Geometric Theory of Neutrinos*

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(Received 24 May 1965)

A theory of neutrinos is constructed within the present scheme of geometrodynamics. First, a special class of c-number Heisenberg fields is considered and a plane-wave-like solution of the covariant neutrino equation is obtained in the two-component limit. It is shown that the special class of neutrinos forms a Rainich null geometry. Conversely, a geometry suitable for the special neutrinodynamics is distinguished from general null geometries by imposing the condition that the null eigenvector of geometry is the gradient of the neutrino complexion. Finally, the geometrical parallelism between gravito-electrodynamics and gravito-neutrinodynamics is discussed.

I. INTRODUCTION

THE view that the physical world is composed of an empty geometry is not quite new. It is perhaps sufficient to recall that Einstein's theory of general relativity is already two score and ten years old. However, it was only a decade ago when Wheeler succeeded for the first time in constructing a complete geometrical model of mass, giving a new interpretation of the concept of mass.[1] In Einstein's theory, masses and fields are still sources foreign to space-time geometry. Instead, Wheeler considered them as products of the dynamics of geometry. He and his collaborators have further developed geometrodynamics dealing only with geometry and its evolution and have discovered previously unexpected rich implications for curved empty space.[2,3] Above all, it is very compelling, despite certain unresolved difficulties,[4] that a Rainich geometry[5–7] accommodates the coupled theory of Einstein's general relativity and Maxwell's electrodynamics, which is characterized by the Rainich algebraic conditions

$$R=0;\ R_{00}\geqslant 0;\ R_{\mu\lambda}R_{\nu}{}^{\lambda}=\tfrac{1}{4}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} \qquad (1)$$

and by the Rainich-Misner-Wheeler differential condition[8]

$$\partial_\mu\alpha_\nu-\partial_\nu\alpha_\mu=0;\ \alpha_\mu=e_{\mu\nu\rho\sigma}R^{\sigma}{}_{\lambda}\nabla^{\rho}R^{\nu\lambda}/R_{\alpha\beta}R^{\alpha\beta}. \qquad (2)$$

This is, in the language of Misner and Wheeler, an already unified theory of gravity and electromagnetism.[6]

Recently, some attention has been paid to solving the Rainich problem for neutrinos.[9] In fact, the neutrino is the simplest field with half-integral spin. It has no mass and no charge; it travels with the speed of light. It appears to be as fundamental as the gravitational and electromagnetic fields. Nevertheless, no attempt has succeeded in providing a proper place for this third field in geometrodynamics.

A question arises as to whether it is possible for the neutrino field to be a primary object of geometrodynamics. Concerning this point, Wheeler has remarked that there is no classical description of the neutrino along correspondence-principle lines, and has added that Pauli's descriptive term "nonclassical two-valuedness" antedated the term "spin.[10]" The natural origin of the two-valuedness could be found in quantum geometrodynamics, and probably it would be necessary to accept quantum considerations from the very beginning in dealing with neutrinos. Nevertheless, in this paper, we consider it worthwhile to reinvestigate the c-number neutrino field as a solution of the general relativistic Dirac equation, and we wish to see if one may say anything about neutrinos within the present scheme of geometrodynamics.

In the null case, where $R_{\alpha\beta}R^{\alpha\beta}$ vanishes, the Rainich vector α_μ is singular, and the Rainich-Misner-Wheeler Eq. (2) fails to hold. When a geometry has the properties

$$R=0;\ R_{00}\geqslant 0;\ R_{\mu\lambda}R_{\nu}{}^{\lambda}=0, \qquad (3)$$

the Ricci tensor can be expressed in the form

$$R_{\mu\nu}=2\epsilon^2\varphi_\mu\varphi_\nu, \qquad (4)$$

* Based on part of a thesis submitted by the first author to Rensselaer Polytechnic Institute in partial fulfillment of the requirements for the Ph.D. degree. Preliminary reports of this work were given in Bull. Am. Phys. Soc. 9, 87 and 450 (1964).

[1] J. A. Wheeler, Phys. Rev. **95**, 511 (1955).

[2] J. A. Wheeler, *Geometrodynamics* (Academic Press Inc., New York, 1962), in which Refs. 1, 6 and other related articles are collected.

