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Kinetic Supermultiplets of $\tilde{U}(12)$

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A further extension of the multiplet structure of broken $\tilde{U}(12)$ is proposed. The "kinetic supermultiplets" are represented by reducible tensors and group together separate nondegenerate multiplets. The example of the lowest kinetic boson supermultiplet is treated in detail. Such a supermultiplet has already been shown by Borchi and Gatto to provide for a classification of the higher boson resonances. The mass relations are derived including first-order SU_3 breaking. Comparison with the data shows a remarkable accuracy. The predicted equidistance relation $\frac{1}{2}[m^2(A_2)+m^2(A_1)]=m^2(B)$ between the squared masses of the resonances A_1, A_2, and B is satisfied to $\lesssim 1.5\%$. A $T=1$, $J^{PC}=0^{++}$ meson at (970 ± 50) MeV and a $T=0$, $J^{PC}=1^{+-}$ meson at (1215 ± 15) MeV are directly predicted. If $K^*(1430)$, $\kappa(725)$, and $f^0(1253)$ are included in the supermultiplet, as suggested by their quantum numbers, and the assumption is made that f^0 has a maximal mixing with another particle of the same T and J^{PC}, one can predict in addition the following resonant masses: (1560 ± 50) MeV with $T=0$, $J^{PC}=2^{++}$; (1270 ± 30) MeV with $T=0$, $J^{PC}=1^{+-}$; (1180 ± 190) MeV and (990 ± 200) MeV both with $T=0$, $J^{PC}=1^{++}$. The supermultiplet also includes two mixed K resonances K' and K'' with $J^P=1^+$. One of them could be $K^*(1175)$ or C (1215), the other resonance being then predicted at (1100 ± 40) MeV or (1050 ± 40) MeV, respectively.

1. INTRODUCTION

UNSUCCESSFUL attempts to employ symmetry theories beyond their reasonable limits of validity have provoked a widespread skepticism about the usefulness of the whole approach. In spite of the unfavorable current views, we shall propose here a further extension of the $\tilde{U}(12)$ multiplet structure[1] by discussing the properties of possible "kinetic supermultiplets" of broken $\tilde{U}(12)$.

We shall consider in particular the lowest boson "kinetic supermultiplet" in view of classifying the higher boson resonances.

The present work is an extension of a previous proposal by Borchi and Gatto.[2] The "kinetic supermultiplet" is described by reducible tensor and groups together separate nondegenerate multiplets.

The quantum number assignment to the component multiplets reflects the properties of a model of bound quarks.[3] The mass formula, derived up to first-order SU_3 breaking, accounts for the possible mixing effects inside the kinetic supermultiplet.

The application of the results to a classification of the higher boson resonances seems to be very promising. For the best established resonances, a mass relation, connecting particles of different component multiplets, is accurately verified, within the reported experimental errors ($\lesssim 1.5\%$). The relation is: $\frac{1}{2}(A_2+A_1)=B$, where A_2, A_1, and B are the squared masses of $A_2(1320)$, $A_1(1090)$, and $B(1215)$. In addition, the various mass relations derived allow for the prediction of resonant masses and for a consistent quantum number assignment. In particular, we predict a $T=1$ scalar meson with negative G-parity with a mass around 970 MeV and two strongly mixed, axial-vector resonant states with $T=0$ and negative G-parity, one of them with a mass around 1215 MeV. More powerful predictions could be derived by including some less established experimental results, but at this time they can only be of rather tentative character.

2. KINETIC SUPERMULTIPLETS

We describe a "kinetic supermultiplet" in terms of a reducible tensor obtained by multiplying a basic $\tilde{U}(12)$ tensor of an irreducible representation with a "kinetic" tensor belonging to the **143**.

Instead of dealing here with the most general case we shall discuss the simplest case of the lowest boson "kinetic supermultiplet." Let us consider the reducible $\tilde{U}(12)$ tensor

$$M_{AB}{}^{CD}=\sum_{\tau}(M_\tau)_A{}^C(\gamma^\tau\otimes 1)_B{}^D =\sum_{iR\tau}\Phi_{R,\tau}{}^i(\gamma^R)_\alpha{}^\gamma(\gamma^\tau)_\beta{}^\delta(T_i)_a{}^c\delta_b{}^d \quad (1)$$

where $(M_\tau)_A{}^C$ belongs for each τ to the **143** representa-

[1] A. Salam, R. Delbourgo, and J. Strathdee, Proc. Roy. Soc. (London) A284, 146 (1965); B. Sakita and K. C. Wali, Phys. Rev. Letters 14, 404 (1965); M. A. B. Bég and A. Pais, Phys. Rev. Letters 14, 267 (1965).
[2] E. Borchi and R. Gatto, Phys. Letters 14, 352 (1965).
[3] M. Gell-Mann, Phys. Letters 8, 214 (1964).

tion of $\tilde{U}(12)$ broken by the Bargmann-Wigner equations of motion. The amplitudes $\Phi_{R,\tau}{}^{i}$ satisfy

$$p^{\tau}\Phi_{R,\tau}{}^{i}=0\,, \tag{2}$$

p^{τ} being the particle four-momentum. The notations are the same as Delbourgo, Salam, and Strathdee[1]: γ^{R} ($R=1,\cdots,16$) are the Dirac covariants; γ^{τ} ($\tau=1,\cdots,4$) are in particular the Dirac matrices; T_i ($i=0,\cdots,8$) are the unitary spin matrices; A, B, C, D are $\tilde{U}(12)$ indices decomposed in $\tilde{U}(4)\otimes U(3)$ according to $A=(\alpha,a)$ etc.

