Modification of a Regge-Pole Representation*

W. J. ABBE, P. KAUS, P. NATH, AND Y. N. SRIVASTAVA
Department of Physics, University of California, Riverside, California
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A recent representation by Cheng of the partial-wave scattering amplitude has been modified to display the large-energy behavior explicitly in order to improve convergence in terms of Regge poles. The resulting representation has the following properties regardless of the number of trajectories included: (a) It is unitary for all l and $s>0$. (b) It gives the correct threshold behavior as $s\to 0$ for the real and imaginary parts of the amplitude. (c) It reproduces the appropriate analytic properties for the total amplitude in the $\cos\theta$ plane. (d) It converges rapidly.

INTRODUCTION

SINCE the introduction of Regge poles, one of the most crucial problems has been the correct contribution of such a pole to the (total) scattering amplitude. Upon assuming certain bounds for the scattering amplitude in the angular-momentum plane, Khuri[1] was able to obtain a representation in which the contribution of a single Regge pole to the total scattering amplitude had the desired analytic behavior. However, in order to write the scattering amplitude as an infinite sum of Regge contributions of the Khuri form he had to assume certain bounds on the partial-wave amplitudes in the left-half-angular momentum plane.

Cheng and Wu[2] have since shown that the bounds on the partial-wave amplitudes assumed by Khuri were incorrect. Here we modify Cheng's's[3] product representation for the S matrix to exhibit explicitly the asymptotic behavior in energy. Since the resulting expression is manifestly unitary for any number of trajectories retained and since the real and the imaginary parts of the amplitude have the correct threshold behavior, as a by-product, we also obtain an amazingly fast convergence in terms of only a few trajectories.

I. THE CHENG REPRESENTATION

Desai and Newton[4] have given heuristic arguments, on the basis of unitarity, that for potentials of the superposition of Yukawa form (with exponentially decreasing weight factor), the S matrix can be written as a product of Regge poles:

$$S(\lambda,s)=S(0,s)e^{-2i\lambda\,\mathrm{Im}a(s)}\times\prod_{n=1}^{\infty}\frac{\lambda_n(s)}{\lambda_n^*(s)}\frac{\lambda-\lambda_n^*(s)}{\lambda-\lambda_n(s)}e^{2i\lambda\,\mathrm{Im}\lambda_n^{-1}(s)},\quad(1.1)$$

for $s=k^2>0$ and $\lambda=l+\frac{1}{2}$. Here $a(s)$ is some unknown function and $\lambda_n(s)$ are the Regge trajectories. For a more restricted class of potentials,[5] Cheng[3] has shown that Eq. (1.1) is rigorously correct.

For this restricted class of potentials, Cheng and Wu[2] have found the following asymptotic form for the S matrix:

$$S(\lambda,s)\xrightarrow[|\lambda|\to\infty]{} e^{2i\pi\lambda}\ \mathrm{Re}\lambda<0.\quad(1.2)$$

By using this asymptotic form for $S(\lambda,s)$, Cheng[3] has been able to eliminate the unknown subtraction constants in (1.1) to obtain for the S-matrix

$$S(\lambda,s)=\prod_n \exp\int_{\lambda_n(s)}^{\lambda_n^*(s)}d\lambda'\frac{e^{(\lambda'-\lambda)\xi(s)}}{\lambda'-\lambda},\quad(1.3)$$

where $\cosh\xi(s)=1+\mu^2/2s$, and the product n runs over all the Regge poles.

The representation (1.3) may alternatively be written in terms of the exponential integral as

$$\ln S(\lambda,s)=2i\delta(\lambda,s)=\sum_n\left[\mathrm{Ei}((\lambda-\lambda_n(s))\xi(s))-\mathrm{Ei}((\lambda-\lambda_n^*(s))\xi(s))\right],\quad(1.4)$$

where

$$\mathrm{Ei}(z)\equiv\int_z^{\infty}\frac{e^{-x}}{x}dx.$$

We may now identify each member in the sum of (1.4) with the contribution of one pole to the phase shift and write

$$S(\lambda,s)=\prod_{n=1}^{\infty}S_n(\lambda,s),\quad s>0,\quad(1.5)$$

where

$$S_n(\lambda,s)=\exp\int_{\lambda_n(s)}^{\lambda_n^*(s)}d\lambda'\frac{e^{(\lambda'-\lambda)\xi(s)}}{(\lambda'-\lambda)}.$$

