Theoretical and Experimental Comments on the Momentum Dependence of the S Matrix for Many-Particle Processes

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Reactions including five or more particles require quadrilinear momentum invariants for their complete kinematical specification, in contrast to those involving less than five particles, where bilinear invariants suffice. This raises the possibility that the structure of the S matrix may be different in the two cases, and even leads one to consider the remote possibility that different conservation laws might operate. We have investigated both the linear and quadratic dependence of the S matrix on the quadrilinear invariants for the five- and six-particle reactions $K^-+p\rightarrow\Lambda+\pi^++\pi^-(+\pi^0)$. We observe no significant linear or quadratic dependence, the former implying that parity or time-reversal invariance holds for the above reactions.

It has been shown[1,2] that reactions involving n particles (including both incident and outgoing ones), in the case $n\geqslant 5$ require quadrilinear momentum invariants for their complete kinematical specification, in contrast to the case $n<5$, where bilinear invariants suffice. This raises the possibility that the structure of the S matrix is different in the two cases, and even leads one to consider the remote possibility that different conservation laws might operate in the two cases. The purpose of this paper is, firstly, to review some relevant theoretical considerations and secondly, to report some experimental results, which (1) confirm parity conservation or time-reversal invariance for five- and six-particle strong interactions,[3] and (2) bear on the question of the dependence of the S matrix on products of quadrilinear invariants.

It is well known that invariant transition probabilities, when averaged over spins, depend only on invariants constructed from the four-momenta of the participating particles. For an $n\geqslant 4$ particle interaction there are only $3n$-10 independent bilinear invariants,[1] $p_{ij}\equiv(p_\mu{}^i-p_\mu{}^j)\cdot(p_i{}^\mu-p_j{}^\mu)$. In addition, for $n\geqslant 5$ there also exist quadrilinear invariants which may be constructed from any four linearly independent momenta. If one chooses three reference momenta p_1, p_2, p_3, there are $n-3$ such invariants, one for every p_i, $i\neq 1, 2, 3$, viz.,

$$\epsilon_i=\operatorname{sgn}\epsilon_{\alpha\beta\gamma\delta}p_i{}^\alpha p_1{}^\beta p_2{}^\gamma p_3{}^\delta. \tag{1}$$

Here $\epsilon_{\alpha\beta\gamma\delta}$ is the completely antisymmetric pseudotensor in Minkowski space. However, only $n-4$ of these are independent because of four-momentum conservation. The invariants (1) are the *only* quadrilinear invariants[4] which cannot be expanded in terms of bilinear ones. Specification of the ϵ_i is both necessary and sufficient to permit the construction of p_i relative to p_1, p_2, p_3, given the $3n-10$ independent p_{ij}.

It must be emphasized that the above remarks imply nothing about the question of whether the S matrix depends on the ϵ_i as well as on the p_{ij}. They do show however that there is an essential difference between the $n<5$ case where there are no ϵ_i, and the $n\geqslant 5$ case where ϵ_i's are needed to specify the system. We are therefore led to inquire about the possible ϵ_i dependence of the S matrix for $n\geqslant 5$.

The product of two quadrilinear invariants, say, $\epsilon_k\epsilon_l$, can be expressed in terms of the bilinear invariants p_{ij}, but the original $3n-10$ are insufficient for this purpose. On the other hand, these products $\epsilon_k\epsilon_l$ together with the $3n-10$ independent p_{ij} *are* sufficient to determine uniquely *all* other p_{ij}. Thus, higher products of the ϵ_i need not be considered; linear and quadratic dependence suffice. Now, it is easily shown that *invariance under space reversal* (P) *or time reversal* (T), *precludes a linear dependence of the cross section on the* ϵ_i for all n. Since the cross section has the form

$$\sigma=A(p_{ij})+\sum_k\epsilon_kB_k(p_{ij})+\sum_{k>l}\epsilon_k\epsilon_lC_{kl}(p_{ij}),$$

and since under P or T inversion $\epsilon_i\rightarrow-\epsilon_i$, while A, B_k, and C_{kl} are invariant for spin averaged processes,[5] invariance of σ requires[6] that all $B_k=0$. Concerning the

[1] F. Rohrlich, Nucl. Phys. **67**, 659 (1965).
[2] F. Rohrlich, Nuovo Cimento **38**, 673 (1965).
[3] For evidence of P conservation, in $n<5$ interactions see M. Block *et al.*, Phys. Rev. **120**, 571 (1960) and references given therein.

