PHYSICAL REVIEW VOLUME 140, NUMBER 6B 20 DECEMBER 1965

Bootstrap Algebra for Mesons

RUDOLPH C. HWA* AND SHARASHCHANDRA H. PATIL
The Institute for Advanced Study, Princeton, New Jersey
(Received 26 July 1965)

The bootstrap of vector and pseudoscalar mesons is considered. The self-supporting dynamics for an arbitrary finite number of these particles is formulated in the analytic S-matrix approach, leading to a set of bootstrap equations. From these, we obtain a set of algebraic relations for the coupling coefficients, called bootstrap algebra. This algebra includes the Lie algebra, whose structure constants are the coefficients of the trilinear coupling of the vector mesons. It is also shown that the pseudoscalar mesons belong to some representation of the associated Lie group. Restrictions on the dimension of this representation are derived; in particular, the pseudoscalar mesons must be in the adjoint representation if there exists a singlet vector meson in addition to the multiplet in the adjoint representation. Physical consequences of the bootstrap algebra, regarding the conservation of G and A parities, are discussed.

I. INTRODUCTION

In the last few years, the philosophy of bootstrap for the strongly interacting particles[1] has led to considerable success in the understanding of certain aspects of the strong-interaction phenomena. The domain of its success falls mainly into two distinct categories. On the one hand, it has contributed,[2–6] qualitatively and sometimes quantitatively, to the understanding of the existence of certain particles, their multiplicities, their masses and the strengths of their interactions with the other particles. On the other hand, it has provided a physical basis for the understanding of the internal symmetries. However, most of the effort in this latter category has been to show that the application of the bootstrap hypothesis yields results that are compatible with those obtained in the studies in which the symmetries are assumed from the outset.[7–12] The deeper question as to how the bootstrap hypothesis could induce the internal symmetry has been left largely unanswered.

The first major step in the direction of the induction of internal symmetry was taken by Cutkosky[7] who succeeded in formalizing the results of a bootstrap model in the language of the algebra of symmetry groups. He considered a world in which only the vector mesons exist. Then, within the framework of the S-matrix theory and under certain simplifying assumptions, the bootstrap condition is imposed by requiring that the forces due to the exchange of the vector mesons in the scattering of two vectors produce the same mesons in the direct channel. This yields the result that the coupling constants for these vector mesons are the structure constants of a compact, semisimple Lie group. This is indeed a striking result, but the model has the obvious shortcoming that it is oversimplified and unrealistic.

In the present study we pursue further the problem of the induction of internal symmetry by considering a more realistic model in which both the vector and the pseudoscalar mesons are assumed to exist and interact. That these vector and psuedoscalar mesons may form a closed system is plausible within the framework of low-mass approximation of the unitarity condition in the S-matrix theory, and is perhaps suggestive of the spin–unitary-spin symmetry. Though the introduction of the pseudoscalars brings into the problem some nontrivial complications, we find that not only are Cutkosky's results retained, but some new and interesting features emerge. The combined results consist of a set of algebraic equations which we call the bootstrap algebra. From this algebra one deduces that in addition to the vectors belonging to the adjoint representation, the pseudoscalars must belong to some representation of the group. We also obtain certain restrictions on the dimension of this representation and the strengths of the various interactions involved. It is found, for example, that the bootstrap algebra demands the vanishing of a certain coupling, a result which is in agreement with the G-parity conservation and which offers an understanding of the empirical A quantum number of Bronzan and low.[13]

In the next section we formulate the bootstrap dynamics of the problem with the view that as few simplifying assumptions are made as possible so as to see the minimum requirements that can lead to exact symmetry. The bootstrap equations are then applied in the following section to the specific scattering system of vector and pseudoscalar mesons, leading to the meson bootstrap algebra. In the last section we discuss some of the physical consequences of the algebra.

* Work supported by the National Science Foundation.

1 G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).
2 F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).
3 G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).
4 E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963).
5 A. W. Martin and K. C. Wali, Nuovo Cimento **31**, 1324 (1964).
6 R. C. Hwa and S. H. Patil, Phys. Rev. **139**, B969 (1965).
7 R. E. Cutkosky, Phys. Rev. **131**, 1888 (1963).
8 E. Abers, F. Zachariasen, and C. Zemach, Phys. Rev. **132**, 1831 (1963).
9 R. H. Capps, Phys. Rev. Letters **10**, 312 (1963); Nuovo Cimento **30**, 340 (1963); Phys. Rev. **134**, B460 (1964).
10 H. M. Chan, P. DeCelles, and J. Paton, Phys. Rev. Letters **11**, 521 (1963); Nuovo Cimento **33**, 70 (1964).
11 R. C. Hwa and S. H. Patil, Phys. Rev. **138**, B933 (1965).
12 R. F. Dashen, *Proceedings of the Second Coral Gables Conference* (W. H. Freeman and Company, San Francisco, 1965).
13 J. B. Bronzan and F. E. Low, Phys. Rev. Letters **12**, 522 (1964).

