S-Matrix Theory and Higher Order Corrections to the Weak Interactions*†

M. B. HALPERN‡
CERN, Geneva, Switzerland and Harvard University, Cambridge, Massachusetts
(Received 7 May 1965)

We have studied the use of dispersive and unitary methods to calculate higher order corrections to the weak interactions, with emphasis on the W-meson theory. As a first step, we have examined the ordinary elastically unitary approximation techniques in the W theory of electron-neutrino scattering (namely, N/D with one W-meson exchange, Mandelstam iteration procedure, and the strip approximation). We find that these are incapable of generating a nonrenormalizable amplitude; instead, they simply neglect the divergences of the perturbation theory at every order. This is a serious weakness of the methods. Our travail leads to a prescription for the dispersive generation of the nonrenormalizability in the W theory: At least some contributions from many-boson intermediate states must be included as input information. On the basis of this prescription, we have made two distinct nonperturbative attacks on the dynamics. In the first, we have summed the leading absorptive parts of the W-meson ladder graphs, using the Cutkosky rules. These absorptive parts correspond to putting all the W mesons on the mass shell, and are finite at each order of perturbation theory (although asymptotically ill-behaved), so that no regulator is needed in the summation. We find an exponentially increasing absorptive part, which contradicts the results of Feinberg and Pais, whose solutions are bounded. In an attempt to discover the source of this discrepancy, we study peratization from a dispersive point of view. This study throws serious doubt on the validity of the Feinberg-Pais program. It is seen explicitly that their non-Hermitian methods (regulator, and an analytic continuation in the coupling) have led them to solutions which violate the positive-definiteness of the mass spectrum. In the second attack, a new family of exact solutions to the N/D equations with singular inputs allows us to propose, and to reduce to quadratures, a systematic, dispersive, unitary and regulator-free program for calculation in nonrenormalizable field theory. We show how to use as input into the N/D equations any set of graphs (regardless of their divergence) whose left-hand absorptive part, although asymptotically ill-behaved, is not itself divergent. The program makes calculation possible in a large variety of nonrenormalizable contexts, including W-meson theory, Fermi theory, derivative coupling theories, and spin-$\frac{3}{2}$ and higher spin theories in general (for example, linearized gravitational theory). This paper is intended as a sketch of the main results of the work, whereas most of the details will appear in a subsequent series of papers.

I. INTRODUCTION

ATTEMPTS to extract information from nonrenormalizable field theories seem to have invariably failed to stand the test of time. Notable among these is the technique of analytic continuation in the coupling proposed by Arnowitt, Deser, and Cooper,[1] which was eventually shown to violate the positive-definiteness of the mass spectrum.[2] Also of note are the conjectures based on perturbation theory and the Lehmann representation of Redmond and others,[3] which have come to be viewed as outright distortions of the perturbation theory. Perhaps the most ambitious attempt thus far is that made by Feinberg and Pais,[4] whose regulator limiting process has remained sufficiently ill-understood to guarantee interest in the theory.

We study the problem here (and in the subsequent series of papers) within the framework of S-matrix theory, using analytic and unitary methods.

We wish to point out that an analytic and unitary context is a very natural one in which to study a theory plagued with infinities.

(1). One is in general concerned with the calculation of transition probabilities. The divergences yielded by perturbation theory for these quantities are glaringly inconsistent with unitarity, which demands essentially that all probabilities come out between zero and one. It is suggested then that one could rid the theory of divergences by calculating with an approximation scheme unitary at every stage.

(2). Although the amplitudes themselves appear divergent in perturbation theory, a great many of the absorptive parts of these amplitudes are quite finite, even in perturbation theory (although usually asymptotically ill-behaved). For example, in the W theory, the absorptive part of any set of ladder graphs (with all the mesons on the mass shell) is finite. (This follows because,

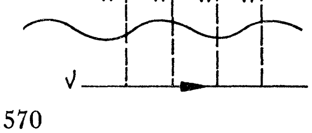

FIG. 1. Finite absorptive parts of the ladder graphs.

* Based in part on Ph.D. thesis, Harvard University, October 1964 (unpublished).
† During this work, the author was a National Science Foundation Pre-Doctoral Fellow.
‡ NATO Post-Doctoral Fellow.

[1] R. Arnowitt and S. Deser, Phys. Rev. **100**, 349 (1955); L. N. Cooper, Phys. Rev. **100**, 362 (1955).

[2] B. M. Barbashov and G. V. Efimov, Zh. Eksperim. i Teor. Fiz. **43**, 1057 (1962) [English transl.: Soviet Phys.—JETP **16**, 748 (1963)].

[3] P. J. Redmond and J. L. Uretsky, Phys. Rev. Letters **1**, 141 (1958). N. N. Bogoliubov *et al.*, Zh. Eksperim. i Teor. Fiz. **37**, 805 (1959) [English transl.: Soviet Phys.—JETP **10**, 574 (1960)]. In Bogoliubov's calculation, e.g., the chain graphs of the Fermi theory are summed by adding the absorptive parts first. We know, however, that the chain graphs themselves have no sum (except of course the trivial vanishing one in a regulator limit—see Ref. 4), so that it is difficult to view the calculation as more than a perhaps suggestive distortion of the original perturbation theory.

[4] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963). G. Feinberg and A. Pais, Phys. Rev. **133**, B477 (1964). These articles will henceforth be referred to as FP I and FP II.

after having cut all the boson lines, there are no more closed loops, and hence no divergence. See Fig. 1.) Having finite absorptive parts to manipulate gives one something of a handle on the problem. For example, one might hope that the amplitude has a form something like

$$A(s,g^2)=\int_0^\infty \cos(g^2s')ds'/s'-s \tag{1}$$

(which has no power-series expansion in the coupling g^2); if such were the case, one would need only to add up the termwise finite perturbation expansion for the absorptive part, and then Hilbert transform to obtain the amplitude. Life does not turn out to be this simple, but this indicates the sort of use to which we intend to put the finite absorptive parts.