[4] The invariants ϵ_i have a simple geometrical significance. They determine the orientation of the four-volume spanned by the four vectors p_i, p_1, p_2, p_3. Moreover, in the rest system of the particle i, ϵ_i takes the special form $\mathbf{p}_1\cdot\mathbf{p}_2\times\mathbf{p}_3/|\mathbf{p}_1\cdot\mathbf{p}_2\times\mathbf{p}_3|$, which is the orientation of the three-volume spanned by the reference momenta.
[5] A, B_k, C_{kl} are functions of the $3n$-10 independent p_{ij}'s only.
[6] It is interesting to note that a linear dependence under ϵ_i would test T invariance in an $n\geqslant 5$ particle *weak* interaction, since P invariance presumably does not hold here. Unfortunately, since the basic Fermi interaction involves exactly four particles, an $n\geqslant 5$ interaction would be of higher order than $g_w{}^2$, which seems experimentally inaccessible at present.

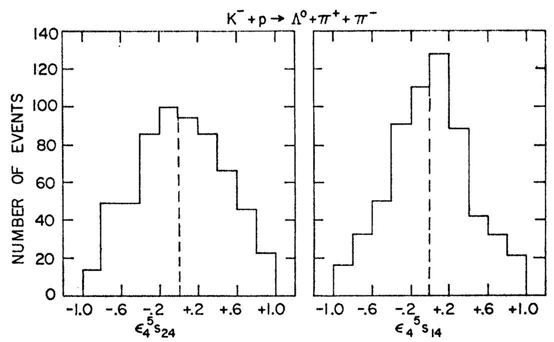

FIG. 1. Typical histograms of the distribution $\epsilon_4{}^5 s_{ij}$ in the reaction $K^-+p\rightarrow\Lambda\pi^+\pi^-$. Events in the $Y^*(1385)^+$ region have been removed, resulting in a sample of 606 events.

quadratic dependence ($\epsilon_k\epsilon_l$) there are no analogous restrictions, so that this dependence remains an open question.

Before we consider the experimental evidence concerning the ϵ_i dependence, it is necessary to note[7] that the phase-space factor depends upon all the bilinear invariants p_{ij}. It is therefore determined by the $3n-10$ independent p_{ij}'s and the products $\epsilon_k\epsilon_l$; there is no ϵ_i dependence in phase space.

Turning now to experiment, we have studied the $n=5$ and $n=6$ reactions

$$K^-+p\rightarrow\Lambda+\pi^++\pi^-, \tag{2}$$

and

$$K^-+p\rightarrow\Lambda+\pi^++\pi^-+\pi^0, \tag{3}$$

(of the form $1+2\rightarrow 3+4+5+6$) using data obtained in an exposure of the 20″ BNL bubble chamber to a 2.24-BeV/c K^- beam. Details of the exposure and analysis procedures are described in detail elsewhere.[8] Effective-mass analysis reveals that the 840-event $\Lambda\pi^+\pi^-$ final state consists essentially of 20% $Y^*(1385)^+\pi^-$ and 80% reaction (2). We use a sample of 606 events whose $\Lambda\pi^+$ effective mass lies outside of the 1380 ± 40 region. The final state $\Lambda\pi^+\pi^-\pi^0$ is more complicated, consisting of $(Y^*)^{+-0}(\pi\pi)^{-+0}$, $\Lambda\omega^0$, and reaction (3) in the ratio ~25%: 30%: 45%. We remove the $\Lambda\omega^0$ events, but accept the Y^* events because they sit on a large reaction (3) background which is of interest here,[9] thereby obtaining an 1182-event sample of the latter.

