Radiative τ Decays and the σ Meson

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The rate for radiative τ decay $\tau^{\pm}\rightarrow\pi^{\pm}+\pi^{+}+\pi^{-}+\gamma$ has been calculated using a linearized form of the Brown-Singer σ-meson model. The corrections to the leading term, which was obtained previously by Dalitz, are of the order of a few percent. The validity of the linear approximation for nonradiative τ decay is discussed in the Appendix.

I. INTRODUCTION

A NUMBER of years age Dalitz[1] calculated the relative rate for radiative τ decay compared to ordinary τ decay using the simplest momentum dependence for the $\tau\rightarrow 3\pi$ vertex.

Recently Brown and Singer[2] have discussed nonradiative τ decay using a model where a hypothetical spin-zero σ meson is introduced so that all vertices in the Feynman diagrams couple only three particles.

The parameters for the σ-meson model have been obtained from an analysis of η-meson decays. The η decay is sensitive to the width of the σ meson because it is possible for the σ meson to be on the mass shell in η decay while this is not possible in τ decay. Brown and Singer have shown that the parameters obtained from the η-decay analysis may also be used to fit the data for τ decay.

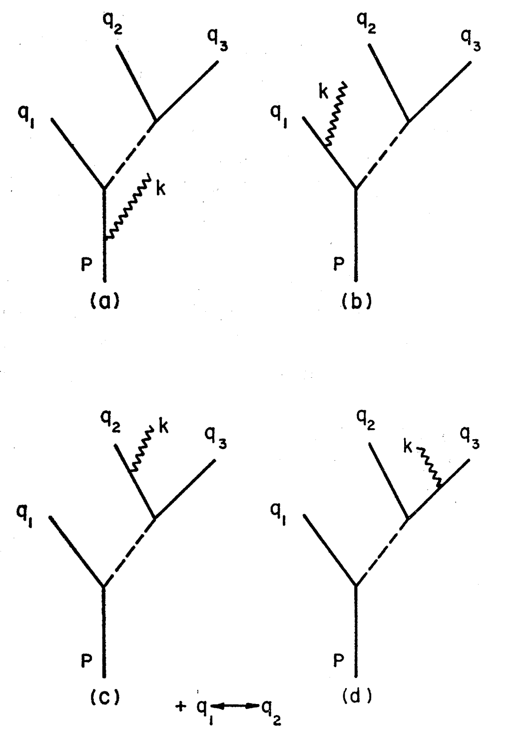


FIG. 1. Feynman diagrams for radiative τ decay. The solid, dotted, and wavy lines represent, respectively, π and τ mesons, σ mesons, and photons. P, q_1, q_2, q_3, and k are four-momenta.

In the present paper we shall apply the Brown-Singer model to a calculation of the radiative τ decay

$$\tau^{\pm}\rightarrow\pi^{\pm}+\pi^{+}+\pi^{-}+\gamma\,.$$

This reaction has been observed a number of times[3] and it is hoped that in the future there will be sufficient data to compare experimentally observed and theoretically predicted spectra as well as the relative rates.

In Sec. II we discuss the theoretical model and carry through the calculation using a linear approximation for the σ-meson propagators.

In Sec. III we compare our results with those obtained by Dalitz and discuss the presently available experimental data.

In the Appendix we discuss the validity of our approximation by comparing the approximate and exact results obtained for nonradiative τ decay.

II. CALCULATION OF THE DECAY RATE

The Feynman diagrams for radiative τ^{+} decay are shown in Fig. 1. The matrix element has the form

$$\mathfrak{M}=\frac{1}{(q_2+q_3)^2-m^2}\left(\frac{q_1\cdot\epsilon}{q_1\cdot k}-\frac{P\cdot\epsilon}{P\cdot k}\right)+\frac{1}{(q_1+q_3)^2-m^2}\left(\frac{q_2\cdot\epsilon}{q_2\cdot k}-\frac{P\cdot\epsilon}{P\cdot k}\right)+\frac{1}{(P-q_1)^2-m^2}\left(\frac{q_2\cdot\epsilon}{q_2\cdot k}-\frac{q_3\cdot\epsilon}{q_3\cdot k}\right)+\frac{1}{(P-q_2)^2-m^2}\left(\frac{q_1\cdot\epsilon}{q_1\cdot k}-\frac{q_3\cdot\epsilon}{q_3\cdot k}\right),\quad(1)$$

where q_1, q_2, and q_3 are the four-momenta of the π^{+}, π^{+}, and π^{-} mesons, respectively; P is the four-momentum

[1] R. H. Dalitz, Phys. Rev. **99**, 915 (1955).