This paper is grouped into three parts. Part II gives a discussion of the main results of the work. Part III gives a brief amplification of some of our statements concerning the Feinberg-Pais (FP) theory. Part IV sketches the method of solution of the singular N/D equations that arise in our dispersive and unitary calculational program.

We emphasize that this paper is meant to give the reader some idea of the nature of the work, and to sketch some of the main results. We apologize to the reader for the cavalier fashion in which we, of necessity, will present most of the material. The bulk of the details will be published in a forthcoming series of articles.

II. DISCUSSION OF THE MAIN RESULTS

S-Matrix Formalism and Preliminaries

We have set up the necessary S-matrix formalism for electron-neutrino scattering, with attention to maximal nonconservation of parity, kinematical singularities in the invariant functions, the crossing matrices, the circle cuts in the partial waves, and the neutrino long-range force. The main results of these essentially kinematical considerations are these:

(1). There is no neutrino infrared divergence, so that one can, in general, hope to use ordinary S-matrix techniques in the problem. This comes about because of the decreased phase space available to a neutrino-antineutrino pair at low energies relative to that available to a photon at low energies. For example, in the neutrino pair "bremsstrahlung" during electron-neutrino scattering (see Fig. 2), the probability of emission of the neutrino pair at low energies (k,k') goes like

$$\ln|k+k'/k-k'|kdk\,k'dk' \tag{2}$$

to be compared with the usual dk/k emission probability for a low-energy photon in bremsstrahlung.

(2). Although the Froissart high-energy upper-bound proof[5] breaks down in the presence of zero-mass particles, we are able, by combining the assumption of Mandelstam representation and considerations like those of Low and Greenberg,[6] to establish bounds only slightly weaker than those of Froissart. One sees explicitly that the long range nature of the neutrino force is responsible for the weakening. The presence of the neutrino force gives a Lehmann ellipse that goes for asymptotic energies like

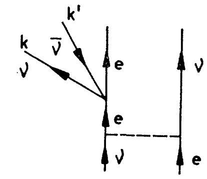

FIG. 2. Neutrino pair "bremsstrahlung."

$$a(s)=1+\lambda/s^2 \tag{3}$$

instead of the usual $a(s)=1+\lambda/s$, which follows from the Mandelstam representation in the absence of massless particles. The relation (3) is, in turn, due to the fact that, in the presence of neutrinos, double spectral functions can be asymptotic to the real s axis (whereas, usually the asymptote is $s=s_0\neq 0$).

(3). Although the Born term in the W theory exhibits essentially the usual q^{2l}-type threshold behavior in the e-ν and e-$\bar{\nu}$ channels, there are serious reasons to believe, even under the assumption of Mandelstam representation, that the full amplitude does not. This comes about because the left cut in the partial waves of the e-ν (e-$\bar{\nu}$) channel from elastic unitarity in the e-$\bar{\nu}$ (e-ν) channel comes all the way up to threshold, wrecking the usual threshold-behavior arguments. This is of possible experimental consequence.

The Elastically Unitary Approximations—Prescription for Dispersive Generation of the Nonrenormalizability

Using this S-matrix formalism, we have discovered that it is not possible to generate a nonrenormalizable electron-neutrino amplitude from the (one-W-exchange) Born term and unitarity statements that involve only leptonic intermediate states (such as, for example, elastic unitarity). In particular, we have found the N/D equations (with elastic unitarity in s) for one-W-meson

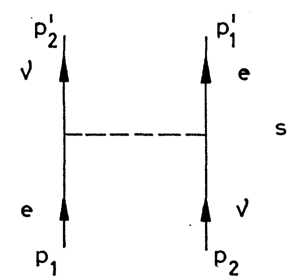

FIG. 3. One-W-meson exchange.

[5] M. Froissart, Phys. Rev. 123, 1053 (1961).

[6] O. W. Greenberg and F. E. Low, Phys. Rev. 124, 2047 (1961).

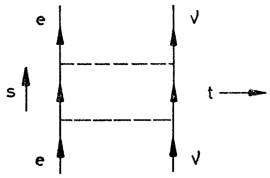

FIG. 4. The box graph in the W theory.

exchange (see Fig. 3), the Mandelstam iteration procedure, and the strip approximation[7] powerless to generate the termwise divergent amplitude characteristic of the problem. By Mandelstam iteration procedure, we mean the single-channel problem assuming the Born term, elastic unitarity in the e-ν channel, and Mandelstam representation.

The methods do yield amplitudes that are unitary and dispersive, but they are regular at $g^2=0$. In fact, we show that these methods simply neglect the divergences of the perturbation theory at every order. In order to have any confidence at all in a solution, we feel that its perturbation expansion must reproduce, at least in part, the divergences of the original perturbation theory; that is, we are interested in using the dispersive and unitary techniques to sum as much of the original perturbation expansion as possible. From this viewpoint, the failure to generate nonrenormalizability on the part of these methods is a very serious weakness, and for this reason, we are forced to reject all the ordinary elastically unitary approximations.