We note that the "kinetic" tensor γ^{τ} plays here the role of an internal orbital angular momentum and therefore is required to carry on one unit of spin in the rest frame of the particles, which is assured by the transversality condition Eq. (2). Furthermore considering only states at rest, we could phrase our model entirely in terms of the nonchiral $U(6)\otimes U(6)$ subgroup of $\tilde{U}(12)$, and the use of γ^{τ} simply implies that such orbital angular momentum belongs to a $(6^*,6)\otimes(6,6^*)$ representation of this group. This is also the behavior of the kinetic term in a simple quark model. The imposition of the equations of motion[1] produces two sets of equations. The first set is

$$p_{\mu}\Phi_{5,\delta}=im\Phi_{\mu5,\delta}\,, \tag{3}$$

$$p^{\mu}\Phi_{5\mu,\delta}=-im\Phi_{5,\delta}\,, \tag{3$'$}$$

with the supplementary conditions

$$p^{\delta}\Phi_{5,\delta}=0\,,\quad p^{\delta}\Phi_{\mu5,\delta}=0\,. \tag{3$''$}$$

From Eqs. (3), (3′), and (3″) we see that $(\Phi_{5,\delta},\Phi_{\mu5,\delta})$ together describe a spin-one field. The second set of equations is

$$p_{\mu}\Phi_{\nu,\delta}-p_{\nu}\Phi_{\mu,\delta}=im\Phi_{\mu\nu,\delta}\,, \tag{4}$$

$$p^{\nu}\Phi_{\nu\mu,\delta}=-im\Phi_{\mu,\delta}\,, \tag{4$'$}$$

with the conditions

$$p^{\mu}\Phi_{\mu,\delta}=0\,,\quad p^{\delta}\Phi_{\mu,\delta}=0\,. \tag{4$''$}$$

We decompose the tensor $\Phi_{\mu,\delta}$ according to

$$\Phi_{\mu,\delta}=S_{\mu\delta}+A_{\mu\delta}+T_{\mu\delta}\,, \tag{5}$$

where (we use the notation

$$\Phi_{\alpha,}{}^{\alpha}=\Phi^{\alpha}{}_{,\alpha}=g^{\alpha\beta}\Phi_{\alpha,\beta})$$

$$S_{\mu\delta}=\tfrac{1}{2}(\Phi_{\mu,\delta}+\Phi_{\delta,\mu})-\tfrac{1}{3}(g_{\mu\delta}-p_{\mu}p_{\delta}/m^2)\Phi_{\alpha,}{}^{\alpha}\,, \tag{6}$$

$$A_{\mu\delta}=\tfrac{1}{2}(\Phi_{\mu,\delta}-\Phi_{\delta,\mu})\,, \tag{6$'$}$$

$$T_{\mu\delta}=\tfrac{1}{3}(g_{\mu\delta}-p_{\mu}p_{\delta}/m^2)\Phi_{\alpha,}{}^{\alpha}\,. \tag{6$''$}$$

The conditions $p^{\mu}S_{\mu\delta}=0$, $p^{\mu}A_{\mu\delta}=0$, and $p^{\mu}T_{\mu\delta}=0$ are each separately satisfied. We define

$$A_{\mu\nu\delta}=\tfrac{1}{2}(\Phi_{\mu\nu,\delta}+\Phi_{\nu\delta,\mu}+\Phi_{\delta\mu,\nu})+i/m(p_{\delta}A_{\mu\nu})\,, \tag{7}$$

and obtain, from Eqs. (4), (4′), and (4″),

$$p_{\mu}A_{\nu\delta}-p_{\nu}A_{\mu\delta}=imA_{\mu\nu\delta}\,, \tag{8}$$

$$p^{\mu}A_{\mu\nu\delta}=-imA_{\nu\delta}\,. \tag{8$'$}$$

Thus $(A_{\mu\nu},A_{\mu\nu\delta})$ together describe a spin-one field. Similarly, with the definition

$$S_{\mu\nu\delta}=\Phi_{\mu\nu,\delta}-A_{\mu\nu\delta}-\tfrac{1}{3}(g_{\nu\delta}\Phi_{\mu\alpha,}{}^{\alpha}-g_{\mu\delta}\phi_{\nu\alpha,}{}^{\alpha}) \tag{9}$$

we obtain

$$p_{\mu}S_{\nu\delta}-p_{\nu}S_{\mu\delta}=imS_{\mu\nu\delta}\,, \tag{10}$$

$$p^{\mu}S_{\mu\nu\delta}=-imS_{\nu\delta}\,. \tag{10$'$}$$

Thus $(S_{\mu\nu},S_{\mu\nu\delta})$ together describe a spin-2 field. Finally defining

$$T_{\mu\nu\delta}=\tfrac{1}{3}(g_{\nu\delta}\Phi_{\mu\alpha,}{}^{\alpha}-g_{\mu\delta}\Phi_{\nu\alpha,}{}^{\alpha}) \tag{11}$$

we obtain

$$p_{\mu}T_{\nu\delta}-p_{\nu}T_{\mu\delta}=imT_{\mu\nu\delta}\,, \tag{12}$$

$$p^{\mu}T_{\mu\nu\delta}=-imT_{\nu\delta}\,. \tag{12$'$}$$

The tensors $(T_{\mu\nu},T_{\mu\nu\delta})$ together describe a spin-zero field. The reducible tensor $M_{AB}{}^{CD}$, satisfying the condition Eq. (2), has thus been decomposed into one spin-zero, two spin-one, and one spin-two components. The normalized wave functions are given in Table I. The