[2] L. M. Brown and P. Singer, Phys. Rev. **133**, B812 (1964).

[3] P. Stamer, T. Huetter, E. L. Koller, S. Taylor, and J. Grauman, Phys. Rev. **138**, B440 (1965). Earlier experimental observations are listed in this paper.

of the τ meson, k and ϵ are the four-momentum and polarization vector of the photon; and $m=m_\sigma-i\frac{1}{2}\Gamma$ is the (complex) mass of the σ meson (m_σ=mass, Γ=full width).

The pion spectra and the relative decay rates are obtained by squaring Eq. (1) and carrying out the integration over the phase space of the pions. In order to carry out the integrations we shall make an approximation to the matrix element given in Eq. (1).

As is well known from studies of nonradiative τ decay, the pion spectra divided by phase space are fit very well by straight lines[4] so that the square of the matrix element may be approximated by an expression which is linear in the kinetic energy of the odd pion.

For the case of radiative τ decay we expect this approximation to be even better than in the nonradiative case because less energy is available to the pions.

We define

$$Q_1=q_2+q_3\,, \tag{2a}$$

$$Q_2=q_1+q_3\,, \tag{2b}$$

$$S_1=p-q_1\,, \tag{2c}$$

$$S_2=p-q_2\,. \tag{2d}$$

Expanding the propagators we have, for example,

$$\frac{1}{Q_1^2-m^2}\approx\frac{1}{\bar{Q}_1^2-m^2}\left(1-\frac{Q_1^2-\bar{Q}_1^2}{\bar{Q}_1^2-m^2}\right), \tag{3}$$

where $\bar{Q}_1$ is an average value for Q_1 which we shall choose later.

We shall also make the simplifying assumption that

$$\bar{Q}_1^2=\bar{Q}_2^2=\bar{S}_1^2=\bar{S}_2^2=\Delta^2(k)\,. \tag{4}$$

Then the matrix element reduces to

$$\begin{aligned}\mathfrak{M}=&\frac{2}{\Delta^2-m^2}\left(\frac{q_1\cdot\epsilon}{q_1\cdot k}+\frac{q_2\cdot\epsilon}{q_2\cdot k}-\frac{q_3\cdot\epsilon}{q_3\cdot k}-\frac{P\cdot\epsilon}{P\cdot k}\right)-\frac{1}{(\Delta^2-m^2)}\left[\left(\frac{Q_1^2-\Delta^2}{\Delta^2-m^2}\right)\left(\frac{q_1\cdot\epsilon}{q_1\cdot k}-\frac{P\cdot\epsilon}{P\cdot k}\right)\right.\\&\left.+\left(\frac{Q_2^2-\Delta^2}{\Delta^2-m^2}\right)\left(\frac{q_2\cdot\epsilon}{q_2\cdot k}-\frac{P\cdot\epsilon}{P\cdot k}\right)+\left(\frac{S_1^2-\Delta^2}{\Delta^2-m^2}\right)\left(\frac{q_2\cdot\epsilon}{q_2\cdot k}-\frac{q_3\cdot\epsilon}{q_3\cdot k}\right)+\left(\frac{S_2^2-\Delta^2}{\Delta^2-m^2}\right)\left(\frac{q_1\cdot\epsilon}{q_1\cdot k}-\frac{q_3\cdot\epsilon}{q_3\cdot k}\right)\right].\end{aligned} \tag{5}$$