Tests of the ϵ_i dependence are carried out by averaging the cross section

$$\sigma_{n=5}=f(p_{12},p_{23},p_{31},p_{14},p_{24},\epsilon_4{}^{n=5}) \tag{4}$$

or

$$\sigma_{n=6}=f(p_{12},p_{23},p_{31},p_{14},p_{15},p_{24},p_{25},p_{34},\epsilon_4{}^{n=6},\epsilon_5{}^{n=6}) \tag{5}$$

over all (nonfixed[10]) bilinear invariants except one, for the entire ensemble of events of type (2) or (3). Here the numbering of the particles corresponds to the order shown in (2) and (3) above. For convenience we shall use linear functions s_{ij} of the p_{ij} which have a range between 0 and 1,

$$s_{ij}=\frac{(p_i+p_j)^2-[(p_i+p_j)^2]_{\min}}{[(p_i+p_j)^2]_{\max}}. \tag{6}$$

The significant quadrilinear invariants[11] ϵ_i are most easily calculated in the rest frame of the proton (i.e., the lab system), where they take the simple form

$$\epsilon_4{}^{n=5}=\operatorname{sgn}(\mathbf{p}_K\cdot\mathbf{p}_\Lambda\times\mathbf{p}_{\pi^+})\quad\text{for}\quad(2),$$

$$\left.\begin{aligned}\epsilon_4{}^{n=6}&=\operatorname{sgn}(\mathbf{p}_K\cdot\mathbf{p}_\Lambda\times\mathbf{p}_{\pi^+})\\ \epsilon_5{}^{n=6}&=\operatorname{sgn}(\mathbf{p}_K\cdot\mathbf{p}_\Lambda\times\mathbf{p}_{\pi^-})\end{aligned}\right\}\quad\text{for}\quad(3).$$

Several typical plots of numbers of events versus the products[10] $\epsilon_k{}^n s_{ij}$, [i, j as in (4) and (5)] are shown in Fig. 1 for $n=5$ and Fig. 2 (a) and (b) for $n=6$. The symmetry of all these distributions is evident. The measured expectation values of $s_{ij}\epsilon_k{}^n$ are given in Table I. They are clearly consistent with the requirements of P or T invariance in five- and six-particle strong interactions.

To investigate the question of a possible quadratic dependence of the S matrix, we study the distributions in $s_{ij}\epsilon_4{}^6\epsilon_5{}^6$ for $n=6$ reactions (there is only one independent ϵ for $n=5$). A typical distribution is shown in Fig. 3. The asymmetry is evident, yielding a measured

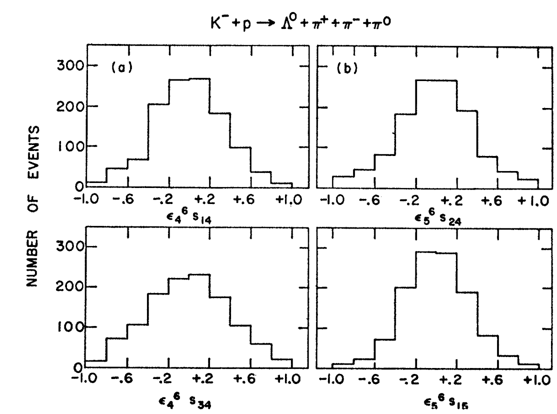

FIG. 2. Typical histograms of the distributions (a) $\epsilon_4{}^6 s_{ij}$ and (b) $\epsilon_5{}^6 s_{ij}$ for 1182 events of the reaction $K^-+p\rightarrow\Lambda\pi^+\pi^-\pi^0$. Events in the ω^0 region have been removed.

[7] N. Byers and C. N. Yang, Rev. Mod. Phys. **36**, 595, (1964).

[8] Bertanza *et al.*, in *Proceedings of the Conference on High-Energy Physics, CERN, 1962* (CERN, Geneva, 1962), p. 279 ff.