TABLE I. Normalized wave functions. Each multiplet is specified by the notation J^{PC}, where J is the spin, P is the parity, and C is the charge conjugation. All amplitudes must satisfy the Klein-Gordon equation.

2^{++}	$S_{\mu\nu}{}^{i}[(1+p/m)\gamma^{\mu}]_{\alpha}{}^{\gamma}(\gamma^{\nu})_{\beta}{}^{\delta}(T_i)_a{}^c\delta_b{}^d$ $S_{\mu\nu}{}^{i}=S_{\nu\mu}{}^{i},\quad p^{\mu}S_{\mu\nu}{}^{i}=0,\quad g^{\mu\nu}S_{\mu\nu}{}^{i}=0$
1^{++}	$A_{\mu\nu}{}^{i}[(1+p/m)\gamma^{\mu}]_{\alpha}{}^{\gamma}(\gamma^{\nu})_{\beta}{}^{\delta}(T_i)_a{}^c\delta_b{}^d$ $A_{\mu\nu}{}^{i}=-A_{\nu\mu}{}^{i},\quad p^{\mu}A_{\mu\nu}{}^{i}=0$
0^{++}	$\sqrt{\tfrac{2}{3}}\Phi^{i}\{[(1+p/m)\gamma^{\mu}]_{\alpha}{}^{\gamma}(\gamma_{\mu})_{\beta}{}^{\delta}$ $-[(1+p/m)]_{\alpha}{}^{\gamma}(p/m)_{\beta}{}^{\delta}\}(T_i)_a{}^c\delta_b{}^d$
1^{+-}	$\sqrt{2}\Phi_{5\tau}{}^{i}[(1+p/m)\gamma_5]_{\alpha}{}^{\gamma}(\gamma^{\tau})_{\beta}{}^{\delta}(T_i)_a{}^c\delta_b{}^d$

notation J^{PC} denotes a multiplet with spin J, parity P, and charge conjugation C (C is the charge conjugation number of the isotopic singlets). All multiplets have positive parity, as can immediately be seen by directly applying the parity operation to the tensors. Application of the charge conjugation operation[4] gives

$$M_{AB}{}^{CD}\to-\eta\sum_{iR\tau}\Phi_{R,\tau}{}^{i}\epsilon_R(\gamma^R)_{\alpha}{}^{\gamma}(\gamma^{\tau})_{\beta}{}^{\delta}(T_i{}^T)_a{}^c\delta_b{}^d\,, \tag{13}$$

where η is a phase-factor and $\epsilon_R=-1$ for V, T, $\epsilon_R=+1$ for S, P, A. With $\eta=+1$, the multiplets described by $(S_{\mu\nu},S_{\mu\nu\delta})$, $(A_{\mu\nu},A_{\mu\nu\delta})$, and $(T_{\mu\nu},T_{\mu\nu\delta})$ are even under charge conjugation, whereas the multiplet described by $(\Phi_{5,\delta},\Phi_{5\mu,\delta})$ is odd. The quantum number assignments are consistent with a model of a bound quark-antiquark system in relative p wave.[2] The resulting multiplets would in fact have $\mathbf{J}=\mathbf{l}+\mathbf{s}$, $P=(-)^l$, $C=(-1)^{l+s}$ (in our case $l=1$). The bound quark-antiquark model thus provides for a simplest interpretation of our formalism.

[4] J. M. Charap and P. T. Matthews, Phys. Letters **13**, 346 (1964).

3. MASS FORMULAS

We shall now derive the mass formula. We first ignore the SU_3 symmetry breaking, that is we assume the (mass)^2 operator to be $\tilde{U}(12)$-invariant. We can form a "central" mass term

$$[0]=M^{\dagger}{}_{BD}{}^{AC}M_{AC}{}^{BD},$$

(which defines the normalization of the wave functions) and the mass terms

$$\begin{aligned}
&[1]=M^{\dagger}{}_{FA}{}^{AB}M_{BE}{}^{EF}, \quad [2]=M^{\dagger}{}_{BD}{}^{AB}M_{FA}{}^{DF},\\
&[3]=M^{\dagger}{}_{EA}{}^{AB}M_{FB}{}^{EF}, \quad [4]=M^{\dagger}{}_{BF}{}^{AB}M_{AE}{}^{EF},\\
&[5]=M^{\dagger}{}_{FE}{}^{AB}M_{AB}{}^{EF}, \quad [6]=M^{\dagger}{}_{EF}{}^{AB}M_{BA}{}^{EF},\\
&[7]=M^{\dagger}{}_{EF}{}^{AB}M_{BA}{}^{FE}, \quad [8]=M^{\dagger}{}_{BA}{}^{AB}M_{FE}{}^{EF}.
\end{aligned}$$