Squaring, again dropping quadratic terms, and summing over the photon polarization we obtain

$$\begin{aligned}|\mathfrak{M}|^2=&\left|\frac{2}{\Delta^2-m^2}\right|^2\left(\frac{\mu^2}{(q_1\cdot k)^2}+\frac{2q_1\cdot q_2}{(q_1\cdot k)(q_2\cdot k)}-\frac{2q_1\cdot q_3}{(q_1\cdot k)(q_3\cdot k)}-\frac{2q_1\cdot P}{(q_1\cdot k)(P\cdot k)}+\frac{\mu^2}{(q_2\cdot k)^2}\right.\\&\left.-\frac{2q_2\cdot q_3}{(q_2\cdot k)(q_3\cdot k)}-\frac{2q_2\cdot P}{(q_2\cdot k)(P\cdot k)}+\frac{\mu^2}{(q_3\cdot k)^2}+\frac{2q_3\cdot P}{(q_3\cdot k)(P\cdot k)}+\frac{M^2}{(P\cdot k)^2}\right)\\&-\mathrm{Re}\frac{4}{(\Delta^2-m^{*2})^3}\left[\left(\frac{Q_1^2-\Delta^2}{\Delta^2-m^2}\right)\left(\frac{\mu^2}{(q_1\cdot k)^2}+\frac{q_1\cdot q_2}{(q_1\cdot k)(q_2\cdot k)}-\frac{q_1\cdot q_3}{(q_1\cdot k)(q_3\cdot k)}\right.\right.\\&\left.-\frac{2q_1\cdot P}{(q_1\cdot k)(P\cdot k)}-\frac{q_2\cdot P}{(q_2\cdot k)(P\cdot k)}+\frac{q_3\cdot P}{(q_3\cdot k)(P\cdot k)}+\frac{M^2}{(P\cdot k)^2}\right)\\&+\left(\frac{Q_2^2-\Delta^2}{\Delta^2-m^2}\right)\left(\frac{\mu^2}{(q_2\cdot k)^2}+\frac{q_1\cdot q_2}{(q_1\cdot k)(q_2\cdot k)}-\frac{q_2\cdot q_3}{(q_2\cdot k)(q_3\cdot k)}-\frac{2q_2\cdot P}{(q_2\cdot k)(P\cdot k)}-\frac{q_1\cdot P}{(q_1\cdot k)(P\cdot k)}+\frac{q_3\cdot P}{(q_3\cdot k)(P\cdot k)}+\frac{M^2}{(P\cdot k)^2}\right)\\&+\left(\frac{S_1^2-\Delta^2}{\Delta^2-m^2}\right)\left(\frac{\mu^2}{(q_2\cdot k)^2}+\frac{q_1\cdot q_2}{(q_1\cdot k)(q_2\cdot k)}-\frac{2q_2\cdot q_3}{(q_2\cdot k)(q_3\cdot k)}-\frac{q_2\cdot P}{(q_2\cdot k)(P\cdot k)}-\frac{q_1\cdot q_3}{(q_1\cdot k)(q_3\cdot k)}+\frac{\mu^2}{(q_3\cdot k)^2}+\frac{q_3\cdot P}{(q_3\cdot k)(P\cdot k)}\right)\\&+\left(\frac{S_2^2-\Delta^2}{\Delta^2-m^2}\right)\left(\frac{\mu^2}{(q_1\cdot k)^2}+\frac{q_1\cdot q_2}{(q_1\cdot k)(q_2\cdot k)}-\frac{2q_1\cdot q_3}{(q_1\cdot k)(q_3\cdot k)}\right.\\&\left.\left.-\frac{q_1\cdot P}{(q_1\cdot k)(P\cdot k)}-\frac{q_2\cdot q_3}{(q_2\cdot k)(q_3\cdot k)}+\frac{\mu^2}{(q_3\cdot k)^2}+\frac{q_3\cdot P}{(q_3\cdot k)(P\cdot k)}\right)\right].\end{aligned} \tag{6}$$

μ is the mass of the charged pion and M is the mass of the τ meson.

[4] T. Huetter *et al.*, Phys. Rev. **140**, B655 (1965).