There are two equivalent ways of understanding this failure. The more mathematical way is as follows: It turns out that the important nonrenormalizability of the ladder graphs is essentially contained in single spectral functions in the cross channels which find their source in many-meson intermediate states. For example, in the case of the box graph of Fig. 4 one can establish the Mandelstam representation

$$F(s,t)=a_0+a_1t+\frac{t^2}{\pi}\int_{4M^2}^{\infty}\frac{\rho_t(t')dt'}{t'^2(t'-t)}+\frac{s}{\pi}\int_{4m^2}^{\infty}\frac{ds'}{s'}\int_{4M^2}^{\infty}\frac{dt'}{\pi}\frac{\rho(s',t')}{(s'-s)(t'-t)},\quad(4)$$

where m is the mass of the electron and M is the mass of the W meson. The constants a_0 and a_1 are divergent in perturbation theory (a_0 like some cutoff mass squared) and $\rho_t(t)$, the single spectral function, goes asymptotically like t. It is clear that elastic unitarity in the s channel, such as is used, for example, in the N/D based on the Born term, or the Mandelstam iteration procedure, will fail in fourth order to calculate the infinite constants, or the asymptotically ill-behaved single spectral functions. (Elastic unitarity in one channel is always inadequate to calculate things like a single spectral function in a cross channel.[8]) To calculate $\rho_t(t)$ in this case from analyticity and unitarity, one would need to use the two-meson intermediate state in the unitarity statement for the t channel. This is essentially the situation in higher orders of the ladder graphs as well. One finds in general that, when N mesons are exchanged in a ladder graph, there are single spectral functions in the cross channels (t and u), from cutting all the boson lines, which go asymptotically like

$$\rho^{(N)}(x)\sim x^{N-1}.\quad(5)$$

(For N even, x should be understood as t, for N odd, as u.) In fact, the "leading" divergences[9] of the perturbation expansion of the ladder can be understood as arising from attempts to do cutoff but unsubtracted Hilbert transforms over these asymptotically pathological single spectral functions

$$\int^{\Lambda^2}\frac{\rho^{(N)}(x')dx'}{x'-x}\sim\Lambda^{2(N-1)}.\quad(6)$$

This is the dispersive mechanism behind the generation of the most singular terms of the ladder. It is clear from this that elastic unitarity in s (the e-ν channel) will be inadequate to generate the nonrenormalizability because it is inadequate to calculate single spectral functions in the cross channels in general. We emphasize that these perturbative contributions to the single spectral functions arise from many-meson intermediate states, so that one would need to include many-meson intermediate states in the unitarity relations in the t and u channels in order to have any hope of calculating the single spectral functions from analyticity and unitarity. Because the single spectral functions are connected so intimately with many-meson intermediate states, it is not likely that they will be generated in a strip approximation either (which is really only an attempt to "add" elastic unitarity in several channels in a consistent way, and actually includes no explicit information about multimeson unitarity).

There is also an equivalent but more intuitive basis for expecting the elastically unitary approximations to fail to generate nonrenormalizability: the nonrenormalizability of the W theory is intimately connected with the asymptotic pathology of the $q_\mu q_\lambda$ term of the W propagator

$$g^2\left[\frac{g_{\mu\lambda}-q_\mu q_\lambda/M^2}{q^2-M^2}\right].\quad(7)$$

(The $q_\mu q_\lambda$ term identifies the W as a transverse vector meson.) In S-matrix techniques, however, we use the Born term on the mass shell, that is (7) sandwiched between spinors. Because

$$ku_{(\nu)}(k)=0,\quad pu_{(e)}=m\quad(8)$$

[7] As a general reference on these techniques, see S. Frautschi, *Regge Poles and S Matrix Theory* (W. A. Benjamin, Inc., New York, 1963).

[8] S. Mandelstam, Phys. Rev. **115**, 1752 (1959).

[9] As defined, for example, in FP I.

the $q_\mu q_\lambda = (p_2' - p_1)_\mu (p_2' - p_1)_\lambda$ part of the Born term (Fig. 3)

$$\bar{u}_{(e)}(p_1')\gamma^\mu u_{(\nu)}(p_2)\bar{u}_{(\nu)}(p_2')\gamma^\lambda u_{(e)}(p_1) q_\mu q_\lambda = m^2 \bar{u}_{(e)}(p_1') u_{(\nu)}(p_2) \bar{u}_{(\nu)}(p_2') u_{(e)}(p_1) \quad (9)$$

is really proportional simply to m^2. On-the-mass shell, all the asymptotic pathology of the $q_\mu q_\nu$ term has been removed. The asymptotic behavior of the Born term is no worse than if we had used a Gupta-Bleuler vector meson

$$g^2[g_{\mu\lambda}/q^2 - M^2]. \quad (10)$$

If all the information we have about the W meson is the mass-shell Born term, we will have essentially failed to distinguish the W, intrinsically a transverse vector meson, from a Gupta-Bleuler vector meson, which in general does not lead to a nonrenormalizable theory. (For example, massive electrodynamics is certainly regular at zero coupling.) Notice that the unitarity statements usually assumed in addition to the Born term do not give any further information about the transverse-vector nature of the W. That is, elastic unitarity in the e-ν channel (see Fig. 5) only gives us more information about the leptons, as only lepton lines are cut. Adding, in a consistent fashion, elastic unitarity in the e-$\bar{\nu}$ channel, as in the strip approximation, does not remedy the situation either. We see that the one-W-exchange Born term and unitarity statements involving only leptonic intermediate states simply do not provide enough input information into a theory to distinguish the transverse W meson from a Gupta-Bleuler meson (and hence to distinguish the W theory from a com-
(and hence to distinguish the W theory from a completely renormalizable one, like massive electrodynamics).

Both of these explanations for the failure of the elastically unitary approximations to generate nonrenormalizability point unequivocally to a prescription for the dispersive generation of nonrenormalizability. To generate dispersively a nonrenormalizable lepton-lepton amplitude, at least some contributions from many-meson intermediate states must be included as input information. This further information about the W meson, obtained by cutting W lines, will then fully specify the transverse nature of the W meson, and automatically generate at least some nonrenormalizability. In particular, to generate the leading ladder divergences in the solution, it is necessary to include the full single spectral functions as input.

We have used this prescription to launch two distinct nonperturbative attacks on the dynamics.

