2}+E^2(2W^2m_A^2)^{-1}]I_1(d_A)+3E(4Wm_A^2)^{-1}I_2(d_A)\}, \tag{A17}$$

$$T_{22}^A=3^{-1}\gamma_A\{[3(4k^2)^{-1}+E(Wm_A^2)^{-1}]I_0(d_A)+[5(4m_A^2)^{-1}+W^{-2}-E^2(Wm_A)^{-2}]I_1(d_A) -[9(4k^2)^{-1}+3E(Wm_A^2)^{-1}]I_2(d_A)-9(4m_A^2)^{-1}I_3(d_A)\}, \tag{A18}$$

where we have defined

$$I_n(x)=-\frac{1}{2}\int_{-1}^{1}dz\frac{z^n}{x+z}, \tag{A19}$$

so that one obtains in terms of the Legendre function of the second kind $Q_l(x)$

$$I_n(x)=\sum_{l=0}^{n}(-)^{l+1}a_lQ_l(x). \tag{A20}$$

The coefficients $a_l$ are given by

$$a_n=2^n(n!)^2/(2n)!, \qquad a_m=[(2m+1)2^m n!(n/2+m/2)!]/[(n/2-m/2)!(n+m+1)!], \tag{A21}$$

when $n-m$ is even and positive, $a_m=0$, when $n-m$ is odd or negative. In writing down (A20), we have used the relation

$$Q_l(-x)=(-)^{l+1}Q_l(x). \tag{A22}$$

As it was pointed out in Sec. VII, $T_i^{J=1}(0\leftrightarrow 0)/s$ is a kinematic singularity-free amplitude. However, $T_i^{J=1}/s$ does not make (A9) completely free from the kinematic singularities. One can easily verify that in our definition $T_i^{J=1}(0\leftrightarrow 2)/s^2k^2$ is free from kinematic singularities.

### *C. 2⁻ State*

As in the case of the $1^+$ amplitude, we can write the scattering amplitude in this state as

$$T_i^{J=2}=\begin{pmatrix} T_{11}^i & T_{12}^i \\ T_{12}^i & T_{22}^i \end{pmatrix} \quad (i=\rho, T), \tag{A23}$$

where

$$T_{11}^i=T_i^{J=2}(1\leftrightarrow 1), \tag{A24}$$

$$T_{12}^i=T_{21}^i=T_i^{J=2}(1\leftrightarrow 3), \tag{A25}$$

$$T_{22}^i=T_i^{J=2}(3\leftrightarrow 3). \tag{A26}$$

The $\rho$-exchange contributions are

$$T_{11}^\rho=5^{-1}\gamma_\rho\{[-M^4(2W^2)^{-1}+4^{-1}(3E^2)]I_0(d_\rho)-[4^{-1}(3k^2)+3M^2EW^{-1}]I_1(d_\rho)+[2M^4W^{-2}-4^{-1}(9E^2)-3M^2]I_2(d_\rho)$$
$$+[3M^2EW^{-1}+4^{-1}(3k^2)-3WE]I_3(d_\rho)+[3M^2-2^{-1}(3W^2)-3M^4(2W^2)^{-1}]I_4(d_\rho)\}, \tag{A27}$$

$$T_{12}^\rho=(6^{1/2}/20)\gamma_\rho\{[E^2+M^4W^{-2}]I_0(d_\rho)+[M^2EW^{-1}-k^2]I_1(d_\rho)$$
$$+[M^2-3E^2-4M^4W^{-2}]I_2(d_\rho)+[k^2-4WE-M^2EW^{-1}]I_3(d_\rho)+[3M^4W^{-2}-2W^2-M^2]I_4(d_\rho)\}, \tag{A28}$$

$$T_{22}^\rho=5^{-1}\gamma_\rho\{4^{-1}[2E^2-3M^4W^2]I_0(d_\rho)+[3M^2EW^{-1}-2^{-1}k^2]I_1(d_\rho)+[3M^2-2^{-1}(3E^2)-3M^4W^{-2}]I_2(d_\rho)$$
$$+[2^{-1}k^2-2WE-3M^2EW^{-1}]I_3(d_\rho)+[3M^2+W^2+9M^4(4W^2)^{-1}]I_4(d_\rho)\}. \tag{A29}$$