This expression may be simplified by noting that there is complete symmetry between the pions in carrying out the integrations. Thus Eq. (6) reduces to

$$|\mathfrak{M}|^2=\left|\frac{2}{\Delta^2-m^2}\right|^2\left(\frac{3\mu^2}{(q_1\cdot k)^2}-\frac{2q_2\cdot q_3}{(q_2\cdot k)(q_3\cdot k)}-\frac{2q_1\cdot P}{(q_1\cdot k)(P\cdot k)}+\frac{M^2}{(P\cdot k)^2}\right)$$
$$-\mathrm{Re}\frac{8}{(\Delta^2-m^{*2})^3}\left[\left(\frac{Q_1^2-\Delta^2}{\Delta^2-m^2}\right)\left(\frac{\mu^2}{(q_1\cdot k)^2}-\frac{2q_1\cdot P}{(q_1\cdot k)(P\cdot k)}+\frac{M^2}{(P\cdot k)^2}\right)+\left(\frac{S_1^2-\Delta^2}{\Delta^2-m^2}\right)\left(\frac{2\mu^2}{(q_2\cdot k)^2}-\frac{2q_2\cdot q_3}{(q_2\cdot k)(q_3\cdot k)}\right)\right]. \quad (7)$$

The first term is just Dalitz' result while the second term represents the first correction using the σ-meson model. Integrating the Dalitz term, we obtain

$$\int\int\int|\mathfrak{M}|^2\frac{d^3q_1}{\omega_1}\frac{d^3q_2}{\omega_2}\frac{d^3q_3}{\omega_3}\frac{d^3k}{k}\delta^4(P-q_1-q_2-q_3-k)=\frac{4}{(\Delta^2-m_\sigma^2+\frac{1}{4}\Gamma^2)^2+m_\sigma^2\Gamma^2}64\pi^3\frac{dk}{k}\left(1-\frac{2k}{M}\right)\Phi(k), \quad (8)$$

where

$$\Phi(k)=\int_\mu^{\omega_{\max}(k)}\{q[1+x^2(k)]\tanh^{-1}x(k)+x(k)\omega\tanh^{-1}(q/\omega)-2qx(k)\}d\omega, \quad (9)$$

and

$$x(k)=[(E_k^2-2\omega E_k-3\mu^2)/(E_k^2-2\omega E_k+\mu^2)]^{1/2}, \quad (10)$$

$$E_k=(M^2-2Mk)^{1/2}, \quad (11)$$

$$\omega_{\max}(k)=(1/2E_k)(E_k^2-3\mu^2), \quad (12)$$

ω is the energy of the third pion in the 3π center-of-mass system.

The second term integrates to

$$\frac{8[(\Delta^2-m_\sigma^2+\frac{1}{4}\Gamma^2)^2-3m_\sigma^2\Gamma^2](\Delta^2-m_\sigma^2+\frac{1}{4}\Gamma^2)}{[(\Delta^2-m_\sigma^2+\frac{1}{4}\Gamma^2)^2+m_\sigma^2\Gamma^2]^3}64\pi^3\frac{dk}{k}\left(1-\frac{2k}{M}\right)\Psi(k), \quad (13)$$

where

$$\Psi(k)=\int_\mu^{\omega_{\max}(k)}\{(Q^2-\Delta^2)(qx(k)-\omega x(k)\tanh^{-1}(q/\omega))+(S^2-\Delta^2)(qx(k)-q[1+x^2(k)]\tanh^{-1}x(k))\}d\omega, \quad (14)$$

and

$$Q^2=E_k^2-2\omega E_k+\mu^2, \quad (15)$$

$$S^2=M^2-2\omega M+\mu^2. \quad (16)$$

We chose

$$\Delta^2=(E_k-\mu)^2-\tfrac{1}{2}(E_k^2-3\mu^2-2E_k\mu)=\tfrac{1}{2}(E_k^2-2E_k\mu+5\mu^2), \quad (17)$$

which is the midpoint of the spectrum in the 3π center-of-mass system.