The Single Spectral Functions and the Feinberg-Pais Programme

In the first attack, we have used the Cutkosky rules to sum the leading absorptive parts of the ladder graphs in order to obtain the single spectral functions mentioned above. These absorptive parts involve cutting all

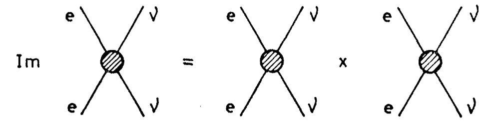

FIG. 5. Elastic unitarity in electron-neutrino scattering.

the W lines, so that, as explained in the Introduction, they are finite even in perturbation theory, and no regulator is needed in the summation. Although, as shown in (5), the perturbative contributions to the single spectral functions are increasingly pathological at high energies for larger N, one might hope that (as discussed in the Introduction) the perturbation series sums into exponentially decreasing or oscillating single spectral functions. If such were the case, one could use these single spectral functions as input into a Mandelstam iteration procedure (in the e-ν channel), finite at every order of the iteration (because of the good asymptotic behavior of the input). This would systematically construct a unitary double spectral representation which would contain all the nonrenormalizability of the ladder graphs, as explained in the section immediately above. Acutally things do not turn out so simply. We find that the single spectral functions are exponentially increasing. This is the first subject discussed in Part III of this paper. We are thus forced to give up any simple hope of constructing a double spectral representation for the amplitude.

We notice, however, that we are really only trying to ask for the absorptive part of the Feinberg-Pais solution, whose modulus, after the regulator limit, is certainly bounded. In the latter section of Part III we discuss this obvious discrepancy. A brief study of the analytic structure of the FP solutions will reveal the following properties:

(a) The perturbation expansion of the imaginary part of the FP solutions does not, in general, agree with the imaginary part of the original perturbation expansion—even though the latter is well defined at every order.

(b) The FP solutions violate the positive-definiteness of the mass spectrum, in that they include a continuous spectrum of imaginary masses.

Further details (beyond Part III) of the calculations, and further calculations, will be given in the forthcoming papers. In particular, we will point out that this breakdown of field-theoretic principle can be thought of as arising from the use of non-Hermitian intermediate steps (regulator, and an analytic continuation in the coupling) in their program. The less one learns is that, once the Hermitian structure of a theory is damaged (such as, for example, with a regulator), one is not necessarily guaranteed the ability to regain a Hermitian theory (for example, on taking the regulator limit). The fact that there is an intrinsic analytic continuation in the coupling in the calculation allows us to draw an

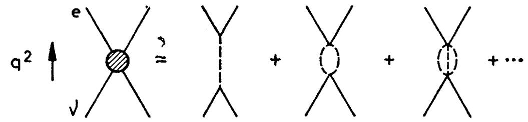

FIG. 6. The Feinberg-Pais leading singularity approximation.

analogy between, on the one hand, the methods and shortcomings of peratization, and, on the other hand, the methods and shortcomings of the previously proposed program of Arnowitt, Deser, and Cooper.

A Unitary, Dispersive and Regulator-Free Calculational Program

From the asymptotic behavior (in perturbation theory) of the single spectral functions, that is, Eq. (5), one can show that, in the partial waves of the e-ν (or e-$\bar{\nu}$) channel, the left-hand cuts associated with multiple-meson exchange increase like powers of the energy far to the left. Moreover, the asymptotic behavior is progressively worse for the cuts associated with more mesons. It turns out that the left-hand imaginary part associated with the exchange of N mesons goes asymptotically like

$$\alpha^{(N)}(-\nu)\sim\lambda\nu^{N-1}, \qquad (11)$$

where ν is the three-momentum squared in the center-of-mass frame. In Part IV of this paper, we show how to use such singular left-hand absorptive parts as inputs into the N/D equations. The solution to such equations can be found in closed form. This allows the proposal, and the reduction to quadratures, of a unitary, dispersive and regulator-free program for calculation in nonrenormalizable field theory. In general, we can use as input into the N/D equations any set of graphs (regardless of their divergence) whose left-hand cuts, although asymptotically ill-behaved, are not themselves divergent. For example, in the W theory, we can use as input the left cuts of the ladder graphs corresponding to any finite number of W exchanges. Some nonladder-type graphs can also be used as input. Similarly, one can calculate in Fermi theory, theories of higher spin in general, and theories with derivative coupling.

III. THE SINGLE SPECTRAL FUNCTIONS AND THE FEINBERG-PAIS PROGRAM

Exponential Increase of the Single Spectral Functions

The first thing we want to do in this part is find the single spectral functions mentioned above.

In their first paper, Feinberg and Pais make a "most singular" approximation on the ladder graphs of Fig. 6. This pictorial representation emphasizes the fact that, in this approximation, the only surviving imaginary parts are those associated with many-meson intermediate states. The fact that it also reduces to a one-dimensional integral equation hints that the approximation is an ideal short-cut to the single spectral functions.[10] To do this, we need only cut all the meson lines (see Fig. 7).

The equation for the trace of the amplitude in this approximation of Fig. 6 is[11]

$$T^{(\pm)}(q^2)=g^2\left[\frac{4-q^2/M^2}{q^2-M^2}\right] \mp\frac{ig^2}{(2\pi)^4M^2}\int\frac{d^4q'\,T^{(\pm)}(q'^2)}{(q-q')^2-M^2} \qquad (12)$$

in which all the propagator denominators, and hence all the absorptive parts, are from W exchange. We are not at present concerned with the solution of Eq. (12). In fact, it was pointed out in FPI that the equation is ill-defined without some regulator limiting process. We are interested in it only as a convenient short-cut to the perturbative contributions of each ladder graph to the single spectral functions.