The important observation to be made here is that not only is $S(\lambda,s)$ unitary, but each Regge-pole contribution $S_n(\lambda,s)$ is also unitary:

$$S_n^*(\lambda^*,s)=S_n^{-1}(\lambda,s).\quad(1.6)$$

* Work supported in part by the U. S. Atomic Energy Commission.
[1] N. N. Khuri, Phys. Rev. **130**, 429 (1963).
[2] H. Cheng and T. T. Wu, Harvard University report (unpublished).
[3] H. Cheng, Harvard University report (unpublished).
[4] B. R. Desai and R. G. Newton, Phys. Rev. **129**, 1445 (1963).
[5] This restricted class of potentials consists of the square well, the cut-off Coulomb potential, the single Yakawa potential, any potential which has a power-series expansion $\sum_{n=-1}^{\infty}a_nr^n$ as long as the potential is cut off at a finite r, and a superposition of Yukawas of the form $V_0\int_\mu^\infty(e^{-\mu'r}/r)e^{-\mu'}d\mu'$. The interested reader is referred to Ref. 2 for details of this proof.

In this respect it differs from the Khuri[1] and modified Khuri[6,7] representations. This fact is of great importance in any approximation based on a finite number of trajectories, such as is proposed in Ref. 7.

With $A(\lambda,s)\equiv(1/2i)s^{-1/2}[S(\lambda,s)-1]$, the residue[8] $\beta_n(s)$ of the amplitude at the nth Regge pole is easily found with a minimum of algebra to be

$$\beta_n(s)=s^{-1/2}\,\mathrm{Im}\alpha_n(s)\times\exp\left\{\int_{\alpha_n(s)}^{\alpha_n^*(s)}dl'\frac{e^{[l'-\alpha_n(s)]\xi(s)}-1}{l'-\alpha_n(s)}\right\}\prod_{m\neq n}S_m(\alpha_n,s) \quad (1.7)$$

where the trajectories $\alpha_n(s)$ are expressed in l space along with the S matrix. This couples the β_n to all the α_m as in Ref. 7, but much more simply.

In an approximation scheme now it is simply a question of how many terms in the infinite product are to be kept. The inclusion of more trajectories in the computation of the $\beta_n(s)$ here is much easier than in Ref. 7, because of the product representation.

The one-trajectory approximation would now read

$$\beta_n(s)=s^{-1/2}\,\mathrm{Im}\alpha_n(s)\times\exp\int_{\alpha_n(s)}^{\alpha_n^*(s)}dl'\frac{e^{[l'-\alpha_n(s)]\xi(s)}-1}{l'-\alpha_n(s)}. \quad (1.8)$$

If we expand the integral in (1.8) in powers of $\mathrm{Im}\alpha_n(s)$, we find that to lowest order

$$\beta_n(s)=s^{-1/2}\,\mathrm{Im}\alpha_n(s)\,, \quad (1.9)$$

and the next term gives

$$\beta_n(s)=s^{-1/2}\,\mathrm{Im}\alpha_n(s)e^{-2i[\mathrm{Im}\alpha_n(s)]\xi(s)}. \quad (1.10)$$

The equations (1.9) and (1.10) correspond, respectively, to what are called "Universal" and "Khuri" one-trajectory approximations in Refs. 7 and 8.

As will be seen in the next section, using (1.8) to compute the top $\beta_n(s)$ with an exact $\alpha_n(s)$ input does not materially improve upon the one-trajectory Khuri approximation (1.10). This provides the primary motivation for the modification discussed in the next section.