The probability of emitting a photon with energy k in the interval dk is then

$$P(3\pi+\gamma)=P_D(3\pi+\gamma)+P_c(3\pi+\gamma), \quad (18)$$

where the probabilities are proportional to Eqs. (8) and (13).

$$P(3\pi+\gamma)dk=4\pi kdk\frac{g_1^2g_2^2e^2}{(2\pi)^4}\int\int\int|\mathfrak{M}|^2\frac{d^3q_1}{\omega_1}\frac{d^3q_3}{\omega_2}\frac{d^3q_3}{\omega_3}\delta^4(P-q_1-q_2-q_3-k), \quad (19)$$

and g_1 and g_2 are the $\pi\pi\sigma$ and $\pi K\sigma$ coupling constants.

III. COMPARISON WITH DALITZ' RESULT AND EXPERIMENTAL DATA

In order to compare the predictions of the linearized σ-meson model and Dalitz' calculation we must also determine the probability for nonradiative τ decay. This calculation is given in the Appendix.

The probability for nonradiative τ decay is given by Eq. (A8). Again

$$P(3\pi)=P_D(3\pi)+P_c(3\pi). \quad (20)$$

We are interested in the integral

$$R(k)=\int_k^{k_{\max}}P(3\pi+\gamma)dk/P(3\pi), \quad (21)$$

which is plotted in Fig. 2(a).

The curve given in Fig. 2(a) is essentially that obtained by Dalitz. In order to study the corrections which arise from the σ-meson hypothesis, it is of interest to compare the magnitudes of the Dalitz term and the correction term.

In Fig. 2(b) we plot $P_c(3\pi+\gamma)/P_D(3\pi+\gamma)$ as a function of the photon energy k. We note that over the entire range of k the corrections are only a few percent. For large k, where the correction exceeds 10%, the rate falls off very rapidly, as seen in Fig. 2(a).

Experimentally, the overwhelming fraction of observed events will be in the energy range $k \lesssim 40$ MeV. In this energy range the correction to the leading term is negative.

The present data[3] consist of only about a dozen events, so that it is not yet possible to examine the spectra of the photons or pions in radiative τ decay. However, it may be possible in the near future to compare the total rates for low-energy photons to those predicted by the Dalitz calculation and the linearized σ-meson model.

Finally, we note that our results are actually insensitive to the assumption of the σ-meson model. Since we have used a linear approximation for the matrix element, one would expect to obtain similar results using either a linear matrix element theory[5] directly, or a linear approximation to a number of other models which have been proposed.[6–12]

IV. CONCLUSIONS

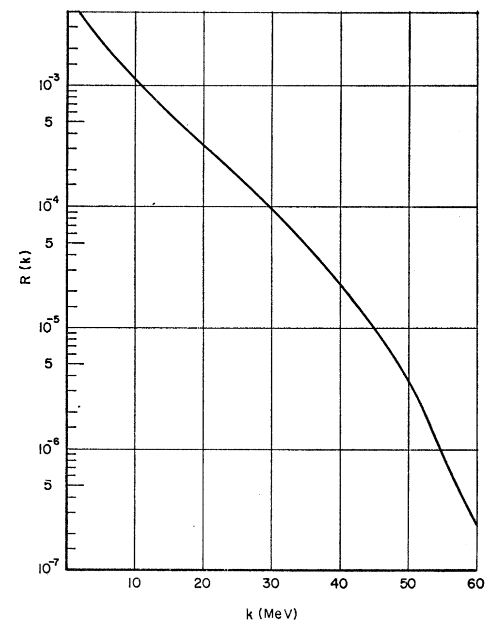
In this paper we have considered the effects of introducing a scalar σ meson to provide a momentum dependence to the nonelectromagnetic part of the radiative τ decay. As in ordinary τ decay, we have made use of a linear approximation to the propagator for the σ meson.

The results of our calculations indicate that the corrections to the radiative rate due to the structure introduced by the σ meson are small, of the order of a few percent. For low-energy photons the correction is negative while for high-energy photons the correction is positive. However, the rate for high-energy photons decreases rapidly with photon energy. In the energy region of experimental interest the contribution of the correction to the leading term will be negative.