Thus, we want to calculate the absorptive part of each term of the (termwise divergent) perturbation expansion

$$T^{(\pm)}(q^2)=g^2\left(\frac{4-q^2/M^2}{q^2-M^2}\right) \mp\frac{ig^4}{(2\pi)^4M^2}\int\frac{d^4q'}{(q-q')^2-M^2}\left(\frac{4-q'^2/M^2}{q'^2-M^2}\right) \mp\left(\frac{ig^2}{(2\pi)^4M^2}\right)^2 g^2\int\frac{d^4q'}{(q-q')^2-M^2}\int\frac{d^4q''}{(q'-q'')^2-M^2} \times\left[\frac{4-(q'')^2/M^2}{(q'')^2-M^2}\right]+\cdots. \qquad (13)$$

The discontinuity across the positive q^2 axis of each of the graphs of Fig. 6 is obtainable from Eq. (13) by the usual Cutkosky rule

$$[q^2-M^2+i\epsilon]^{-1}\to-(2\pi i)\theta(q_0)\delta(q^2-M^2). \qquad (14)$$

This gives us the (finite) discontinuities (above axis

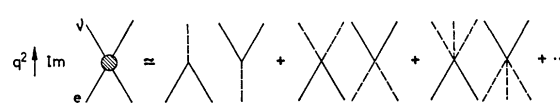

FIG. 7. The finite-imaginary parts of the perturbation series in the Feinberg-Pais approximation.

[10] That the most singular approximation of FP should lead to the single spectral functions is not surprising. Remember we have shown that the leading divergences of the ladder can be thought of as Hilbert transforms over the single spectral functions.

[11] Our notation, to be given fully in the forthcoming papers, does not differ appreciably from that of FPI.

minus below axis)

$$\mathfrak{D}[T^{(\pm)}(q^2)]$$
$$=3g^2(-2\pi i)\theta(q_0)\delta(q^2-M^2)$$
$$\mp\frac{ig^2}{(2\pi)^4M^2}(3g^2)(-2\pi i)^2\int d^4q'\,\theta(q_0-q_0')$$
$$\times\delta[(q-q')^2-M^2]\theta(q_0')\delta(q'^2-M^2)+\cdots. \quad (15)$$

We can sum the discontinuities in the integral equation

$$\operatorname{Im}T^{(\pm)}(q^2)=-3g^2\theta(q_0)\delta(q^2-M^2)\mp\frac{g^2}{(2\pi)^3M^2}\int d^4q'$$
$$\times\theta(q_0-q_0')\delta[(q-q')^2-M^2]\operatorname{Im}T^{(\pm)}(q'^2)\,, \quad (16)$$

where we have introduced the imaginary part on top of the q^2 axis $\mathfrak{D}[T^{(\pm)}(q^2)]=2i\operatorname{Im}T^{(\pm)}(q^2)$. In the function $\operatorname{Im}T^{(\pm)}(q^2)$ we are summing, simultaneously, the single spectral functions in both the e-$\bar{\nu}$ and ν-$\bar{\nu}$ channels. Substituting

$$\int d^4q'\to\int d^4q'\int_{-\infty}^{+\infty}d\lambda^2\delta(\lambda^2-q^2) \quad (17)$$
$$\operatorname{Im}T^{(\pm)}(q^2)=\theta(q_0)[y_\pm(s)-3g^2\pi\delta(s-M^2)],$$

where $s=q^2$, we obtain a one-dimensional integral equation for $y_\pm(s)$, the high-energy form of which is

$$sy_\pm(s)=\mp3g^4\kappa\pi s\mp g^2\kappa\int_0^s ds'\,(s-s')y_\pm(s');$$
$$\kappa\equiv\pi^2/(2\pi)^4M^2. \quad (18)$$

This can be solved immediately by Laplace transform and we obtain

$$y_\pm(s)=3g^2\pi(g^2\kappa)^{1/2}s^{-1/2}\begin{Bmatrix}J_1\\I_1\end{Bmatrix}[2(2g^2\kappa s)^{1/2}]. \quad (19)$$

We find, in particular, that $\operatorname{Im}T^{(-)}$ exhibits exponential increase at high energies.

Actually, the integral equation for the absorptive part [Eq. (16)] can be solved exactly at all energies. The equation is algebraic in (4-dimensional) Fourier transform space, and we obtain simply[12]

$$\operatorname{Im}T^{(\pm)}(q^2)=\frac{-3g^2i}{2}\int\frac{d^4y\,e^{iqy}\Delta^{(+)}(y,M^2)}{1\pm(g^2i/M^2)\Delta^{(+)}(y,M^2)}. \quad (20)$$

Putting $M=0$ in $\Delta^{(+)}(y,M^2)$, i.e., taking

$$\Delta^{(+)}(y,M^2)\to\frac{i}{4\pi^2}\frac{1}{y^2-i\epsilon y_0} \quad (21)$$

gives the solutions Eq. (19). (Putting $M=0$ in $\Delta^{(+)}$ is equivalent to asking for the high-energy behavior.)

We want to emphasize that the perturbation expansion of the solutions (20) reproduces exactly the leading absorptive parts of the perturbation theory [i.e., the iterative solution of Eq. (16)]. The structures (20) each contain a δ function at M^2 and an infinite series of branch points, the first at $4M^2$, the second at $9M^2$, etc., with the cuts going to the right. There are no contributions from $s<M^2$. For reasons which will become clear below, we make the following obvious statement: the fact that there is no imaginary part for $s<0$ is simply the positive-definiteness of the mass spectrum. All this is just what one would expect of the absorptive parts of such many-meson intermediate states; in this simple approximation, the scattering amplitude has one-dimensional, propagator-like analyticity properties. In the approximation where we take $M=0$ in the meson propagators (high energy), all the cuts begin at $s=0$ and go to the right. Of particular note is that the iterative solution of the equation for $\operatorname{Im}T^{(-)}$, Eq. (16) is, at least in the high-energy ($M=0$) approximation, a series of positive-definite terms, and could hardly help but add to an increasing exponential.