[3] See also J. G. Fletcher's review article in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962).

[4] A fundamental difficulty is that the required initial data may correspond to more than one Maxwell field; see L. Witten, Phys. Rev. **120**, 635 (1960). Whether or not such a difficulty is fatal to geometrodynamics is not yet known. An optimistic observation on this matter is seen in Ref. 3, p. 416. Another serious problem is the singular case of the Rainich vector α_μ, which this paper discusses in part.

[5] G. Y. Rainich, Trans. Am. Math. Soc. **27**, 106 (1925).

[6] C. W. Misner and J. A. Wheeler, Ann. Phys. (N. Y.) **2**, 525 (1957).

[7] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill Book Company, Inc., New York, 1965), p. 418.

[8] In Eq. (2), $e_{\mu\nu\rho\sigma}$ is the permutation tensor; see, e.g., J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960), p. 18.

[9] O. Bergmann, J. Math. Phys. **1**, 172 (1960); J. R. Klauder, *ibid.* **5**, 1204 (1964), and see also Ref. 11.

[10] This remark is given in Ref. 2, p. 88. Other important problems on the geometrization of neutrinos are also extensively discussed there.

where ϵ is an undetermined scalar and φ_μ a null vector that satisfies

$$\varphi_\mu \varphi^\mu = 0;\ \varphi_0{}^2 \geqslant 0. \tag{5}$$

If this geometry entails any physical law at all, there should exist a physical correspondent to the null vector φ_μ. In contrast with the non-null field geometry, it is as yet uncertain whether the null field geometry has a rigorous physical correspondent. In effect, one can construct out of geometrical quantities an antisymmetric tensor which is adequate for describing the Maxwell null field.[6,11,12] However, the null electromagnetic field so defined is not always unique. By a nonconstant duality rotation, another null field can be found satisfying Maxwell's equations.[13] According to Peres' investigation,[14] such arbitrariness occurs when the field possesses a wave front; i.e., when, for a certain value of ϵ, the curl of φ_μ vanishes

$$\partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu = 0. \tag{6}$$

We will call this differential relation for the null vector φ_μ the Peres condition.

It is now natural to ask if there is any alternative way of understanding the null field geometry. To examine this possibility, let us accept the conjecture that the null vector φ_μ could be constructed out of bilinear covariants of a c-number Dirac field ψ. There are, however, no bilinear covariants of the four-component field that can fulfill the properties (5). As is seen in Sec. II, a unique bilinear null vector may be formed from the Dirac field subject to the two-component constraint $\psi = \gamma_5 \psi$. This fact makes it clear that the c-number Heisenberg nonlinear equation turns out to be the neutrino equation in the two-component limit. Section III deals with a special class of Heisenberg fields and derives a plane-wave-like solution of the covariant Dirac equation for a massless field in the two-component limit. Section IV is devoted to showing that the special class of neutrinos forms a Rainich null geometry characterized by (3). In Sec. V, a geometry suitable for the special neutrino physics is distinguished from general null geometries by imposing the condition that the null vector is the gradient of the neutrino complexion. As a result of this additional restriction, the gravito-neutrino geometry thus formulated belongs to Peres' exceptional case of null geometries. The Peres condition (6), though a little weaker for neutrinos, is comparable with the Rainich-Misner-Wheeler equation in the non-null case. Finally, Sec. VI summarizes the parallelism between null and non-null field geometries and discusses the self-consistency of the theory in some detail.

[11] L. Witten, Phys. Rev. **115**, 206 (1959).
[12] A. Peres, Phys. Rev. **118**, 1105 (1960).
[13] See L. Witten's article in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), p. 395.
[14] A. Peres, Ann. Phys. (N. Y.) **14**, 419 (1961).

Throughout this paper we shall employ natural units in which $8\pi G = 1$ as well as $\hbar = c = 1$.