Note that, since we are dealing with a reducible tensor, saturation of an upper with a lower index in the same tensor does not always give a vanishing expression. The couplings [1] and [2] do not contribute, [3] and [4] by Hermiticity and C invariance must be summed with equal coefficients, and similarly for [5] and [6]. The "central" term [0] gives a contribution proportional to

$$\tfrac{1}{2}(S^{\dagger}{}_{i}{}^{\mu\nu}S_{\mu\nu}{}^{i})+\tfrac{1}{2}(A^{\dagger}{}_{i}{}^{\mu\nu}A_{\mu\nu}{}^{i})+(\Phi_i{}^{\dagger}\Phi^{i})+(\Phi^{\dagger}{}_{i}{}^{5\mu}\Phi_{5\mu}{}^{i})\,; \quad (14)$$

the sum [3]+[4] gives

$$\tfrac{1}{2}(A^{\dagger}{}_{i}{}^{\mu\nu}A_{\mu\nu}{}^{i})+\tfrac{3}{2}(\Phi_i{}^{\dagger}\Phi^{i})+\tfrac{1}{2}(\Phi^{\dagger}{}_{i}{}^{5\mu}\Phi_{5\mu}{}^{i})\,; \quad (15)$$

the sum [5]+[6] gives

$$\tfrac{1}{2}(S^{\dagger}{}_{i}{}^{\mu\nu}S_{\mu\nu}{}^{i})-\tfrac{1}{2}(\Phi_i{}^{\dagger}\Phi^{i})+\tfrac{1}{2}(\Phi^{\dagger}{}_{i}{}^{5\mu}\Phi_{5\mu}{}^{i})\,; \quad (16)$$

the coupling [7] gives

$$\tfrac{1}{2}(S^{\dagger}{}_{0}{}^{\mu\nu}S_{\mu\nu}{}^{0})-\tfrac{1}{2}(A^{\dagger}{}_{0}{}^{\mu\nu}A_{\mu\nu}{}^{0})+\tfrac{1}{2}(\Phi_0{}^{\dagger}\Phi^{0})\,; \quad (17)$$

and, finally, [8] gives

$$(\Phi_0{}^{\dagger}\Phi^{0})\,. \quad (18)$$

We now include first-order breaking of SU_3 assuming that the complete (mass)^2 operators may be written as $m^2=(a\mathbf{1}+b\lambda_8)\otimes\mathbf{1}$ in its $SU(3)\otimes\tilde{U}(4)$ decomposition. It is easy to see that, in correspondence to the "central" term [0], one can form two SU_3-breaking couplings

$$[\alpha_0]=M^{\dagger}{}_{CD}{}^{AB}(\lambda_8)_A{}^{E}M_{EB}{}^{CD}, \qquad [\beta_0]=M^{\dagger}{}_{CD}{}^{AB}(\lambda_8)_E{}^{C}M_{AB}{}^{ED}.$$

Furthermore, corresponding to each of the couplings [1], [2], $\cdots$[8], one can form four SU_3-breaking couplings:

$$\begin{aligned}
&[\alpha_1]=M^{\dagger}{}_{FA}{}^{AB}(\lambda_8)_C{}^{F}M_{BE}{}^{EC}, &&[\beta_1]=M^{\dagger}{}_{FA}{}^{AB}(\lambda_8)_B{}^{C}M_{CE}{}^{EF},\\
&[\gamma_1]=M^{\dagger}{}_{FC}{}^{AB}(\lambda_8)_A{}^{C}M_{BE}{}^{EF}, &&[\delta_1]=M^{\dagger}{}_{FA}{}^{AB}(\lambda_8)_C{}^{E}M_{BE}{}^{CF},\\
&[\alpha_2]=M^{\dagger}{}_{BD}{}^{AB}(\lambda_8)_C{}^{D}M_{EA}{}^{CE}, &&[\beta_2]=M^{\dagger}{}_{BD}{}^{AB}(\lambda_8)_A{}^{C}M_{EC}{}^{DE},\\
&[\gamma_2]=M^{\dagger}{}_{CD}{}^{AB}(\lambda_8)_B{}^{C}M_{EA}{}^{DE}, &&[\delta_2]=M^{\dagger}{}_{BD}{}^{AB}(\lambda_8)_E{}^{C}M_{CA}{}^{DE},\\
&[\alpha_3]=M^{\dagger}{}_{EC}{}^{AB}(\lambda_8)_A{}^{C}M_{FB}{}^{EF}, &&[\beta_3]=M^{\dagger}{}_{EA}{}^{AC}(\lambda_8)_C{}^{B}M_{FB}{}^{EF},\\
&[\gamma_3]=M^{\dagger}{}_{CA}{}^{AB}(\lambda_8)_E{}^{C}M_{FB}{}^{EF}, &&[\delta_3]=M^{\dagger}{}_{EA}{}^{AB}(\lambda_8)_C{}^{F}M_{FB}{}^{EC},\\
&[\alpha_4]=M^{\dagger}{}_{BF}{}^{AB}(\lambda_8)_A{}^{C}M_{CE}{}^{EF}, &&[\beta_4]=M^{\dagger}{}_{CF}{}^{AB}(\lambda_8)_B{}^{C}M_{AE}{}^{EF},\\
&[\gamma_4]=M^{\dagger}{}_{BF}{}^{AB}(\lambda_8)_E{}^{C}M_{AC}{}^{EF}, &&[\delta_4]=M^{\dagger}{}_{BF}{}^{AB}(\lambda_8)_C{}^{F}M_{AE}{}^{EC},\\
&[\alpha_5]=M^{\dagger}{}_{FE}{}^{AB}(\lambda_8)_A{}^{C}M_{CB}{}^{EF}, &&[\beta_5]=M^{\dagger}{}_{FE}{}^{AB}(\lambda_8)_B{}^{C}M_{AC}{}^{EF},\\
&[\gamma_5]=M^{\dagger}{}_{FE}{}^{AB}(\lambda_8)_C{}^{E}M_{AB}{}^{CF}, &&[\delta_5]=M^{\dagger}{}_{FE}{}^{AB}(\lambda_8)_C{}^{F}M_{AB}{}^{EC},\\
&[\alpha_6]=M^{\dagger}{}_{EF}{}^{AB}(\lambda_8)_A{}^{C}M_{BC}{}^{EF}, &&[\beta_6]=M^{\dagger}{}_{EF}{}^{AB}(\lambda_8)_B{}^{C}M_{CA}{}^{EF},\\
&[\gamma_6]=M^{\dagger}{}_{EF}{}^{AB}(\lambda_8)_C{}^{E}M_{BA}{}^{CF}, &&[\delta_6]=M^{\dagger}{}_{EF}{}^{AB}(\lambda_8)_C{}^{F}M_{BA}{}^{EC},\\
&[\alpha_7]=M^{\dagger}{}_{FE}{}^{AB}(\lambda_8)_A{}^{C}M_{BC}{}^{EF}, &&[\beta_7]=M^{\dagger}{}_{FE}{}^{AB}(\lambda_8)_B{}^{C}M_{CA}{}^{EF},\\
&[\gamma_7]=M^{\dagger}{}_{FE}{}^{AB}(\lambda_8)_C{}^{F}M_{BA}{}^{EC}, &&[\delta_7]=M^{\dagger}{}_{FE}{}^{AB}(\lambda_8)_C{}^{E}M_{BA}{}^{CF},\\
&[\alpha_8]=M^{\dagger}{}_{BC}{}^{AB}(\lambda_8)_A{}^{C}M_{FE}{}^{EF}, &&[\beta_8]=M^{\dagger}{}_{CA}{}^{AB}(\lambda_8)_B{}^{C}M_{FE}{}^{EF},\\
&[\gamma_8]=M^{\dagger}{}_{BA}{}^{AB}(\lambda_8)_E{}^{C}M_{FC}{}^{EF}, &&[\delta_8]=M^{\dagger}{}_{BA}{}^{AB}(\lambda_8)_F{}^{C}M_{CE}{}^{EF}.
\end{aligned}$$

The explicit computation is simplified by noting that (i) the couplings $[\alpha_1]\cdots[\delta_1]$ and $[\alpha_2]\cdots[\delta_2]$ cannot contribute and (ii) the following pairs are related to each other by Hermiticity and charge conjugation: (α_0,β_0), (α_3,β_4), (β_3,δ_4), (γ_3,α_4), (δ_3,γ_4), (α_5,γ_6), (β_5,δ_6), (γ_5,β_6), (δ_5,α_6), (α_7,β_7), (β_7,δ_7), (α_8,β_8), (γ_8,δ_8). The terms $[\alpha_0]$, $[\beta_0]$ simply give a contribution

$$\alpha_0\,\mathrm{Tr}(\lambda_8\{\lambda_i,\lambda_j\})\,. \quad (19)$$

The terms $[\alpha_3]\cdots[\delta_3]$ and $[\alpha_4]\cdots[\delta_4]$ are found to give the following contribution to the mass matrix:

$$\begin{aligned}
&(\alpha_3+\beta_3+\gamma_3+\delta_3)[\tfrac{1}{2}(A^{\dagger}{}_{i}{}^{\mu\nu}A_{\mu\nu}{}^{j})+\tfrac{3}{2}(\Phi_i{}^{\dagger}\Phi^{j})\\
&\quad+\tfrac{1}{2}(\Phi^{\dagger}{}_{i}{}^{5\mu}\Phi_{5\mu}{}^{j})]\,\mathrm{Tr}[\lambda_8\{\lambda_i,\lambda_j\}]\\
&\quad+\tfrac{1}{4}\sqrt{2}(\alpha_3+\beta_3-\gamma_3+\delta_3)[\epsilon_{\rho\lambda\beta\delta}(p^{\delta}/m)g^{\beta\tau}A_j{}^{\rho\lambda}\Phi^{\dagger}{}_{5\tau}{}^{i}\\
&\qquad+\epsilon^{\mu\delta\alpha\lambda}g_{\tau\lambda}(p_{\alpha}/m)\Phi_j{}^{5\tau}A^{\dagger}{}_{\mu\nu}{}^{i}]\,\mathrm{Tr}(\lambda_8[\lambda_i,\lambda_j])\,, \quad (20)
\end{aligned}$$

where we have indicated by α_3 the coefficient of the coupling $[\alpha_3]$, etc. The terms $[\alpha_5]\cdots[\delta_5]$ and