We emphasize that, at least in the case of $T^{(-)}$, there is an obvious discrepancy between our results and those of FP. Their $T^{(-)}$, after the regulator limit, is bounded at high energy and yet our calculation certainly sums the absorptive parts of the perturbation theory, each term taken in the FP leading divergence approximation. What then are the imaginary parts that FP obtain in place of the correct (20)?

Analytic Structure of the FP Solution

The equation for the trace (12) has the immediate solution by Fourier transform[13]

$$T^{(\pm)}(q^2)=\frac{g^2}{2i}\int\frac{d^4y'\,e^{iqy}[4+(\Box^2/M^2)]\Delta_F(y)}{1\pm(g^2/2M^2)\Delta_F(y)}. \quad (22)$$

This is not well defined, but we formally introduce, with FP, the identity $(\Box^2+M^2)\Delta_F(x,M^2)=-2i\delta^{(4)}(x)$ in order to rewrite

$$T^{(\pm)}(q^2)=\frac{3g^2}{2i}\int\frac{e^{iqy}\Delta_F(y)d^4y}{1\pm(g^2/2M^2)\Delta_F(y)} \quad (23)$$

where we have set, with evident lack of rigor,

$$\int\frac{e^{iqy}\delta^{(4)}(y)d^4y}{1\pm(g^2/2M^2)\Delta_F(y)}=0. \quad (24)$$

We know from FPI that this is equivalent to their limiting process. For simplicity, we make the zero-

[12] $\Delta^{(+)}(y,M^2)$ is defined as in S. S. Schweber, *Relativistic Quantum Field Theory* (Harper and Row, New York, 1961).

[13] $\Delta_F(y,M^2)$ is also defined as in Schweber.

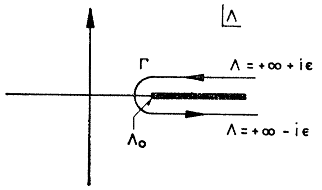

FIG. 8. The peratization path around the branch point in the complex Λ plane.

boson-mass approximation

$$\Delta_F(x,M^2) \rightarrow -(1/2\pi^2)1/x^2 - i\epsilon \quad (25)$$

and use the FP reduction formulas to rephrase (23) as

$$T^{(\pm)}(q^2) = \frac{3g^2}{2q}\int_0^\infty \frac{yd(y^2)\,J_1(yq)}{y^2 \mp \lambda^2 - i\epsilon}, \qquad q^2>0$$

$$= \frac{-3g^2}{2\bar{q}}\int_0^\infty \frac{yd(y^2)\,J_1(y\bar{q})}{-y^2 \mp \lambda^2 - i\epsilon}, \quad q^2<0 \quad (26)$$

where $q=(q^2)^{1/2}$ when $q^2>0$, $\bar{q}=(-q^2)^{1/2}$ when $q^2<0$, and $\lambda = g^2/(2\pi)^4M^2$. The imaginary parts of the solutions are then simply

$$\mathrm{Im}T^{(+)}(s) = -3g^2\pi\delta(s) + \frac{3g^2\pi\lambda}{2}s^{-1/2}J_1(\lambda s^{1/2})\theta(s)\,,$$

$$\mathrm{Im}T^{(-)}(s) = -3g^2\pi\delta(s) \quad (27)$$

$$-\frac{3g^2\pi\lambda}{2}(-s)^{-1/2}\cdots J_1[\lambda(-s)^{1/2}]\theta(-s)\,,$$

where again, $s=q^2$. We see that the imaginary part of $T^{(-)}$ for positive s, found above to be a series of positive terms which summed to an exponential increase, has been replaced by zero after the regulator limit. [The imaginary part of $T^{(+)}$ agrees with what we found above.] Worse still, we know that the original perturbation expansion had, in both cases, imaginary parts only for $s>0$. This reflected the positive-definiteness of the mass spectrum. Thus the vanishing of the right-hand imaginary part and the appearance of one on the left in $T^{(-)}(s)$, indicates not only a serious distortion of the original perturbation series, but also a violation of the positive-definiteness of the mass spectrum.

It can be shown that, also in the case where we keep nonzero M in the propagator, an imaginary part arises on the left in $T^{(-)}$.

It is clear that in finding solutions, FP have (a) distorted the original absorptive parts and (b) ended up in a space without a positive-definite mass spectrum. How can such things come about? In the forthcoming papers, we will show that the violation of the mass-spectrum constraint arises through the use of non-Hermitian steps (regulator, etc.). As we shall show, once the Hermitian structure of the theory is broken (a regulator introduces particles with imaginary coupling), one has, in general, no guarantee of being able to regain it (e.g., by taking the regulator limit, etc.).

In particular, we shall show that one can obtain the (nonexponential but mass-spectrum-violating) FP absorptive part from our (exponentially increasing) absorptive part by an analytic continuation in the complex regulator plane. The two absorptive parts are each other's analytic continuation in complex regulator. Another way of phrasing the situation is that the regulator-limiting process exploits a lack of analyticity in the complex regulator plane. Roughly speaking, there is a branch point on the real axis in the complex regulator plane (see Fig. 8). That is, the ladder graphs start at $\Lambda = +\infty + i\epsilon$ on top of the cut where the absorptive part is exponentially increasing (and the real part is ill-defined). The peratization prescription turns out to be equivalent to the following: lower the regulator mass along Γ past the branch point at Λ_0, solve the integral equation there, and then come out of the regulator plane along the *bottom* of the cut. Unfortunately, although the limit on the bottom of the cut is defined and bounded, it violates the mass spectrum constraint. We learn how dangerous regulators can be. The fact that a limit exists on the bottom of the cut does not automatically guarantee Hermiticity of the resulting theory.