[9] We have investigated the ϵ_i dependence using samples which include the resonance events as well as samples which do not, and find no appreciable difference in the results as one would expect, since resonance events are just particle interactions.

[10] The invariant p_{12} is fixed by the nature of the experiment.

[11] The choice ϵ_{π^+} for (2) and ϵ_{π^-}, ϵ_{π^+} for (3) is only one of various possibilities corresponding to permutation of the particles. As a check we repeated the study using other choices for the ϵ_i and found no significant difference in the results.

TABLE I. Measured expectation values and errors of the quantities, $\epsilon_k{}^n s_{ij}$ for the reactions $K^- + p \to \Lambda\pi^+\pi^-(n=5)$ $K^- + p \to \Lambda\pi^+\pi^-\pi^0(n=6)$.

i	j	n	$\langle\epsilon_4{}^n s_{ij}\rangle$	$\langle\epsilon_5{}^n s_{ij}\rangle$
1	3	5	0.02±0.02	...
2	3	5	0.01±0.02	...
1	4	5	0.01±0.02	...
2	4	5	0.02±0.02	...
1	3	6	−0.01±0.01	0.00±0.01
2	3	6	0.01±0.01	−0.01±0.01
1	4	6	0.00±0.01	−0.01±0.01
2	4	6	0.00±0.01	0.00±0.01
3	4	6	0.00±0.01	−0.01±0.01
1	5	6	0.00±0.01	0.01±0.01
2	5	6	0.00±0.01	0.01±0.01

value of $\epsilon_4{}^6\epsilon_5{}^6 = -0.30\pm0.03$, which differs from 0 by 10 standard deviations. As emphasized earlier, however, much of this dependence comes from the phase space factor,[12] which predicts $\epsilon_4{}^6\epsilon_5{}^6 = -0.34$. The agreement between the measured value and the phase space prediction is quite good implying that the $\epsilon_4\epsilon_5$ dependence of the matrix element is small. This is perhaps not so surprising in view of the fact that the two-particle invariant mass distributions[8] (not shown), agree rather well with phase-space predictions, indicating that the matrix element does not play an essential role in the interaction.

We summarize our conclusions as follows:

(i) There is no linear ϵ_i dependence of the S matrix for five- and six-particle reactions to an accuracy of a few percent, confirming P or T invariance for these processes.

(ii) Within experimental error the $\epsilon_k\epsilon_l$ dependence of reaction (3) is entirely due to the phase space dependence. Very high accuracy experiments may be required to establish[13] the (expected) $\epsilon_k\epsilon_l$ dependence of the S matrix.

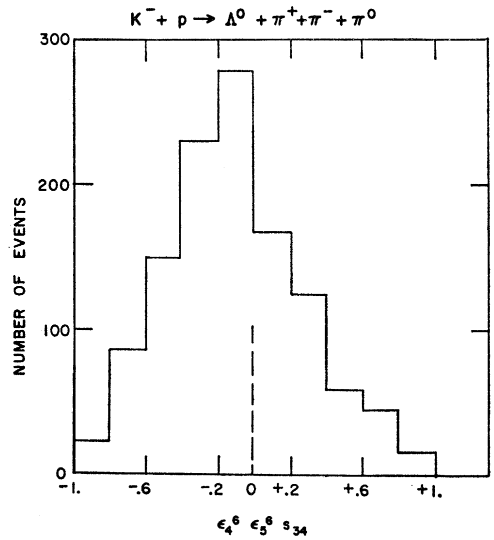

FIG. 3. A sample histogram of the distribution $\epsilon_4{}^6\epsilon_5{}^6 s_{ij}$ for 1182 events in the reaction $K^- + p \to \Lambda\pi^+\pi^-\pi^0$.

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[12] This result was obtained by means of a standard Monte Carlo program in which 10 000 events were generated.

[13] One could entertain the remote possibility that a new invariance principle forbids all $\epsilon_i\epsilon_j$ dependence.