$[\alpha_6]\cdots[\delta_6]$ similarly give a contribution

$$(\alpha_5+\beta_5+\gamma_5+\delta_5)[\tfrac{1}{2}(S^{\dagger}{}_i{}^{\mu\nu}S_{\mu\nu}{}^j)-\tfrac{1}{2}(\Phi_i{}^{\dagger}\Phi^j)+\tfrac{1}{2}(\Phi^{\dagger}{}_i{}^{5\mu}\Phi_{5\mu}{}^j)]\,\mathrm{Tr}(\lambda_8\{\lambda_i,\lambda_j\})-\tfrac{1}{4}\sqrt{2}(\alpha_5-\beta_5-\gamma_5-\delta_5)[\epsilon_{\rho\alpha\tau\lambda}g^{\delta\tau}(p^\alpha/m)\Phi^{\dagger}{}_{5\delta}{}^iA_j{}^{\lambda\rho}-\epsilon^{\mu\delta\rho\alpha}(p_\alpha/m)g_{\lambda\rho}A^{\dagger}{}_{\mu\delta}{}^i\Phi_j{}^{5\lambda}]\,\mathrm{Tr}(\lambda_8[\lambda_i,\lambda_j])\,. \tag{21}$$

The terms $[\alpha_7]\cdots[\delta_7]$ give

$$\alpha_7[\tfrac{1}{2}(S^{\dagger}{}_j{}^{\mu\nu}S_{\mu\nu}{}^i)-\tfrac{1}{2}(A^{\dagger}{}_j{}^{\mu\nu}A_{\mu\nu}{}^i)+(\Phi_j{}^{\dagger}\Phi^i)]\times[\mathrm{Tr}(\lambda_j\lambda_8)\,\mathrm{Tr}(\lambda_i)+\mathrm{Tr}(\lambda_i\lambda_8)\,\mathrm{Tr}(\lambda_j)]\,. \tag{22}$$

Finally the terms $[\alpha_8]\cdots[\delta_8]$ give

$$\alpha_8(\Phi_i{}^{\dagger}\Phi^j)[\mathrm{Tr}(\lambda_i\lambda_8)\,\mathrm{Tr}(\lambda_j)+\mathrm{Tr}(\lambda_j\lambda_8)\,\mathrm{Tr}(\lambda_i)]\,. \tag{23}$$

From the above expressions (14)–(23) we can derive the complete form of the mass matrix. The following parameterization turns out to be the very convenient:

$$\langle 8,2|\mathfrak{M}^2|8,2\rangle=M_0+3\alpha+(N+3\beta)\lambda\,, \tag{24a}$$
$$\langle 8,1^-|\mathfrak{M}^2|8,1^-\rangle=M_0+2\alpha+(N+2\beta)\lambda\,, \tag{24b}$$
$$\langle 8,1^+|\mathfrak{M}^2|8,1^+\rangle=M_0+\alpha+(N+\beta)\lambda\,, \tag{24c}$$
$$\langle 8,0|\mathfrak{M}^2|8,0\rangle=M_0+N\lambda\,, \tag{24d}$$
$$\langle 8,2|\mathfrak{M}^2|1,2\rangle=\sqrt{2}(N+3\beta+3c)\,, \tag{24e}$$
$$\langle 8,1^-|\mathfrak{M}^2|1,1^-\rangle=\sqrt{2}(N+2\beta)\,, \tag{24f}$$
$$\langle 8,1^+|\mathfrak{M}^2|1,1^+\rangle=\sqrt{2}(N+\beta-3c)\,, \tag{24g}$$
$$\langle 8,0|\mathfrak{M}^2|1,0\rangle=\sqrt{2}(N+3d)\,, \tag{24h}$$
$$\langle 1,2|\mathfrak{M}^2|1,2\rangle=M_0+3\alpha+D\,, \tag{24i}$$
$$\langle 1,1^-|\mathfrak{M}^2|1,1^-\rangle=M_0+2\alpha\,, \tag{24j}$$
$$\langle 1,1^+|\mathfrak{M}^2|1,1^+\rangle=M_0+\alpha-D\,, \tag{24k}$$
$$\langle 1,0|\mathfrak{M}^2|1,0\rangle=M_0+D+E\,, \tag{24 }$$
$$\langle 8,1^-|\mathfrak{M}^2|8,1^+\rangle=\delta Y\,. \tag{24m}$$