Further details, parallel calculations, and further comments will be given in the forthcoming papers. In particular, we shall show that the breakdown of the mass spectrum constraint can alternatively be thought of as arising from another type of non-Hermitian intermediate step, an intrinsic analytic continuation in the coupling. (This is the small imaginary part FP are forced to give to the coupling in order to find a path around the pole they find near the light cone in coordinate space.) In this way we will be able to draw an analogy between the methods and shortcomings of, on the one hand, peratization, and, on the other hand, the methods and shortcomings of the previously proposed program of Arnowitt, Deser, and Cooper. We shall also discuss the rotation of the FP solutions into Euclidean space where they turn out to be complex rather than real as they should be.[14] This is another manifestation of the breakdown of field-theoretic postulate after the regulator limit. Finally, we will make some comment on the breakdown of field-theoretic postulate in calculations which have obtained the FP low-energy modification in an ostensibly regulator-free manner (with the Bethe-Salpeter equation). One is led to wonder whether in fact any of these solutions is really free of field-theoretic-postulate breakdown (and non-Hermitian intermediate steps).

IV. A DISPERSIVE, UNITARY, AND REGULATOR-FREE PROGRAM FOR CALCULATION IN NONRENORMALIZABLE FIELD THEORY

It was mentioned in Part II that the left cuts (in the partial waves of the e-ν and/or the e-$\bar{\nu}$ channel) corre-

[14] J. Schwinger, Proc. Natl. Acad. Sci. 44, 956 (1958).

sponding to multiple-meson exchange, grow like powers of the energy far to the left [see Eq. (11)]. We want to indicate briefly here how to go about solving N/D equations with such singular left-hand discontinuities as input.

Arguments will be presented in the following articles for taking the N/D equations in the particular form,

$$N(\nu)=\frac{\nu}{\pi}\int_{-\infty}^{-\nu_L}\frac{\alpha(\nu')D(\nu')d\nu'}{\nu'(\nu'-\nu)},\qquad D(\nu)=1-\frac{\nu}{\pi}\int_0^{\infty}\frac{N(\nu')\rho(\nu')d\nu'}{\nu'(\nu'-\nu)},\tag{28}$$

where ν is the 3-momentum squared in the center-of-mass frame, $\alpha(\nu)$ is the input imaginary part of $A=N/D$ on the left, and $\rho(\nu)$ is the density-of-states factor. This latter has the property $\rho(\nu)\to 1$ as $\nu\to\infty$. We are concerned with the solution of these equations when

$$\alpha(-\nu)\sim\lambda\nu^m \quad \text{as} \quad \nu\to\infty \tag{29}$$

(m a positive integer and λ a constant). It is quite clear that solutions to (28) cannot be regular at $\lambda=0$, as the "iterative" solution diverges term by term. For simplicity here, we will take

$$\alpha(-\nu)=\lambda\nu^m\theta(\nu)(\nu_L=0);\quad \rho(\nu)=1. \tag{30}$$

This clearly corresponds to a "most singular" approximation to the N/D Eqs. (28), in that we keep the most singular (large ν) parts of $\alpha(\nu)$ and $\rho(\nu)$. In the forthcoming article on this technique, we will show how to use this "most singular" solution as the first step in an iterative approximation scheme to solve (28) in general [i.e., for inputs like (29) in general rather than only the restricted (30)].

In this "most singular" approximation, then, the equation for the D function is

$$D(-\nu)=1+\frac{\nu}{\pi^2}\int_0^{\infty}\frac{\alpha(-\nu')D(-\nu')d\nu'}{\nu'(\nu'-\nu)}\ln\left(\frac{\nu'}{\nu}\right). \tag{31}$$

The substitutions

$$\alpha(-\nu)=\lambda\nu^m;\quad D(-\nu)=\Phi(\xi),\quad \nu=e^{\xi},\quad \nu>0 \tag{32}$$

yield

$$\Phi(\xi)=1+\frac{\lambda}{\pi^2}\int_{-\infty}^{+\infty}\frac{e^{\eta m}\Phi(\eta)(\eta-\xi)}{e^{\eta-\xi}-1}d\eta. \tag{33}$$

In Fourier transform space, this goes over into the finite difference equation

$$\Phi(\omega)=\delta(\omega)-\lambda\,\mathrm{csch}^2\pi(\omega-i\epsilon)\Phi(\omega+im);\qquad \Phi(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-i\xi\omega}\Phi(\xi)d\xi. \tag{34}$$

A necessary condition that the inverse Fourier transform of a solution to the difference equation $\Phi(\omega)$ actually solve the integral equation (33) is that $\Phi(\omega)$ be analytic in the "period strip" $0<\mathrm{Im}\omega<m$. We shall develop a technique for solving (34) that guarantees this.

Toward this end, we first consider the homogeneous equation

$$\Phi_H(\omega)=-\lambda\,\mathrm{csch}^2\pi(\omega-i\epsilon)\Phi_H(\omega+im). \tag{35}$$

If we look for a solution in the form

$$\Phi_H(\omega)=\exp F(\omega), \tag{36}$$

we find the difference equation for $F(\omega)$

$$F(\omega+im)-F(\omega)=-\ln[-\lambda\,\mathrm{csch}^2\pi(\omega-i\epsilon)]. \tag{37}$$

In Fourier transform space this is algebraic and we obtain

$$F(x)=\frac{\int_{-\infty}^{+\infty}d\omega\,\ln[-\lambda\,\mathrm{csch}^2\pi(\omega-i\epsilon)]e^{i\omega x}}{1-e^{mx}};\qquad F(x)=\int_{-\infty}^{+\infty}e^{ix\omega}F(\omega)d\omega. \tag{38}$$