II. DYNAMICAL CONSIDERATIONS

As is usual in strong-interaction physics, we assume Lorentz invariance and the invariance under C, P, and T. The mesons are represented by Hermitian fields, and the coupling constants are real. The language of fields is used only for the convenience of discussing some properties of the coupling coefficients. It is not necessary in the pure S-matrix theory.

In our model of a world consisting of only vector (V) and pseudoscalar (P) mesons, there are only three types of coupling: VVV, VPP, and VVP. Parity and angular-momentum conservation forbid the PPP coupling. Let us use Latin indices a, b, c, etc., to denote the vector mesons and Greek indices α, β, γ, etc., to denote the pseudoscalar mesons. We do not fix the number of V's or P's in the beginning. Let $a, b, c=1, 2, \cdots, N$ and $\alpha, \beta, \gamma=1, 2, \cdots M$, where N and M are some finite positive integers.

It follows from space-time symmetry and Bose statistics that there are actually two kinds of VVV coupling: $F_{ab}{}^c$ and $F_{ab}{}'^c$. The former is totally antisymmetric under the interchange of any two of the three indices, while the latter is antisymmetric in only two indices. In our present investigation, we shall consider only $F_{ab}{}^c$ and ignore $F_{ab}{}'^c$ completely. We use $G_{\alpha\beta}{}^a$ to denote the VPP coupling and $D_{ab}{}^\alpha$, the VVP coupling. The same consideration of space-time symmetry and Bose statistics leads to the properties that $G_{\alpha\beta}{}^a$ is antisymmetric under the interchange of α and β, while $D_{ab}{}^\alpha$ is symmetric under the interchange of a and b.

We consider in our model only those angular-momentum and parity states that communicate with V and P, i.e., 1^- and 0^-. The problem involves three channels: VV, VP, and PP. The orbital angular-momenta l of these channels in the 1^- state can be 1 or 3 for VV, but just 1 for VP and PP. In the 0^- state l can only be 1 for VV and VP; the PP channel cannot be in the 0^- state. We shall neglect the f wave VV interaction with the consequence that all channels are in the p wave.

Thus in each channel the phase space vanishes as k^{2l+1} $(=k^3)$ at the threshold where k is the momentum in the c.m. system. There are, however, additional kinematical factors in some channels due to the spin of the vector mesons. We shall absorb these factors in the definition of the amplitudes and coupling constants such that the unitary condition can be written in the form

$$T_{ij}-T_{ij}^*=2i\rho T_{ki}^*T_{kj}, \tag{2.1}$$

where the phase-space factor $\rho=k^3/s^{1/2}$ is independent of the channel indices. Here $s=4(k^2+m^2)$, m being the mass of any of the mesons. It is an assumption in our model that all the mesons have the same mass. The consequence of this definition of the amplitudes T_{ij} is that some of them acquire kinematical branch points or zeros (but not poles) at $s=0$. These complications, along with some other dynamical singularities, are not expected to manifest in an essential way under the approximation which we shall adopt.

The channel indices i, j, and k stand for (ab), $(a\alpha)$, or $(\alpha\beta)$ according as to whether the channel is VV, VP, or PP. For $J^P=1^-$, we consider all three channels, so T_{ij} is an element of a matrix of dimension $r=(M+N+1)(M+N)/2$. For $J^P=0^-$, we do not consider PP system since it does not communicate with P; hence the matrix $\mathbf{T}$ is of dimension $r=MN+N(N+1)/2$.

We use the usual multichannel ND^{-1} method to describe the analyticity and unitarity properties of the scattering amplitudes within the framework of the analytic S-matrix theory. We write in matrix form

$$\mathbf{T}(s)=\mathbf{N}(s)\mathbf{D}^{-1}(s). \tag{2.2}$$

Let $\mathbf{P}(s)$ be the potential term which specifies the discontinuity across the left-hand cut. Then, as usual, the integral equations for $\mathbf{N}(s)$ and $\mathbf{D}(s)$ are[14,15]

$$N_{ij}(s)=P_{ij}(s)+\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'-s}\times\left[P_{ik}(s')-\frac{(s-s_0)}{(s'-s_0)}P_{ik}(s)\right]\rho(s')N_{kj}(s'), \tag{2.3}$$

$$D_{ij}(s)=\delta_{ij}-\frac{(s-s_0)}{\pi}\int_{4m^2}^{\infty}\frac{ds'\,\rho(s')N_{ij}(s')}{(s'-s)(s'-s_0)}. \tag{2.4}$$

The asymptotic behavior of $P_{ij}(s)$ is assumed to be such that the integrals converge.