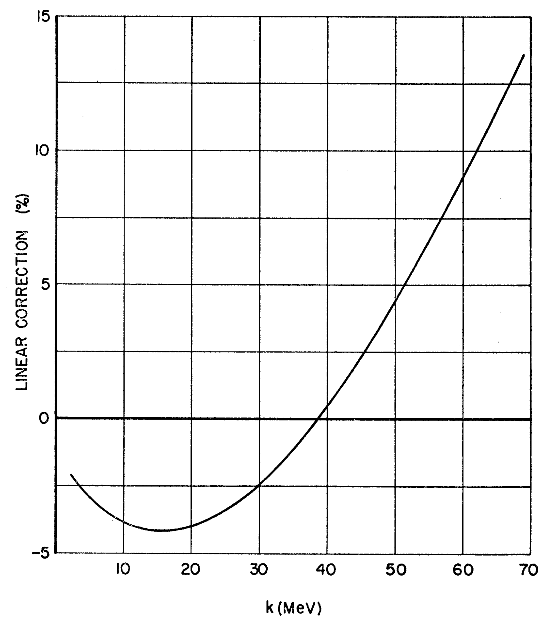
We have obtained our results using a linear approximation to the matrix element. As shown in the Appendix, this approximation is excellent even for nonradiative τ decays. Since the energy available to the pions decreases with increasing photon energy, the approximation improves with increasing values of k.

We also note that a wide range of σ-meson parameters m_σ and Γ may be used to fit the available nonradiative τ-meson decay data.[4] Hence, our result should not be sensitive to the particular values of the parameters which we have chosen.

Furthermore, since we have used a linear approximation, our results should be valid for either a linear matrix element theory[5] or a linearized form of other models which have been proposed.[6–12]



(a)



(b)

FIG. 2. (a) The probability $R(k)$ for emission of a photon exceeding k MeV. (b) Ratio of the linear correction to the Dalitz term as a function of the photon energy.

[5] S. Weinberg, Phys. Rev. Letters **4**, 87 (1960); **4**, 585(E) (1960).

[6] G. Barton, C. Kacser, and S. P. Rosen, Phys. Rev. **130**, 783 (1963).

[7] G. Barton and S. P. Rosen, Phys. Rev. Letters **8**, 414 (1962).

[8] A. N. Mitra and S. Ray, Phys. Rev. **135**, B146 (1964).

[9] C. Kacser, Phys. Rev. **130**, 355 (1963).

[10] S. Hori, S. Oneda, S. Chiba, and A. Wakasa, Phys. Letters **6**, 238 (1963).

[11] Riazuddin and A. Zimmerman, Phys. Rev. **135**, B1211 (1964).

[12] S. Oneda, Y. S. Kim, and L. M. Kaplan, Nuovo Cimento **34**, 655 (1964).

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APPENDIX: NONRADIATIVE τ DECAY

The matrix element for nonradiative τ decay using the σ-meson model is

$$\mathfrak{M}=1/(Q_1^2-m^2)+1/(Q_2^2-m^2). \tag{A1}$$

We expand about $Q_1^2=Q_2^2=\Delta^2$ and obtain

$$\mathfrak{M}=\frac{1}{\Delta^2-m^2}\left[2-\left(\frac{Q_1^2-\Delta^2}{\Delta^2-m^2}\right)-\left(\frac{Q_2^2-\Delta^2}{\Delta^2-m^2}\right)\right]. \tag{A2}$$

Then

$$|\mathfrak{M}|^2=\left|\frac{2}{\Delta^2-m^2}\right|^2\operatorname{Re}\left[1-\left(\frac{Q_1^2-\Delta^2}{\Delta^2-m^2}\right)-\left(\frac{Q_2^2-\Delta^2}{\Delta^2-m^2}\right)\right]$$

$$=\left|\frac{2}{\Delta^2-m^2}\right|^2\operatorname{Re}\left[1-2\left(\frac{Q^2-\Delta^2}{\Delta^2-m^2}\right)\right], \tag{A3}$$

where we have made use of the symmetry among the pions in the last step. The integration of Eq. (A3) may be carried out directly and we obtain