II. THE MODIFIED CHENG REPRESENTATION

In a phenomenological fit in terms of Regge trajectories at high energy,[9] or in a self-consistent calculation, as is proposed by Chew and Jones[10] or Frautschi *et al.*,[7] it is of paramount importance to have the "full" contribution of the poles which are included. In other words, a representation is needed which converges rapidly in terms of the trajectories close to the physical right-hand l plane. Rapid convergence is likely only when the contribution from each Regge pole reflects the analytic properties of the total amplitude, so that it is not necessary to depend on infinitely many terms to produce the correct cuts, unitarity, and asymptotic behavior.

The question of convergence is the motivation in obtaining a modification of the Cheng representation in the same spirit as the previous modification[6,7] of the Khuri representation.[1]

We consider for the case of a Yukawa potential, the integral

$$I=\left(\frac{1}{2\pi i}\right)\int_C\frac{d\lambda'\exp[\lambda'\bar{\xi}(s)]}{\lambda'-\lambda}\times[\ln S(\lambda',s)-ig^2s^{-1/2}Q_{\lambda'-\frac{1}{2}}(\cosh\xi)], \quad (2.1)$$

where $\cosh\xi(s)=1+\mu^2/2s$ and $\cosh\bar{\xi}(s)=1+2\mu^2/s$, with g^2 the strength of the Yukawa coupling. For a distribution of Yukawa potentials, the single Q function is to be replaced by the integral over the weight factor. The contour C is an infinite circle in the λ plane.

Defining a function F, we note that for $\mathrm{Re}\lambda>0$,

$$F=[\ln S(\lambda,s)-ig^2s^{-1/2}Q_{\lambda-\frac{1}{2}}(\cosh\xi)] \quad (2.2)$$

is bounded by $\lambda^{-3/2}\exp[-\lambda\bar{\xi}(s)]$. For $\mathrm{Re}\lambda<0$, from (1.2) and

$$\lim_{|\lambda|\to\infty}Q_\lambda(z)=\left(\frac{2}{\pi\sinh\eta}\right)^{1/2}\frac{e^{-\lambda\eta}}{\lambda^{1/2}}, \quad (2.3)$$

where $z=\cosh\eta$, we have F bounded by $\lambda^{-1/2}e^{-\lambda\xi(s)}$. Hence the integral taken along C vanishes.

The integral (2.1) then yields the following representation:

$$\ln S(l,s)=\sum_n\left\{\int_{\alpha_n(s)}^{\alpha_n^*(s)}dl'\frac{\exp[(l'-l)\bar{\xi}(s)]}{l'-l}-ig^2s^{-1/2}\frac{\exp[-(l+n)\bar{\xi}(s)]}{l+n}P_{n-1}(\cosh\xi)\right\}+ig^2s^{-1/2}Q_l(\cosh\xi). \quad (2.4)$$

In the product form the modified Cheng representation can be written as

$$S(l,s)=[\exp\{ig^2s^{-1/2}Q_l(\cosh\xi)\}]\prod_n S_n(l,s)\,, \quad (2.5)$$

where

$$S_n(l,s)=\exp\left[\int_{\alpha_n(s)}^{\alpha_n^*(s)}dl'\frac{\exp[(l'-l)\bar{\xi}(s)]}{l'-l}-ig^2s^{-1/2}\frac{\exp[-(l+n)\bar{\xi}(s)]}{l+n}P_{n-1}(\cosh\xi)\right]. \quad (2.6)$$

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[6] A. Ahmadzadeh, Phys. Rev. **133**, B1074 (1964).
[7] S. C. Frautschi, P. E. Kaus, and F. Zachariasen, Phys. Rev. **133**, B1607 (1964).
[8] The notation here is the same as in Ref. 7 and D. Hankins, P. E. Kaus, and C. J. Pearson, Phys. Rev. **137**, B1034 (1965).
[9] T. O. Binford and B. R. Desai, Phys. Rev. **138**, B1167 (1965); B. R. Desai, *ibid.* **138**, B1174 (1965); N. N. Khuri and B. M. Udgaonkar, Phys. Rev. Letters **10**, 172 (1963); R. Majumdar, Nuovo Cimento **23**, 1734 (1964).
[10] G. F. Chew and C. E. Jones, Phys. Rev. **135**, B208 (1964).