In the above equations $\lambda=[T(T+1)-\frac{1}{4}Y^2-1]$; M_0, α, N, β, c, d, D, E, and δ are unknown parameters and the matrix elements of the (mass)^2 operator are between octet (8) or singlet (1) states of the multiplets 2^{++}, 1^{++}, 0^{++}, 1^{+-}, briefly denoted as 2, 1^+, 0, and 1^-. For each multiplet 2, 1^+, 0, and 1^- there will be possible mixing between the isospin singlet of the SU_3 octet and the SU_3 singlet. In addition there will be a possible mixing among the $T=\frac{1}{2}$ states of 1^+ and 1^-. The multiplets 1^+ and 1^- have opposite behavior under charge conjugation. However, for the $T=\frac{1}{2}$ states (K resonances) the opposite behavior under C does not prevent possible mixing effects. We also note the following general relations satisfied by the matrix elements of the (mass)^2 operator:

$$\langle\lambda,1^-|\mathfrak{M}^2|\lambda',1^-\rangle=\tfrac{1}{2}[\langle\lambda,2|\mathfrak{M}^2|\lambda',2\rangle+\langle\lambda,1^+|\mathfrak{M}^2|\lambda',1^+\rangle]\,, \tag{25}$$

where λ, $\lambda'=1$ or 8 and

$$\tfrac{1}{3}[2\langle 8,0|\mathfrak{M}^2|8,0\rangle+\langle 8,2|\mathfrak{M}^2|8,2\rangle]=\langle 8,1^+|\mathfrak{M}^2|8,1^+\rangle\,. \tag{26}$$

Before exploiting the detailed consequences of the above equations we introduce a suitable nomenclature. We call $\eta_0(J^C)$, $\eta_8(J^C)$, $\pi(J^C)$, $K(J^C)$ the components of the singlet + octet with spin J and charge conjugation C, before any mixing. The mixing between $\eta_0(J^C)$ and $\eta_8(J^C)$ leads for any J^C to the physical particles

$$X(J^C)=(\cos\theta)\eta_8(J^C)-(\sin\theta)\eta_0(J^C)\,, \tag{27}$$
$$\eta(J^C)=(\sin\theta)\eta_8(J^C)+(\cos\theta)\eta_0(J^C)\,. \tag{27'}$$

The mixing between $K(1^+)$ and $K(1^-)$ leads to the physical particles

$$K'=(\cos\theta')K(1^+)+(\sin\theta')K(1^-)\,, \tag{28}$$
$$K''=(-\sin\theta')K(1^+)+(\cos\theta')K(1^-)\,. \tag{28'}$$

We note that the states $\pi(2)$, $\pi(1^+)$, $\pi(1^-)$, $\pi(0)$, $K(2)$, $K(0)$ do not undergo any mixing and they can thus be identified with the physical particles. The following relations must however be verified, as a consequence of Eqs. (24):

$$\tfrac{1}{2}[\pi(2)+\pi(1^+)]=\pi(1^-)\,, \tag{29}$$
$$\pi(1^+)+\pi(1^-)=\pi(0)+\pi(2)\,. \tag{30}$$

In Eqs. (29) and (30) and in the following, the particle symbol is used for its (mass)^2. The parameters M_0, α, β, N in Eqs. (24) can be expressed in terms of the two independent masses and of $K(2)$ and $K(0)$. We next discuss the states which undergo mixing. We first note the relation

$$K'+K''=K(2)+K(0)\,, \tag{31}$$

quite analogous to Eq. (30). The $\eta_0-\eta_8$ mixing in the 1^- nonet has a peculiar feature. It is uniquely predicted, from Eqs. (24), to be of the ω-φ type, i.e., Eqs. (27) and (27′) are solved, for $J^C=1^-$, with $\cos\theta=\sqrt{\frac{1}{3}}$ and $\sin\theta=\sqrt{\frac{2}{3}}$, for any value of the parameters. The squared masses of $\eta(1^-)$ and $X(1^-)$ must furthermore satisfy the relations:

$$\tfrac{1}{2}[X(1^-)+\eta(1^-)]=\tfrac{2}{3}K(2)+\tfrac{1}{3}K(0)\,, \tag{32}$$
$$\eta(1^-)=\pi(1^-)\,. \tag{33}$$

The mass matrices for the $\eta_0-\eta_8$ mixing in the nonets 1^+ and 2 are obtainable, from Eqs. (24), in terms of the parameters M_0, α, β, N [which are linearly related to the two independent π-masses and to $K(2)$ and $K(0)$] and of c and D. Elimination of D (by comparing the two equations for the traces) gives the relation

$$\tfrac{1}{4}[\eta(1^+)+X(1^+)+\eta(2)+X(2)]=\tfrac{1}{3}[2K(2)+K(0)]\,. \tag{34}$$

Elimination of the parameter c from the quadratic equations for the determinants leads to an additional nonlinear relation. The verification of such a complicated nonlinear relation is unfortunately much more subject to experimental errors than is the verification of the linear relations (29)–(34).