In our model we shall consider only the potential that arises from the exchange of single particles in the various allowable crossed channels. Let us denote the exchanged particle by η and write $P_{ij}(s)=\sum_\eta P_{ij}{}^\eta(s)$. $P_{ij}{}^\eta(s)$ is generally a complicated function of s on account of the kinematical factors. We now make the basic assumption of our dynamical model; that is, the potential $P_{ij}{}^\eta(s)$ can be approximated in the low-energy physical region by the separated form

$$P_{ij}{}^\eta(s)\approx\Gamma_{ij}{}^\eta\phi(s), \tag{2.5}$$

where $\Gamma_{ij}{}^\eta$ depends on the coupling coefficients F, G, and D, and the spin crossing-matrix elements, and is independent of s, while $\phi(s)$ is the universal function independent of the channel indices. This assumption is reasonable since the masses of all particles are assumed to be equal and all (direct and crossed) channels are in p wave, so that the locations of the dynamical singularities are the same in all amplitudes. Furthermore, (2.5) has been found to be approximately valid in some specific examples which we have examined.

Using (2.5) we now have

$$P_{ij}(s)=\textstyle\sum_\eta P_{ij}{}^\eta(s)=V_{ij}\phi(s), \tag{2.6}$$

[14] J. D. Bjorken, Phys. Rev. Letters 4, 473 (1960).

[15] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961); Phys. Rev. **129**, 2363 (1963).

where

$$V_{ij}=\sum_\eta \Gamma_{ij}{}^\eta. \tag{2.7}$$

Since $P_{ij}(s)$ is symmetric on account of the time-reversal invariance, the reality of V_{ij} guarantees that $\mathbf{V}$ can be diagonalized by a real orthogonal transformation. Let $\mathbf{R}$ be the constant orthogonal matrix that effects this diagonalization:

$$\mathbf{R}^{-1}\mathbf{V}\mathbf{R}=\mathbf{\Lambda},$$
$$\Lambda_{ij}=\lambda_i\delta_{ij}. \tag{2.8}$$

Then the columns of $\mathbf{R}$ form the eigenvectors of $\mathbf{V}$, i.e.,[10]

$$\mathbf{R}=(\boldsymbol{\psi}^1,\boldsymbol{\psi}^2,\cdots,\boldsymbol{\psi}^r) \tag{2.9}$$

$$\mathbf{V}\boldsymbol{\psi}^l=\lambda_l\boldsymbol{\psi}^l,\quad 1\leq l\leq r. \tag{2.10}$$

We now show that the same matrix $\mathbf{R}$ diagonalizes $\mathbf{N}(s)$ and $\mathbf{D}(s)$. It is clear from (2.6) that $\mathbf{R}$ diagonalizes $\mathbf{P}(s)$. Thus $\mathbf{N}^d(s)$, defined as $\mathbf{R}^{-1}\mathbf{N}(s)\mathbf{R}$, must satisfy, according to (2.3), an integral equation whose inhomogeneous term and kernel are diagonal. This implies necessarily that the solution $\mathbf{N}^d(s)$ is also diagonal. By inspection of (2.4), it is obvious that $\mathbf{D}(s)$ can similarly be brought to the diagonal form. Let us write

$$\mathbf{R}^{-1}\mathbf{D}(s)\mathbf{R}=\mathbf{1}-\mathbf{E}(s), \tag{2.11}$$

where

$$\mathbf{E}(s)=\frac{(s-s_0)}{\pi}\int_{4m^2}^{\infty}\frac{ds'\,\rho(s')\mathbf{N}^d(s')}{(s'-s)(s'-s_0)}. \tag{2.12}$$

Then the diagonalized form of $\mathbf{T}(s)$ is

$$\mathbf{T}^d(s)\equiv\mathbf{R}^{-1}\mathbf{T}(s)\mathbf{R}=\mathbf{N}^d(s)[\mathbf{1}-\mathbf{E}(s)]^{-1}. \tag{2.13}$$

Now we apply the bootstrap condition which requires that N (or M) of the eigenamplitudes $T_i{}^d(s)$, for scattering in the 1^- (or 0^-) state, have degenerate poles at $s=m^2$. Moreover, we require that no other poles corresponding to other composite states of the system can occur at any higher mass. If we use q to denote N (M) according as $J^P=1^-$ (0^-), then the bootstrap conditions may be stated as follows:

$$E_l(m^2)=1,\quad 1\leq l\leq q, \tag{2.14a}$$

$$E_l(s)<1,\quad q+1\leq l\leq r,\quad s>0. \tag{2.14b}$$

Using (2.13) and (2.14) we have

$$\begin{aligned}\lim_{s\to m^2}(m^2-s)T_l{}^d(s)&=N_l{}^d(m^2)/E_l'(m^2),\quad 1\leq l\leq q,\\ &=0,\quad q+1\leq l\leq r,\end{aligned} \tag{2.15}$$