Using the identity

$$\int_{-\infty}^{+\infty}d\omega\,e^{ix\omega}\ln[\mathrm{csch}\pi(\omega-i\epsilon)]=\frac{P2\pi}{x(1-e^{-x})}+(\pi\ln 4+\pi^2 i)\delta(x) \tag{39}$$

we find in the end

$$\Phi_H(\omega)=\exp\left\{\frac{i\omega}{m}\ln(4\lambda)+2P\int_{-\infty}^{+\infty}\frac{dx\,e^{-ix\omega}}{x(1-e^{-x})(1-e^{mx})}\right\}. \tag{40}$$

Now we turn to the inhomogeneous equation. To solve this, we guess a variation of parameter solution of the form

$$\Phi(\omega)=\Phi_H(\omega)G(\omega). \tag{41}$$

Substituting this into Eq. (34), and using Eq. (35), we find that $G(\omega)$ satisfies the "Green's function" difference equation

$$G(\omega+im)-G(\omega)=-\Phi_H^{-1}(0)\delta(\omega). \tag{42}$$

As it turns out, one can show that

$$\Phi_H^{-1}(0)=\exp\left\{-2P\int_{-\infty}^{+\infty}\frac{dx}{x(1-e^{-x})(1-e^{mx})}\right\}=m. \tag{43}$$

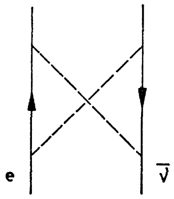

FIG. 9. Simplest input for a possible W-meson bootstrap.

Equation (42) can be solved immediately by Fourier transform, and one obtains

$$G(\omega)=\frac{m}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\omega x}}{1-e^{mx}}. \tag{44}$$

We can take, of course, any of three prescriptions at the pole $x=0$. We choose the prescription in the following way: In order to guarantee that $D(\nu)$ is real on the left, as it should be, $\Phi(\omega)$ must have the symmetry property

$$\Phi(\omega)=\Phi^*(-\omega) \tag{45}$$

for real ω. It is clear on inspection that $\Phi_H(\omega)$ already has this symmetry, so to guarantee Eq. (45) we must choose the Green's function with this symmetry as well. Thus we use the principal value prescription in evaluating Eq. (44). We obtain finally, doing the inverse Fourier transform,

$$D(-\nu)=\tfrac{1}{2}i\int_{-\infty}^{+\infty} e^{i\omega\,\ln\nu}d\omega\,\coth\frac{\pi}{m}(\omega+i\epsilon)$$
$$\times\exp\left\{\frac{i\omega}{m}\ln 4\lambda+2P\int_{-\infty}^{+\infty}\frac{dx\,e^{-ix\omega}}{x(1-e^{-x})(1-e^{mx})}\right\}. \tag{46}$$

Notice that the final $\Phi(\omega)$, that is Eq. (40) times Eq. (44), is transparently analytic in the period strip, as required. The exponentiated Fourier transform in Eq. (46) can be evaluated in terms of logarithms, dilogarithms, and hypergeometric functions, so that, really, only the final inverse Fourier transform remains to be done.

In the forthcoming papers on this technique we will prove that Eq. (46) is a solution of Eq. (31) in a less cavalier fashion and explore in detail the properties of the solutions. Of particular interest will be the lack of uniqueness of the solution and the fact that the solutions contain ghosts. In the case of the former, we will show that the solution exhibited here [Eq. (46)] has the distinction of being the (smooth) regulator limit of the (regulated) perturbation expansion of (31), and we will advance reasons why we feel, for this reason, it is to be preferred over any other solution.

Because of the ghosts in the solutions, our program is evidently having some of the same sort of trouble as we have found in the FP program (and which was previously found in the Arnowitt program). We shall advance arguments as to why our ghosts may be less serious than the failures in the other theories, and even try to interpret the ghosts as indicating an oscillatory left-hand cut in the full theory. However, when faced with repeated breakdown of field-theoretic postulate like this, one cannot help but also wonder if it is even possible to require all the postulates of local-field theory in the presence of nonrenormalizability. We will briefly discuss this point as well in the coming papers. Of interest also will be the discussion of whether or not one has a principle of the dominance of nearby singularities in nonrenormalizable field theories.

The method of solution outlined here allows one to use any asymptotically ill-behaved set of left-hand discontinuities as input into the N/D equations. This makes possible dispersive, unitary, and regulator-free calculation in a wide variety of nonrenormalizable contexts. For example, in the W theory, the left cuts (corresponding to multiple-meson exchange) of any finite number of ladder graphs may be used as input. Some nonplanar graphs can also be included. In much the same way, we can also calculate in the Fermi theory (e.g., by exchange of some number of two-lepton bubbles), theories of higher spin in general, and theories with derivative coupling. For example, the program makes possible the use as input of one (or multiple) gravitaton exchange, or one (or more) spin $\frac{3}{2}$ particle(s). The natural advantage of this approach over that of summing the ladder graphs is that we obtain mass-shell solutions for all energies. Possible applications include, e.g., low-energy (elastic) lepton-lepton scattering with exchange of some finite number of W mesons or lepton bubbles. A W-meson bootstrap is possible in principle with the technique, using, for example, the graph of Fig. 9 as input in the $J=1$ wave of the $e\bar{\nu}$ channel. These applications are presently in progress.

We emphasize that, in this program, one calculates the weak interactions on the same footing as the strong. The program provides a natural framework in which to study strong and weak forces together and their mutual interaction. To do this, one need only use inputs including both strong and weak forces. For example, one might study the higher order weak corrections to the lowest order π to μ decay coupling ratio. This calculation is also in progress.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor W. Gilbert for many invaluable discussions and constant encouragement, Professor D. Amati for reading the manuscript, and Professor L. Van Hove for the hospitality extended to the author at CERN.