$$\iiint|\mathfrak{M}|^2\frac{d^3q_1}{\omega_1}\frac{d^3q_2}{\omega_2}\frac{d^3q_3}{\omega_3}\delta^4(P-q_1-q_2-q_3)$$

$$=\frac{32\pi^2}{(\Delta^2-m_\sigma^2+\frac{1}{4}\Gamma^2)^2+m_\sigma^2\Gamma^2}\int_\mu^{\omega_{\max}}\phi(\omega)\left[1-2\frac{(Q^2-\Delta^2)(\Delta^2-m_\sigma^2+\frac{1}{4}\Gamma^2)}{(\Delta^2-m_\sigma^2+\frac{1}{4}\Gamma^2)^2+m_\sigma^2\Gamma^2}\right]d\omega, \tag{A4}$$

where

$$\phi(\omega)=(\omega^2-\mu^2)^{1/2}\left(\frac{M^2-2M\omega-3\mu^2}{M^2-2M\omega+\mu^2}\right)^{1/2}, \tag{A5}$$

$$Q^2=M^2-2M\omega+\mu^2, \tag{A6}$$

and

$$\omega_{\max}=(1/2M)(M^2-3\mu^2). \tag{A7}$$

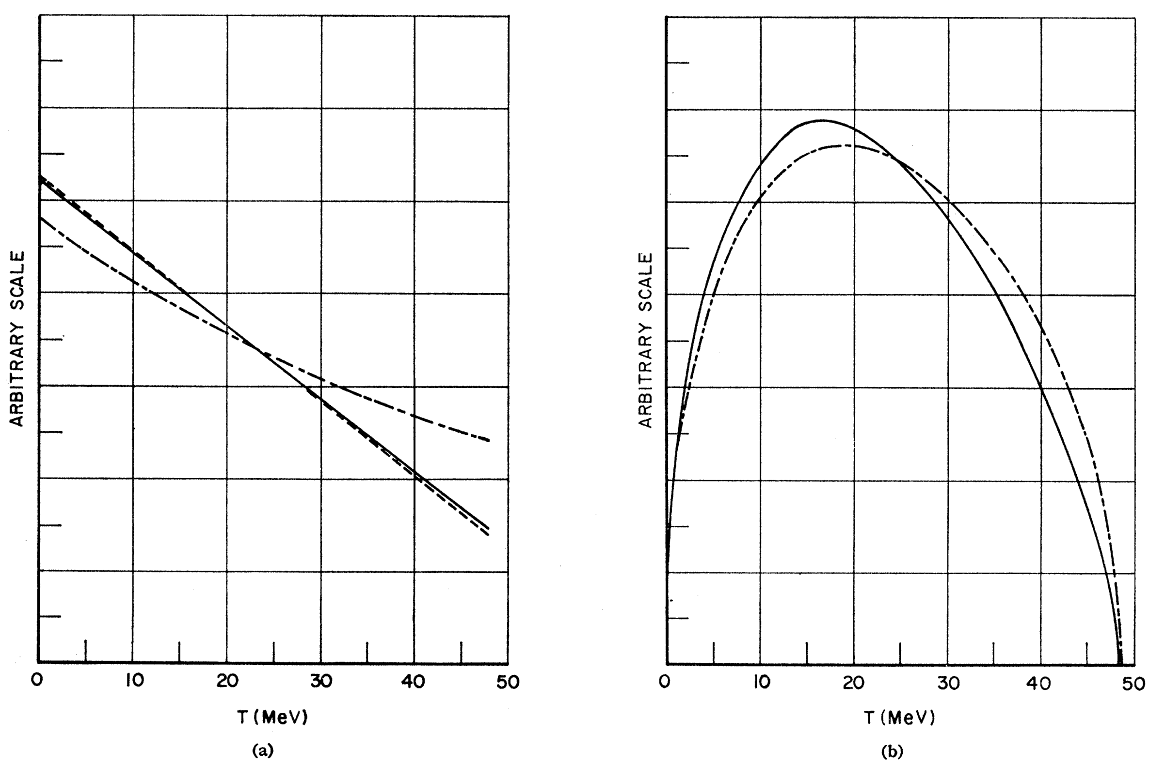

FIG. 3. (a) Comparison of the exact and linear approximation calculations for the reduced π^+ spectrum in τ^+ decay. The solid (short-dashed) line is obtained using a value of $\Delta^2=319$ (322) MeV. The long-and-short-dashed line is the exact result. (b) The same comparison including the phase space.