where $E_l'(s)$ is the first derivation of $E_l(s)$. The right-hand side of (2.15) is, by definition, the residue of the pole term in each of the eigenamplitudes. Because of the identity of the particles, these poles must not only be at the same position but also have the same residue, for otherwise these particles would interact with different strengths to the other particles. Thus, the residue must be independent of the index l. Let us then write

$$\gamma=N_l{}^d(m^2)/E_l'(m^2),\quad 1\leq l\leq q. \tag{2.16}$$

From (2.9), (2.15) and the real orthogonality of $\mathbf{R}$, it follows that

$$\lim_{s\to m^2}(m^2-s)T_{ij}(s)=\gamma\sum_{l=1}^{q}\psi_i{}^l\psi_j{}^l. \tag{2.17}$$

If we use $g_i{}^l$ to stand generically for coupling constants proportional to $F_{ab}{}^c$, $G_{\alpha\beta}{}^a$, or $D_{ab}{}^\alpha$, then we have

$$\sum_{l=1}^{q}g_i{}^lg_j{}^l=\gamma\sum_{l=1}^{q}\psi_i{}^l\psi_j{}^l. \tag{2.18}$$

This implies, in fact,

$$\psi_j{}^l=\gamma^{-1/2}g_j{}^l,\quad 1\leq l\leq q, \tag{2.19}$$

so that, from (2.10), the coupling constants $g_j{}^l$ form the first q eigenvectors of $\mathbf{V}$

$$V_{ij}g_j{}^l=\lambda_lg_i{}^l,\quad 1\leq l\leq q. \tag{2.20}$$

This result has also been obtained by others[7,10] under more restrictive assumptions.

We now develop further consequences of (2.16). Using (2.12), we have

$$\gamma^{-1}=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{ds'\,\rho(s')}{(s'-m^2)^2}\left[\frac{N_l{}^d(s')}{N_l{}^d(m^2)}\right]. \tag{2.21}$$

Although, in general, this integral equation has more than one solution, i.e., the quantity inside the square bracket may depend on l, we choose to consider here only the case for which the integrand is independent of l. This amounts to making the reasonable assumption that all the eigenamplitudes, which have bound-state poles at the same position with the same residue, are in fact the same throughout the complex plane. Thus we have[16]

$$N_l{}^d(s)=N_l{}^d(m^2)n(s), \tag{2.22a}$$

for $1\leq l\leq q$, where $n(s)$ is the universal function independent of the channel indices. Now, using (2.22a) in (2.12), we see that $E_l(s)$ for $1\leq l\leq q$ is of the form $N_l{}^d(m^2)$ multiplied by a universal function. Equation (2.14a) then implies that $N_l{}^d(m^2)$ is independent of l; let us denote it by $\tilde{N}$. To guarantee that (2.14b) be satisfied for all positive s, it is necessary that $N_l{}^d(s)\ll\tilde{N}n(s)$ for $q+1\leq l\leq r$. We make the approximation

$$N_l{}^d(s)=0,\quad q+1\leq l\leq r. \tag{2.22b}$$

Thus from this and (2.22a), we get

$$N_{ij}(s)=N_{ij}(m^2)n(s).$$

It is now easy to see from (2.3) that, in order to have this separable form for $\mathbf{N}(s)$, the potential $\mathbf{P}(s)$ must be such that

$$\mathbf{V}\mathbf{V}=\lambda\mathbf{V}, \tag{2.23}$$

[16] We note that the form (2.22) for $N_{ij}(s)$ includes the results of certain special assumptions such as the determinantal approximation [where the integral in (2.3) is ignored] and the linear approximation for the D functions.

where λ is a constant. This implies that the eigenvalues λ_l of $\mathbf{V}$ are either λ or 0. This result must, of course, be consistent with bootstrap. The application of (2.23) to (2.3) and then to (2.12) yields the result that $\mathbf{E}(s)=\mathbf{\Lambda}e(s)$, where $e(s)$ is some universal function. The bootstrap conditions (2.14) demand that

$$\lambda_l=\lambda=e^{-1}(m^2)\,,\quad 1\leq l\leq q\,, \tag{2.24a}$$

$$\lambda_l=0\,,\quad q+1\leq l\leq r\,, \tag{2.24b}$$

the latter being necessary to guarantee that $E_l(s)$ is small for all positive values of s.