The residue of a Regge pole in the amplitude analogous to (1.8) now is

$$\beta_n(s)=s^{-1/2}\,\mathrm{Im}\alpha_n(s)[\exp\{ig^2s^{-1/2}Q_{\alpha_n(s)}(\cosh\xi)\}]$$
$$\times\left[\exp\left\{\int_{\alpha_n(s)}^{\alpha_n^*(s)}dl'\frac{\exp[(l'-\alpha_n(s))\xi(s)]-1}{l'-\alpha_n(s)}\right.\right.$$
$$\left.\left.-ig^2s^{-1/2}\frac{\exp[-(\alpha_n+1)\xi(s)]}{\alpha_n+1}\right\}\right]\prod_{m\neq n}S_m(\alpha_m(s),s). \quad (2.7)$$

The order of approximation again depends simply on how many terms in the product of (2.7) are to be kept. In the one-trajectory approximation, the product is put equal to unity.

The representation (2.5) has several remarkable properties:

(a) $S(l,s)$ is unitary at each level of approximation;

$$S^*(l^*,s)=S^{-1}(l,s) \quad (2.8)$$

independent of how many terms are kept in the infinite product.

(b) With the correct threshold behavior of the $\alpha_n(s)$ assumed,

$$\delta(l,s)\underset{s\to0}{\longrightarrow}s^{l+1/2}, \quad (2.9)$$

it follows that both $\mathrm{Re}A(l,s)$ and $\mathrm{Im}A(l,s)$ will have the correct threshold behavior s^l and $s^{2l+1/2}$,[11] respectively [independent of the approximation. The multiplicative factor in (2.9) is suppressed.] This implies the correct boundary for the double spectral function.[12]

(c) For the single Yukawa potential, the amplitude given by (2.7) has a pole in the $\cos\theta$ plane at $\cos\theta=1+\mu^2/2s$ from the Q function and a cut starting at $\cos\theta=1+2\mu^2/s$ from the $\xi(s)$, contributed by each trajectory. This is the correct analytic structure of the total amplitude $A(s,\cos\theta)$. For a distribution of Yukawas with maximum range $1/\mu_0$, the Born pole becomes a cut, which will be correctly given by the generalization of (2.5) from $\cos\theta=1+\mu_0^2/2s$ to $\cos\theta=1+2\mu_0^2/s$.

It may at first seem surprising that the Born term in (2.7), being of the exponential form, does not introduce a cut or a different residue at the Born pole position. However, consider the "zero-pole" approximation in (2.5):

$$S^0(l,s)=\exp\{ig^2s^{-1/2}Q_l(\cosh\xi)\}$$
$$=1+ig^2s^{-1/2}Q_l(\cosh\xi)$$
$$-\tfrac{1}{2}g^4s^{-1}Q_l^2(\cosh\xi)+\cdots, \quad (2.10)$$

$$A^0(l,s)=\tfrac{1}{2}g^2s^{-1}Q_l(\cosh\xi)$$
$$+\tfrac{1}{4}ig^4s^{-2}Q_l^2(\cosh\xi)+\cdots. \quad (2.11)$$

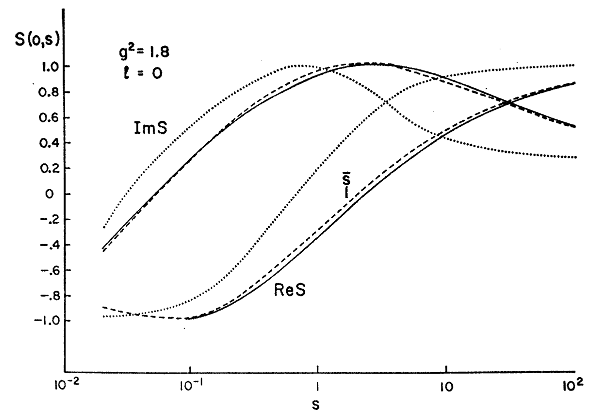

FIG. 1. The S matrix $S(0,s)$ for $l=0$ and $g^2=1.8$ is plotted as a function of the energy s. Exact (——) (Ref. 6), modified Cheng (one trajectory) representation (- - -) and the unmodified Cheng representation (· · ·). The point marked $\bar{s}$ is the energy at which $\mathrm{Re}\alpha_1(s)=-0.5$.