We note that the term $\sim\int\phi(\omega)d\omega$ is Dalitz' result, while the second term in the bracket in Eq. (A4) represents the first correction using the σ-meson model.

The probability for nonradiative τ decay is given by

$$P(3\pi)=\frac{g_1{}^2g_2{}^2}{2\pi}\int\int\int|\mathfrak{M}|^2\frac{d^3q_1}{\omega_1}\frac{d^3q_2}{\omega_2}\frac{d^3q_3}{\omega_3}\delta^4(P-q_1-q_2-q_3)\,, \tag{A8}$$

where g_1 and g_2 are the $\pi\pi\sigma$ and $\pi K\sigma$ coupling constants.

We note in Eq. (A4) that the correction term is of the order of 10% of the Dalitz term. However, since we have chosen Δ^2 at the midpoint of the range of Q^2, the correction term has positive and negative values. Thus when the integral is carried out, the correction term is less than 2% of the Dalitz term.

It is of interest to compare our approximate answer with the exact answer obtained by Brown and Singer.

Squaring Eq. (A1) we obtain

$$|\mathfrak{M}|^2=2/|Q_1{}^2-m^2|^2+2\,\mathrm{Re}1/(Q_1{}^2-m^{*2})(Q_2{}^2-m^2)\,, \tag{A9}$$

where we have made use of the symmetry between the pions.

Integrating,

$$\int\int\int|\mathfrak{M}|^2\frac{d^3q_1}{\omega_1}\frac{d^3q_2}{\omega_2}\frac{d^3q_3}{\omega_3}\delta^4(P-q_1-q_2-q_3)$$

$$=16\pi^2\int_\mu^{\omega_{\max}}\frac{\phi(\omega)d\omega}{(Q^2-m_\sigma{}^2+\frac{1}{4}\Gamma^2)^2+m_\sigma{}^2\Gamma^2}-16\pi^2\int_\mu^{\omega_{\max}}\frac{(Q^2-m_\sigma{}^2+\frac{1}{4}\Gamma^2)/4M}{(Q^2-m_\sigma{}^2+\frac{1}{4}\Gamma^2)^2+m_\sigma{}^2\Gamma^2}\ln\left[\frac{(h+\phi)^2+m_\sigma{}^2\Gamma^2/M^2}{(h-\phi)^2+m_\sigma{}^2\Gamma^2/M^2}\right]d\omega$$

$$+16\pi^2\int_\mu^{\omega_{\max}}\frac{m_\sigma\Gamma/2M}{(Q^2-m_\sigma{}^2+\frac{1}{4}\Gamma^2)^2+m_\sigma{}^2\Gamma^2}\tan^{-1}\left[\frac{2m_\sigma\Gamma\phi/M}{h^2-\phi^2+m_\sigma{}^2\Gamma^2/M^2}\right]d\omega\,, \tag{A10}$$

where $h=-(M\omega+\mu^2-m_\sigma{}^2+\frac{1}{4}\Gamma^2)/M$.

In Fig. 3 we plot the integrands of Eqs. (A4) and (A10) and also the reduced spectra obtained by dividing by the phase-space factor $\phi(\omega)$. Writing $\Delta^2=(m-\mu)^2-2MT$, $T=T_{\mathrm{av}}=24.1$ MeV, we have $\Delta^2=(319\text{ MeV})^2$. We have chosen $m_\sigma=400$ MeV and $\Gamma=100$ MeV. [For simplicity we have expanded about the midpoint of the distribution. Numerically this differs by about 3 MeV from the best fit[13] to the matrix element, i.e., for $\Delta^2=(322\text{ MeV})^2$. In the case of radiative τ decay, for reasonably large values of k, this difference will be even smaller.]

[13] The best fit is obtained by minimizing the mean square deviation.