A $2^-$ exchange of the $B$ meson gives, on the other hand,

$$T_{11}^T=-10^{-1}\gamma_T\{[W^{-2}A_2-sm_T^4W^{-2}-E^2W^{-2}A_1+s^{1/2}EW^{-2}A_3+2^{-1}(3A_1)]I_0(d_T)+[5k^{-2}A_2+4EW^{-1}A_1$$
$$-2s^{1/2}W^{-1}A_3]I_1(d_T)+[-3W^{-2}A_2+3sm_T^4W^{-2}+3E^2W^{-2}A_1-3s^{1/2}EW^{-2}A_3+2^{-1}A_1]I_2(d_T)\}, \tag{A30}$$

$$T_{12}^T=-(6^{1/2}/10)\gamma_T\{2^{-1}[A_1-W^{-2}A_2+sm_T^4W^{-2}+E^2W^{-2}A_1-s^{1/2}EW^{-2}A_3]I_0(d_T)$$
$$+4^{-1}[2EW^{-1}A_1-s^{1/2}W^{-1}A_3]I_1(d_T)-(3/2)[-W^{-2}A_2+sW^{-2}m_T^4+E^2W^{-2}A_1-s^{1/2}EW^{-2}A_3+A_1]I_2(d_T)$$
$$+[5(4W)^{-1}s^{1/2}A_3-5(2W)^{-1}EA_1]I_3(d_T)\}, \tag{A31}$$

$$T_{22}^T=-10^{-1}\gamma_T\{[-(3/2)(-W^{-2}A_2+sm_T^4W^{-2}+E^2W^{-2}A_1-s^{1/2}EW^{-2}A_3)+A_1]I_0(d_T)$$
$$+[9s^{1/2}(2W)^{-1}A_3-9EW^{-1}A_1-12(2k^2)^{-1}A_2]I_1(d_T)$$
$$+[(9/2)(-W^{-2}A_2+sm_T^4W^{-2}+E^2W^{-2}A_1-s^{1/2}EW^{-2}A_3)-(21/2)A_1]I_2(d_T)$$
$$+[25(2k^2)^{-1}A_2+15EW^{-1}A_1-15s^{1/2}E(2W)^{-1}A_3]I_3(d_T)+2^{-1}(25A_1)I_4(d_T)\}. \tag{A32}$$

Here, $A_1$ and $A_2$ are given by (A7) and (A8) and

$$A_3=m_T^2(m_T^2+m^2-M^2). \tag{A33}$$

Again we mention that although $T_i^{J=2}(1\leftrightarrow 1)/s^2k^2$ is free from kinematic singularities, the same factor does not eliminate all the kinematical singularities in (A23). One can see that all of these partial-wave projections have the correct threshold behavior.

### II. A Model for the $\pi\omega$ Resonance

By making the threshold approximation of the driving forces given above, we get

$$T_\rho^{1-}(1\leftrightarrow 1)\simeq(\gamma_\rho/3)(2m^2+M^2)[m_\rho^2-(M-m)^2]k^2, \tag{A34}$$

$$T_A^{1-}(1\leftrightarrow 1)\simeq-(\gamma_A/3)\{m_A^{-2}+2[m_A^2-(M-m)^2]^{-1}\}[m_A^2-(M-m)^2]^{-1}k^2, \tag{A35}$$

$$T_T^{1-}(1\leftrightarrow 1)\simeq-(\gamma_T/3)\{A_1(k^2=0)+2A_2(k^2=0)[m_T^2-(M-m)^2]^{-1}\}[m_T^2-(M-m)^2]^{-1}k^2, \tag{A36}$$

$$T_\rho^{J=1}\simeq(\gamma_\rho/3)Mm[m_\rho^2-(M-m)^2]^{-1}k^2\begin{pmatrix} 4 & 2^{1/2} \\ 2^{1/2} & O(k^2) \end{pmatrix}\approx O(k^2), \tag{A37}$$

$$T_A^{J=1}\simeq\gamma_A[m_A^2-(M-m)^2]^{-1}\begin{pmatrix} 1 & -O(k^2) \\ -O(k^2) & O(k^4) \end{pmatrix}, \tag{A38}$$

$$T_\rho^{J=2}\simeq\gamma_\rho[m_\rho^2-(M-m)^2]^{-1}\begin{pmatrix} 3^{-1}M^2k^2 & O(k^4) \\ O(k^4) & O(k^6) \end{pmatrix}, \tag{A39}$$

and

$$T_T^{J=2} \simeq (\gamma_T/3)[m_T^2-(M-m)^2]^{-1}\begin{pmatrix}[A_1(k^2=0)-2[m_T^2-(M-m)^2]^{-1}A_2(k_2=0)]k^2 & O(k^4)\\ O(k^4) & O(k^6)\end{pmatrix}. \quad \text{(A40)}$$

One observes that an axial vector exchange of the $B$-meson gives a repulsive force to the $1^-$ state and a $2^-$ exchange of the $B$ gives an attraction to the $1^-$ state, i.e., $T_A^{1-}(1\leftrightarrow 1)<0$ while $T_T^{1-}(1\leftrightarrow 1)>0$. The $\rho$ exchange gives $T_\rho^{J=1}\sim O(k^2)$ compared to $T_A^{J=1}$. The $2^-$ state gets attractive forces from both the $\rho$ exchange and the $2^-$ exchange of the $B$ meson. Thus we may conclude that the $2^-$ state is more favorable quantum number for the $B$ meson.