From (2.8) and (2.9) we obtain

$$V_{ij}=\sum_{k,l} R_{ij}\Lambda_{kl}R_{jl}=\sum_{l=1}^{r}\lambda_l\psi_i^l\psi_j^l \tag{2.25}$$

whereupon, using (2.18) and (2.24), we get

$$V_{ij}=(\lambda/\gamma)\sum_{l=1}^{q} g_i^l g_j^l. \tag{2.26}$$

Finally, combining this with (2.7), we have

$$\sum_{l=1}^{q} g_i^l g_j^l=-\frac{\gamma}{\lambda}\sum_{\eta}\Gamma_{ij}^{\eta}. \tag{2.27}$$

This will be our basic equation from which the bootstrap algebra is to be derived.

The orthogonality of $\mathbf{R}$ implies that

$$\sum_{j=1}^{r}\psi_j^k\psi_j^l=\delta_{kl}\,, \tag{2.28}$$

$$\sum_{l=1}^{r}\psi_i^l\psi_j^l=\delta_{ij}. \tag{2.29}$$

The use of (2.19) then yields

$$\sum_{j=1}^{r} g_j^k g_j^l=\delta_{kl}\gamma \tag{2.30}$$

for $1\leq k,l\leq q$.

The main assumptions that have been used in the derivation of (2.27), apart from the bootstrap conditions, are those embodied in (2.5) and (2.22). We do not pretend to imply that they are necessarily good approximations of the actual situation. The sole reason for making these assumptions is to show, in the next section, that they lead to *exact* symmetry. They appear to us to be the weakest assumptions possible, though we have no proof for this. It is reasonable to expect that as the dynamical calculation is improved, such as when these assumptions are eliminated and the mass degeneracy is lifted, what emerges will be a broken symmetry as observed. Our purpose here is only to establish the "minimum" requirements for exact symmetry.

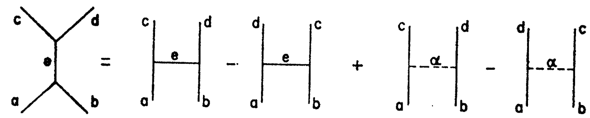

FIG. 1. Diagram for bootstrap equation of VV scattering in the 1^- state.

III. BOOTSTRAP ALGEBRA

In this section we consider more specifically the one-particle exchange potentials for the scattering of vector and pseudoscalar mesons in the 0^- and 1^- states. We shall show that, when applied to the bootstrap equation (2.27), they lead to a set of algebraic relations on the coupling coefficients free from undetermined dynamical constants.

In the following we shall write the potential term $(\gamma/\lambda)\Gamma_{ij}^{\eta}$ arising from the exchange of a particular particle in a particular crossed channel (η), in the form

$$r_\nu g_i^\eta g_j^\eta\,, \tag{3.1}$$

where g_i^η is the generic symbol for $F_{ab}^{\;c}$, $G_{\alpha\beta}^{\;a}$, and $D_{ab}^{\;\alpha}$. The index η does not necessarily correspond to the superscripts of F, G, and D. The constants r_ν, which vary with each triplet (i,j,η), depend on such quantities as kinematic factors, spin crossing-matrix elements, etc. They can, in principle, be calculated from the dynamical considerations of the problem. However, in our investigation we shall not have need to calculate them explicitly.

Consider first the vector-vector scattering in the 1^- state. We have

$$F_{ab}^{\;e}F_{cd}^{\;e}=r_1(F_{ac}^{\;e}F_{bd}^{\;e}-F_{ad}^{\;e}F_{bc}^{\;e})+r_2(D_{ac}^{\;\alpha}D_{bd}^{\;\alpha}-D_{ad}^{\;\alpha}D_{bc}^{\;\alpha})\,, \tag{3.2}$$

where repeated indices are summed. The diagrammatic presentation of this equation is shown in Fig. 1. The relative signs of the terms in the parenthesis are dictated by the antisymmetric properties of the F's on the left-hand side. Similarly, for VV scattering in the 0^- state, we have

$$D_{ab}^{\;\alpha}D_{cd}^{\;\alpha}=s_1(F_{ac}^{\;e}F_{bd}^{\;e}+F_{ad}^{\;e}F_{bc}^{\;e})+s_2(D_{ac}^{\;\alpha}D_{bd}^{\;\alpha}+D_{ad}^{\;\alpha}D_{bc}^{\;\alpha})\,. \tag{3.3}$$

There are seven other equations for the remaining scattering processes.