4. COMPARISON WITH EXPERIMENTAL DATA

The experimental status of the higher boson resonances is still very incomplete.[5] However, some remarkable features already seem to emerge giving support to our predictions. Table II summarizes some possible assignments. The strongest support comes from the column of the $T=1$ mesons (π) in Table II. Our prediction, Eq. (29), becomes $\frac{1}{2}(A_2+A_1)=B$, and it turns out to be very accurately verified. From Eq. (30) we can predict the existence of $\pi(0)$ around 970 MeV (± 50 MeV). The existence of such a meson will be a crucial test of the theory presented here. Such a meson, with $J^{PG}=0^{+-}$ and $T=1$, cannot decay strongly into less than 5 pions. It may be observable as an η-π resonance. Its production cross section may be rather low because of the smallness of the η-nucleon coupling which would account for production via η exchange. The rather preliminary stage of the experiments prevents us from definitely deciding on a complete particle assignment. The assignments of the better known particles $f(1250)$, $A_2(1320)$, $B(1215)$, and $A_1(1090)$, within the proposed multiplet structure, appear to be on a more secure ground than the other assignments. The other suggested assignments in Table II will have to be reviewed when more complete data are available. We note that, in contrast to the almost complete lack of experimental information about possible candidates for the $T=0$ spin-one mesons, there are a number of possible candidates among the K resonances. Accurate study of these resonances, and definite confirmations of their existence, would be very useful. In spite of the rather unclear experimental situation we have decided—*dira necessitas*—to make full use of the available information and of our mass formulas to arrive at a tentative complete scheme of predictions. Such a scheme is exhibited in Table III. The input data relevant for the predictions are underlined. Arrows indicate the predicted resonant masses. Also indicated is a possible completion of the 0^{++} multiplet by including $\sigma^0(390)$ and $\epsilon^0(730)$. The additional assumption essential to the tentative predictions of Table III is that the mixing between the two $T=0$ particles of the 2^{++} multiplet is maximal (like the ω-φ mixing). On this assumption the otherwise dominant mode of decay into 2π of the predicted $T=0$, 2^{++} particle at (1560 ± 50) MeV would be weakly suppressed. As for the $T=\frac{1}{2}$, 1^{+-} and 1^{++} resonances, we consider in Table III two independent possibilities, according to the choices of $K^*(1175)$ or $C(1215)$ as possible inputs. The predictions of Table III are obtained by using the mass relations that we have derived. We note that, apart from the (970 ± 50) MeV $T=1$, 0^{++} particle, directly predicted from the assignments of A_2, B, and A_1, and of the two strongly mixed (mixing of ω-φ kind) $T=0$, 1^{+-} particles at (1270 ± 30) MeV and (1215 ± 15) MeV, there is now a definite prediction of a $T=0$, 2^{++} meson at (1560 ± 50) MeV and of the two $T=0$, 1^{++} mesons at (1180 ± 190) MeV and (990 ± 200) MeV. However, our hypothesis of maximal mixing in the 2^{++} nonet is essential for these last predictions, and the predicted mass spectrum could be modified by removing such a hypothesis.

TABLE II. Possible assignment of higher meson resonances.

J^{PC}	X, η	π	K
2^{++}	$f(1250)$?	$A_2(1310)$	$K^*(1430)$?
1^{+-}	? ?	$B(1215)$	$K_c^*(1220)$? $K_{\pi\pi}(1175)$?
1^{++}	? ?	$A_1(1090)$	? ?
0^{++}	$\sigma^0(390)$? $\epsilon^0(730)$?	?	$\kappa(725)$

TABLE III. Tentative predictions for higher boson resonances. The predictions in this table contain an additional assumption: that the mixing between the two $T=0$, 2^{++} mesons is maximal (like ω-φ). The masses are in MeV. The input data are underlined. Arrows indicate the predicted masses. A possible completion of the 0^{++} nonet with σ^0 and ϵ^0 is also indicated.

J^{PC}	$T=0$	$T=1$	$T=\frac{1}{2}$
2^{++}	$(1560\pm 50) \leftarrow$ $\underline{f^0(1253\pm 20)}$	$\underline{A_2(1310\pm 15)}$	$\underline{K^*(1430\pm 15)}$
			Two possibilities:
1^{+-}	$(1270\pm 30) \leftarrow$ $(1215\pm 15) \leftarrow$	$\underline{B(1215\pm 15)}$	(i) $\underline{K'=K^*(1175)}$ $K''(1100\pm 40)$
1^{++}	$(1180\pm 190) \leftarrow$ $(990\pm 200) \leftarrow$	$\underline{A_1(1090\pm 15)}$	(ii) $\underline{K'=C(1215)}$ $K''(1050\pm 40)$
0^{++}	$\sigma^0(390)$? $\epsilon^0(730)$?	$(970\pm 50) \leftarrow$	$\underline{\kappa(725)}$

Note added in proof. Since the completion of this work a number of new resonances has been reported which seem to fit quite well the multiplet proposed here. The predicted $T=0$ meson at (1560 ± 50) MeV decaying into $K\bar{K}$ may be identified with the $f'(1520)$ [see V. Barnes *et al.*, Phys. Rev. Letters **15**, 322 (1965)].

The D meson at 1280 MeV [D. Miller *et al.*, Phys. Rev. Letters **14**, 1074 (1965); Ch. D'Andlau *et al.*, Phys. Letters **17**, 347 (1965)] could be identified, according to the suggested quantum numbers, with the $J^{PC}=1^{++}$, $T=0$ meson predicted at 1180 ± 190 MeV. Furthermore there seems to be evidence about the existence of the $J^{PC}=0^{++}$, $T=1$ meson predicted at (970 ± 50) MeV [W. Kienzle *et al.*, Oxford Conference on Elementary Particles, Oxford, England, 1965, Abstr. A. 96 (unpublished); CERN–Collège de France–Institut du Radium–University of Liverpool Collaboration *ibid.*, Abstr. A. 143 (unpublished).]

[5] A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and M. Roos, UCRL-8030, 1964 (unpublished) R. Armenteros, in *Proceedings of the International Conference on High Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965); S. Y. Nikitin, *ibid.*; G. Goldhaber, Report of the Second Coral Gables Conference on Symmetry Principles, 1965 (to be published).