The total amplitude

$$A^0(s,\cos\vartheta)=\sum_{l=0}^{\infty}(2l+1)A^0(l,s)P_l(\cos\vartheta).$$

With the help of[13]

$$Q_l^2(z)=\int_0^\infty Q_l[z^2+(z^2-1)\cosh\varphi]d\varphi,$$

and

$$\sum_{l=0}^{\infty}(2l+1)Q_l(z)P_l(\cos\vartheta)=(z-\cos\vartheta)^{-1},$$

we can now evaluate the zero-trajectory amplitude to be

$$A^0(s,\cos\vartheta)=\frac{g^2}{2s}\frac{1}{(1+\mu^2/2s)-\cos\vartheta}-\frac{ig^4}{4s^2}$$
$$\times\int_0^\infty\frac{d\varphi}{\cos\vartheta-[\cosh^2\xi-\sinh^2\xi\cosh\varphi]}+\cdots. \quad (2.12)$$

The first term has the usual Born pole and the second term has a cut starting at $\cos\theta=1+2\mu^2/s+\mu^4/2s^2$. In an approximation with trajectories included, the cut will start at $\cos\theta=1+2\mu^2/s$ because of the $\xi(s)$ in the trajectory terms.

Properties (a) and (b) are shared with the unmodified Cheng representation.[3] Property (c) is shared with the modified Khuri representation.[6,7]

The final test of the usefulness of the modified Cheng representation (2.5) is the convergence to the exact amplitude in terms of number of trajectories included. For Yukawa potential trajectories,[6,14] $g^2=1.8$ and $\mu=1$, we compare the one trajectory approximation with the

[11] The behavior of $\mathrm{Im}A(l,s)$ is not correctly given by the Khuri representation; see discussion in Ref. 7.

[12] S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1962), p. 47.

[13] V. DeAlfaro, T. Regge, and C. Rosetti, Nuovo Cimento **26**, 1029 (1962).

[14] A. Ahmadzadeh, P. Burke, and C. Tate, Phys. Rev. **131**, 1315 (1963).

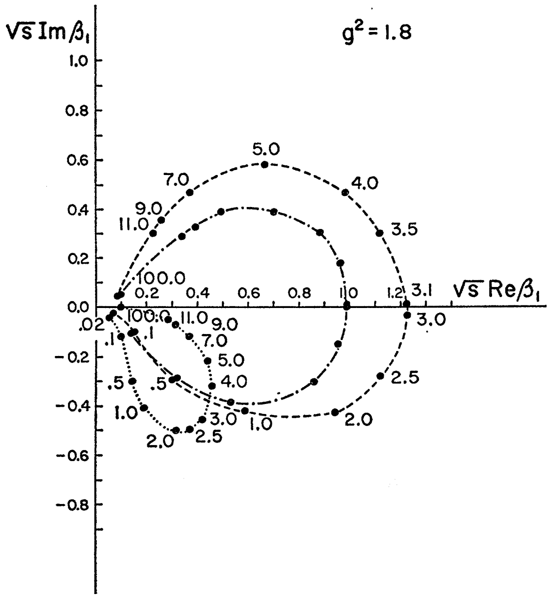

FIG. 2. The residue for the top trajectory $\beta_1(s)$ for $g^2=1.8$ is plotted in the β plane with the energy as a running parameter. Modified Cheng with one trajectory (- - -), modified Khuri (Ref. 8) (-·····) and unmodified Cheng (· · ·). Corresponding points on the modified Cheng and modified Khuri curves are taken at the same energy.