$PV\rightarrow PV$ (1^-):

$$D_{ac}^{\;\alpha}D_{bc}^{\;\beta}=r_3G_{\alpha\beta}^{\;c}F_{ab}^{\;c}+r_4D_{bc}^{\;\alpha}D_{ac}^{\;\beta}+r_5G_{\alpha\gamma}^{\;b}G_{\beta\gamma}^{\;a} \tag{3.4}$$

$PV\rightarrow PV$ (0^-):

$$G_{\alpha\gamma}^{\;a}G_{\beta\gamma}^{\;b}=s_3G_{\alpha\beta}^{\;c}F_{ab}^{\;c}+s_4D_{bc}^{\;\alpha}D_{ac}^{\;\beta}+s_5G_{\alpha\gamma}^{\;b}G_{\beta\gamma}^{\;a} \tag{3.5}$$

$PV\rightarrow VV$ (1^-):

$$D_{ad}^{\;\alpha}F_{bc}^{\;d}=r_6(D_{bd}^{\;\alpha}F_{ac}^{\;d}-D_{cd}^{\;\alpha}F_{ab}^{\;d})+r_7(G_{\alpha\beta}^{\;b}D_{ac}^{\;\beta}-G_{\alpha\beta}^{\;c}D_{ab}^{\;\beta}) \tag{3.6}$$

$PV \to VV$ (0^-):

$$G_{\alpha\beta}{}^a D_{bc}{}^\beta = s_6(D_{bd}{}^\alpha F_{ac}{}^d + D_{cd}{}^\alpha F_{ab}{}^d) + s_7(G_{\alpha\beta}{}^b D_{ac}{}^\beta + G_{\alpha\beta}{}^c D_{ab}{}^\beta) \quad (3.7)$$

$PP \to PP$ (1^-):

$$G_{\alpha\beta}{}^a G_{\gamma\delta}{}^a = r_8(G_{\alpha\gamma}{}^a G_{\beta\delta}{}^a - G_{\alpha\delta}{}^a G_{\beta\gamma}{}^a) \quad (3.8)$$

$PP \to PV$ (1^-):

$$G_{\alpha\beta}{}^b D_{ab}{}^\gamma = r_9(G_{\alpha\gamma}{}^b D_{ab}{}^\beta - G_{\beta\gamma}{}^b D_{ab}{}^\alpha) \quad (3.9)$$

$PP \to VV$ (1^-):

$$G_{\alpha\beta}{}^c F_{ab}{}^c = r_{10}(D_{ac}{}^\alpha D_{bc}{}^\beta - D_{bc}{}^\alpha D_{ac}{}^\beta) + r_{11}(G_{\alpha\gamma}{}^a G_{\beta\gamma}{}^b - G_{\alpha\gamma}{}^b G_{\beta\gamma}{}^a). \quad (3.10)$$

We now emphasize that although the constants r_ν and s_ν are dynamical quantities which can in principle be calculated, but are in practice here undetermined, many of them are, however, restricted to special values by the stringent requirement that the coupling coefficients have definite symmetry properties under the interchange of their indices. These properties, we repeat, are as follows: $F_{ab}{}^c$ totally antisymmetric, $G_{\alpha\beta}{}^a$ antisymmetric under $\alpha \leftrightarrow \beta$, and $D_{ab}{}^\alpha$ symmetric under $a \leftrightarrow b$. We now show that these properties facilitate a reduction of Eqs. (3.2)–(3.10) to simpler algebraic relations.

Consider, for example, Eq. (3.2). Interchange of the indices b and c in that equation leads to

$$F_{ac}{}^e F_{bd}{}^e = r_1(F_{ab}{}^e F_{cd}{}^e - F_{ad}{}^e F_{cb}{}^e) + r_2(D_{ab}{}^\alpha D_{cd}{}^\alpha - D_{ad}{}^\alpha D_{cb}{}^\alpha), \quad (3.11)$$

whereas the interchange of a and c yields

$$F_{cb}{}^e F_{ad}{}^e = r_1(F_{ca}{}^e F_{bd}{}^e - F_{cd}{}^e F_{ba}{}^e) + r_2(D_{ca}{}^\alpha D_{bd}{}^\alpha - D_{cd}{}^\alpha D_{ba}{}^\alpha). \quad (3.12)$$

Making use of the (anti)symmetry properties of F's and D's, we obtain from (3.2), (3.11), and (3.12)

$$(1+2r_1)(F_{ab}{}^e F_{cd}{}^e + F_{ac}{}^e F_{db}{}^e + F_{ad}{}^e F_{bc}{}^e) = 0. \quad (3.13)$$

Since r_1 depends upon the spins and masses of the particles and need not be fixed at $-\frac{1}{2}$, we have, in general,

$$F_{ab}{}^e F_{cd}{}^e + F_{ac}{}^e F_{db}{}^e + F_{ad}{}^e F_{bc}{}^e = 0. \quad (3.14)$$

From Eq. (3.3) and by similar considerations we get

$$D_{ab}{}^\alpha D_{cd}{}^\alpha + D_{ac}{}^\alpha D_{bd}{}^\alpha + D_{ad}{}^\alpha D_{bc}{}^\alpha = 0. \quad (3.15)$$