Since the $p$-wave driving forces in (A39) and (A40) are dominant to the other $f$-wave mixing, we shall represent $T_i^{J=2}(i=\rho, T)$ by their $p$-wave interactions and solve effectively a one-channel problem. We shall ignore the $t$-channel contribution. We solve for $A^{J=2}(s)/s^2k^2$ by employing an approximate form of (131), i.e., one in which we neglect the second integral that is over $C_L$. The function $L(s)$ and $\rho(s)$ are given by

$$L(s)=(sk)^{-2}(T_\rho^{J=2}+T_T^{J=2})=\alpha\gamma_\rho s^{-2}, \quad \text{(A41)}$$

where

$$\alpha=3^{-1}M^2[m_\rho^2-(M-m)^2]^{-1}+(\gamma_T/\gamma_\rho)[m_T^2-(M-m)^2]^{-1} \times\{3^{-1}A_1(k^2=0)-(2/3)A_2(k^2=0)[m_T^2-(M-m)^2]^{-1}\} \quad \text{(A42)}$$

and

$$\rho(s)=(s^{3/2}k^3/8\pi). \quad \text{(A43)}$$

Thus we obtain

$$s^2k^2[A^{J=2}(s)]^{-1}=(s^2/\alpha\gamma_\rho)[1-(\alpha\gamma_\rho/8\pi^2)s^2I(s)], \quad \text{(A44)}$$

where

$$I(s)=\int_{(M+m)^2}^{\infty} ds'k^3(s')s'^{-5/2}(s'-s)^{-1}. \quad \text{(A45)}$$

In order to evaluate the integral $I(s)$ algebraically, we approximate $k^3(s')$ by

$$k^3(s')\simeq 8^{-1}s'^{-3/2}(s'-M^2)^2[1-3m^2(s'+M^2)(s'-M^2)^{-2}] \quad \text{(A46)}$$

which is sufficiently well enough for our purpose. [At $s=m_B^2$, $k^3=14.626m^{3/2}$ while (A46) gives 14.545 $m^{3/2}$.] From (A46), the integral (A45) becomes

$$I(s)=8^{-1}s^{-3}(s-M^2)^3[1-3m^2(s+M^2)(s-M^2)^{-2}]H(s)+8^{-1}s^{-1}(M+m)^{-2} \times\{3(M^2+m^2)-3s^{-1}m^2M^2-\tfrac{3}{2}m^2M^2(M+m)^{-2}-(3m^2M^4-M^6)[s^{-2}+2^{-1}s^{-1}(M+m)^{-2}+3^{-1}(M+m)^{-4}]\}, \quad \text{(A47)}$$

where

$$H(s)=s^{-1}\{\ln[(M+m)^2/(s-(M+m)^2)]+i\pi\} \quad \text{for} \quad s>(M+m)^2 \quad \text{(A48)}$$

and therefore

$$(s^{3/2}k^3/8\pi)\cot\delta^{2-}(s)=(s^2/\alpha\gamma_\rho)[1-(\alpha\gamma_\rho/8\pi^2)s^2\,\mathrm{Re}I(s)]. \quad \text{(A49)}$$

By adjusting (A49) to give a resonance at $s=m_B^2$, one obtains

$$\alpha(\gamma_\rho/4\pi)=2\pi m_B^{-4}[\mathrm{Re}I(m_B^2)]^{-1}. \quad \text{(A50)}$$

From (A42) and (A50), we obtain a relation between $\gamma_\rho$ and $\gamma_T$ when the experimental masses of the $\rho$, $\omega$, $\pi$, and $B$ mesons are used for $m_\rho$, $M$, $m$, and $m_T=m_B$, respectively. The relation (A50) gives $\gamma_T$ in reasonable agreement with the experimental width of the $B$ meson for $(\gamma_\rho/4\pi)=0.49692$. Moreover we see that

$$(\gamma_\rho/4\pi)<7\pi[m_\rho^2-(M-m)^2][M^2m_B^4\,\mathrm{Re}I(m_B^2)]^{-1} \quad \text{(A51)}$$

in order to have a positive $\gamma_T$.