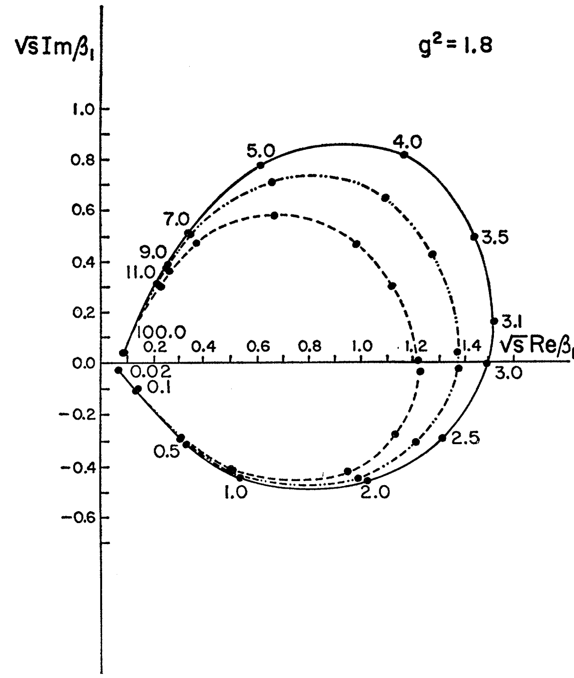

FIG. 3. The residue for the top trajectory $\beta_1(s)$ for $g^2=1.8$ is plotted in the β plane with the energy as a running parameter. Exact (——), modified Cheng with one trajectory (- - -), and modified Cheng with two trajectories (-·· ····· ·). Corresponding points on all three curves are taken at the same energy.

exact S matrix, using the exact top trajectory as input for the Cheng representation (1.5) and the modified Cheng representation (2.5). The coupling is just strong enough to cause one bound S state. The results are given in Fig. 1. For reference, the energy at which the top trajectory retreats to the left-hand λ plane is marked by $\bar{s}$. It is seen that at $\bar{s}$ the S matrix is not close to the Born limit, which would set $\mathrm{Re}S=1$. It therefore seems plausible that if only the low-s part of the trajectories is to be kept (strip approximation), the large-s part of the amplitude should be represented by the relativistic analog of (2.10), rather than by the nonunitary Born term.

In a suggested method[7] of bootstrapping trajectories, as well as a new proposal,[15] in which the background term in a Sommerfeld-Watson transform is to be explicitly evaluated, the residues β must be eliminated in favor of the trajectories α. Equations (1.8) and (2.7) do just that, as did (3.10) in Frautschi *et al.*[7] The first two equations are for the Cheng and modified Cheng representations, respectively; the third, (3.10) in Ref. 7, is for the modified Khuri representation and is obtained by demanding somewhat arbitrarily that unitarity be satisfied at $l=\alpha$ in a representation otherwise nonunitary at any level of approximation. Again success of the bootstrap method will depend largely on rapid convergence of β_n to the correct value in terms of trajectories α_m.

Some results for a single Yukawa potential are given in Figs. 2 and 3. The residue $\beta_1(s)$ is compared to the one trajectory approximation with exact $\alpha_1(s)$ (top trajectory) input, using modified Khuri [(3.0) in Ref. 7], unmodified Cheng (1.8), and modified Cheng (2.7). The resulting $\beta_1(s)$ is given in the β plane in Fig. 2 with s as a running parameter.

The convergence of the modified Cheng representation is shown in Fig. 3. There the exact $\alpha_1(s)$ and $\alpha_2(s)$ input were used for the second curve. It should be mentioned that calculating β is a very severe test as it is most sensitive to input. Amplitudes, at least in the physical region, are not nearly as sensitive. This can be seen by comparing Fig. 3 with Fig. 1.

In summary we can say that the modified Cheng representation is unitary, gives us very fast convergence to the "exact" result and has all the desired asymptotic and analytic properties. It, therefore, may provide a powerful tool for bootstrapping trajectories (or, particle systems) in a more realistic manner,[15] without being forced to mutilate either unitarity[7] or to abandon a trajectory after a certain point (the "strip-width").

The computations were carried out at the Computer Laboratory, University of California, Riverside.

[15] W. J. Abbe, P. E. Kaus, Pran Nath, and Y. N. Srivastava (to be published).