A different consideration arises in the case of (3.8). There the permutation of the indices and the antisymmetry of the G's demand that $r_8 = 1$. In cases such as this, we allow the constants to take on special values, as they are required by general arguments. The consequence of r_8 being unity is

$$G_{\alpha\beta}{}^a G_{\gamma\delta}{}^a + G_{\alpha\gamma}{}^a G_{\delta\beta}{}^a + G_{\alpha\delta}{}^a G_{\beta\gamma}{}^a = 0. \quad (3.16)$$

We state below the results of the reduction of the remaining equations in the set (3.2)–(3.10). From (3.4), (3.5), and (3.10), we obtain

$$G_{\alpha\gamma}{}^a G_{\gamma\beta}{}^b - G_{\alpha\gamma}{}^b G_{\gamma\beta}{}^a = t_1 F_{ab}{}^c G_{\alpha\beta}{}^c, \quad (3.17)$$

$$D_{ac}{}^\alpha D_{cb}{}^\beta - D_{ac}{}^\beta D_{cb}{}^\alpha = t_2 F_{ab}{}^c G_{\alpha\beta}{}^c, \quad (3.18)$$

where t_1 and t_2 are some constants depending on r_ν and s_ν. Since we have never defined the normalization of the coupling coefficients, we may redefine G and D so as to absorb t_1 and t_2 (if nonzero) in these equations, getting

$$G_{\alpha\gamma}{}^a G_{\gamma\beta}{}^b - G_{\alpha\gamma}{}^b G_{\gamma\beta}{}^a = F_{ab}{}^c G_{\alpha\beta}{}^c, \quad (3.17a)$$

$$D_{ac}{}^\alpha D_{cb}{}^\beta - D_{ac}{}^\beta D_{cb}{}^\alpha = F_{ab}{}^c G_{\alpha\beta}{}^c. \quad (3.18a)$$

Finally, from (3.6), (3.7), and (3.9), we obtain

$$D_{ad}{}^\alpha F_{oc}{}^d + D_{bd}{}^\alpha F_{ca}{}^d + D_{cd}{}^\alpha F_{ab}{}^d = 0, \quad (3.19)$$

$$G_{\alpha\beta}{}^a D_{bc}{}^\beta + G_{\alpha\beta}{}^b D_{ca}{}^\beta + G_{\alpha\beta}{}^c D_{ab}{}^\beta = 0, \quad (3.20)$$

$$G_{\alpha\beta}{}^b D_{ab}{}^\gamma + G_{\beta\gamma}{}^b D_{ab}{}^\alpha + G_{\gamma\alpha}{}^b D_{ab}{}^\beta = 0. \quad (3.21)$$

We call these equations, (3.14)–(3.21), the *bootstrap algebra*.

Since $F_{ab}{}^c$ is totally antisymmetric, (3.14) can be written in the form of Jacobi identity

$$F_{cd}{}^e F_{eb}{}^a + F_{db}{}^e F_{ec}{}^a + F_{bc}{}^e F_{ed}{}^a = 0, \quad (3.22)$$

or in the matrix form [where $(F_e)_b{}^a$ is regarded as the element at the ath row, bth column of the matrix F_e]

$$[F_c, F_d]_b{}^a = F_{cd}{}^e (F_e)_b{}^a. \quad (3.23)$$

It is clear then that the F's are the structure constants of a Lie algebra. It defines a Lie group up to a local isomorphism. The vector mesons being coupled by the F's must therefore be in the adjoint representation of the group. The number of vector mesons N is thus fixed, depending upon the group. If it is $SU(n)$, then $N = n^2 - 1$. Just which local Lie group it is, the bootstrap algebra cannot determine uniquely.

We should add that the rank of the internal symmetry group is not expected to be specified by bootstrap, no more than the dimension of space-time can be expected to be determined by any physical theory. The question here is, of course, whether the bootstrap algebra for a more complete system might select a particular group from the many of a given rank.

Another parenthetical remark to be made is that the problem of nonuniqueness of bootstrap[11] does not enter here. The possibility of ambiguity was eliminated from the very beginning when we assumed that the exchange of V and P produces V and P.

Equation (3.17a) indicates that the G's are also elements of the Lie algebra characterized by $F_{ab}{}^c$

$$[G^a, G^b]_{\alpha\beta} = F_{ab}{}^c G_{\alpha\beta}{}^c. \quad (3.24)$$

The matrix representation of G is of dimension M, since there are M pseudoscalar mesons. Thus the pseudoscalar mesons also belong to some representation of the same Lie group. The discussion on the dimension of the representation will be deferred until the next section.

The fact that the bootstrap algebra includes the Lie algebra and that the vector and pseudoscalar mesons both belong to some representation of the associated Lie group strongly suggests that the bootstrap dynamics may provide the physical basis for the existence of internal symmetry.

We close this section with some remarks on the nature of the Lie group characterized by $F_{ab}{}^{c}$. Our model assumes that the number N of vector mesons is finite, so that the Lie group must be compact. Also, we tacitly assumed that every meson interacts with some other mesons, for otherwise that particle would have no place in the bootstrap dynamics. (There is one exception which we shall come to presently.) Thus there is no value of a (apart from that exception), for which $F_{ab}{}^{c}=0$ for all other values of b and c. This implies that the Lie group must be semisimple. The exception is, of course, associated with the identity element of the group, which is the only permissible invariant subgroup of a semisimple Lie group. This allows for the existence of a singlet vector meson, which we shall consider in the next section. We see, therefore, that our bootstrap model can only induce Lie groups that are compact and semisimple.

IV. PHYSICAL CONSEQUENCES

In this section we consider the consequences of the bootstrap algebra when the internal symmetry is $SU(2)$ or $SU(3)$.

The vector mesons, being in the adjoint representation, form a triplet in $SU(2)$, which we may identify with the ρ meson. The coupling coefficient $F_{ab}{}^{c}$ is proportional to the totally antisymmetric tensor of rank 3, ϵ_{abc}. Although the dimension of the representation to which the pseudoscalar mesons belong is as yet unspecified, it is bounded from above on account of (3.16). This can be seen if we transform the Hermitian fields to non-Hermitian ones so that (3.16) may be written in the basis of charge states. Then the resultant algebraic equation, when combined with the requirement of charge conservation, restricts the pseudoscalar mesons to carry charges ≤ 2. Thus the dimension of the representation for P in $SU(2)$ is ≤ 5.

The same considerations applied to $SU(3)$ place the vector mesons in the octet representation and the pseudoscalar mesons in a representation no higher than the 27-dimensional one.

Concerning the coefficients $D_{ab}{}^{\alpha}$, it is easy to see from (3.15) that all the D's must vanish. This follows mainly from the fact that $D_{ab}{}^{\alpha}=D_{ba}{}^{\alpha}$, so that if the lower indices on the left-hand side of (3.15) are all taken to be the same, say, then $D_{aa}{}^{\alpha}=0$. Other cases can similarly be established. The vanishing of D requires that t_2 in (3.18) be zero. Thus the condition for rewriting (3.18) in the form (3.18a) is not satisfied. All the remaining relations of the bootstrap algebra are consistent with $D=0$.

For V in the adjoint representation the implication of $D=0$ in $SU(2)$ is in agreement with the requirement of G-parity conservation. Thus, for example, the coupling between $\rho\rho\pi$ is forbidden. In $SU(3)$, this result gives the theoretical foundation for some aspects of the empirical A-parity conservation of Bronzan and Low.[13] The problem of introducing a singlet vector meson into the bootstrap will be discussed shortly. We note here that under the equal-mass assumption the ω-ϕ mixing is arbitrary. The result $D=0$ forbids only the coupling of P to the two V's in the octet representation.

We now turn to the question of whether our bootstrap allows for the existence of a singlet vector meson V_1. Since the coupling of V_1VV is symmetric under the interchange of the two V's, it does not affect the antisymmetric coefficient $F_{ab}{}^{c}$. Furthermore, V_1 does not couple to PP, though it does to PV. Thus it can readily be established that the introduction of V_1 in the intermediate states of both the direct and crossed channels of the bootstrap does not alter the Lie algebra (3.23) and (3.24). Hence the symmetry group characterized by $F_{ab}{}^{c}$ admits the introduction of the singlet vector meson. The requirement that V_1 is, in fact, bootstrapped in the scattering system of V and P gives rise to new algebraic relations involving the V_1VP coupling. Since V_1 is a singlet while V is in the adjoint representation, it follows that P must also be in the adjoint representation. We see, therefore, that in $SU(2)$ the π meson is an isotopic triplet if there exists a singlet vector meson ω beside the ρ meson. In $SU(3)$ there is an octet of pseudoscalar mesons if the vector mesons form a nonet. All these results are in accord with the physical situation.

In conclusion, we remark that the bootstrap algebra for mesons not only leads to a Lie algebra describing the symmetry of the system, but also restricts the representation to which the mesons can belong. It has consequences that agree with the conservations of G parity and A parity. Although the results do not justify the dynamical assumptions made, they suggest the usefulness of studying further, as a guide to S-matrix calculations, the relationsip between certain dynamical approximations and the resultant symmetry, exact or broken. Whether or not it is possible to show eventually that (almost) exact dynamics leads to the broken symmetry as observed, it is clear that the bootstrap philosophy cannot be entirely empty in its claim to provide the dynamical origin of internal symmetry of strongly interacting particles.

ACKNOWLEDGMENTS

We are grateful to Professor J. R. Oppenheimer for his encouragement and interest in this problem and for his kind hospitality at the Institute for Advanced Study. We wish to thank Dr. J. M. Cornwall and Dr. Y. P. Yao for helpful discussions.