II. NEUTRINOS IN CURVED SPACE

For convenience, we start with the *four-component* spinor field ψ defined in curved space in the usual manner.[15] Here the γ matrices are related to the space-time metric $g_{\mu\nu}$ by

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 g_{\mu\nu}, \tag{7}$$

and the covariant derivative of the spinor field is given by $\nabla_\mu \psi = (\partial_\mu - \Gamma_\mu)\psi$, where Γ_μ is the Fock-Ivanenko spin connection. The adjoint field of ψ is defined by $\bar{\psi} = \psi^\dagger \eta$ with a Hermitian matrix η such that

$$\gamma_\mu{}^\dagger = -\eta \gamma_\mu \eta^{-1} \tag{8}$$

and

$$\partial_\mu \eta + \Gamma_\mu{}^\dagger \eta + \eta \Gamma_\mu = 0. \tag{9}$$

Out of the γ matrices we form an involutory constant matrix,

$$\gamma_5 = \frac{1}{4!} i e_{\kappa\lambda\mu\nu} \gamma^\kappa \gamma^\lambda \gamma^\mu \gamma^\nu,$$

which anticommutes with all γ_μ. From (8) follows

$$\gamma_5{}^\dagger = -\eta \gamma_5 \eta^{-1}. \tag{10}$$

According to these definitions, the vector bilinear $V_\mu = i\bar{\psi}\gamma_\mu\psi$ and the axial-vector bilinear $A_\mu = i\bar{\psi}\gamma_\mu\gamma_5\psi$ are both real. In particular,

$$V_0{}^2 \geqslant 0;\ A_0{}^2 \geqslant 0. \tag{11}$$

For the field ψ one may also verify that the Pauli-Kofink relation holds[16]:

$$(\bar{\psi} Q \gamma_\lambda \psi)\gamma^\lambda \psi = (\bar{\psi} Q \psi)\psi - (\bar{\psi} Q \gamma_5 \psi)\gamma_5 \psi, \tag{12}$$

with $Q = I$, γ_μ, $\gamma_{\mu\nu}$, $\gamma_\mu\gamma_5$, or γ_5, where $\gamma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$.

If the field equation for ψ is invariant under the chirality transformation

$$\psi \to e^{i\beta_0 \gamma_5} \psi, \tag{13}$$

with a constant β_0, then the field constrained by

$$\psi = \gamma_5 \psi \tag{14}$$

can certainly be a particular solution. By means of the projection operators $\Lambda_\pm = \frac{1}{2}(1 \pm \gamma_5)$, the constrained field and its adjoint are expressed as $\chi = \Lambda_+ \psi$, and $\bar{\chi} = \bar{\psi}\Lambda_-$, respectively. It is possible to select a suitable representation such that only two components of the constrained field remain nonvanishing everywhere. Of all possible bilinear covariants of the two-component field, $\bar{\chi} Q \chi$, the only nonvanishing one is the vector

[15] See, e.g., D. R. Brill and J. A. Wheeler, Rev. Mod. Phys. **29**, 465 (1957). The γ_5 matrix adopted here differs from theirs by a multiple $\sqrt{(-1)}$.
[16] W. Kofink, Ann. Physik **30**, 91 (1937).

(axial-vector) bilinear.[17] From the Pauli-Kofink identity (12), one deduces the important null relation for χ,

$$(\bar{\chi}\gamma_\mu\chi)\gamma^\mu\chi=0. \tag{15}$$

The last expression indicates that χ is the null eigenspinor of the vector $\bar{\chi}\gamma_\mu\chi$. Evidently, the nonvanishing bilinear is null:

$$(\bar{\chi}\gamma_\mu\chi)(\bar{\chi}\gamma^\mu\chi)=0. \tag{15'}$$

As an example of a chirality-invariant field equation for ψ, we consider the Heisenberg nonlinear equation in covariant form[18]

$$\gamma^\mu\nabla_\mu\psi+3\epsilon(\bar{\psi}\gamma^\mu\gamma_5\psi)\gamma_\mu\gamma_5\psi=0, \tag{16}$$

where ϵ is a real constant. The chirality invariance permits the equation to possess a particular solution which is subject to the constraint (14). For such a constrained field, however, the second term disappears regardless of the value of ϵ because of the null property (15), and we are left with the Dirac equation in the form

$$\gamma^\mu\nabla_\mu\psi=0. \tag{17}$$

Although the Heisenberg equation would be of little interest in c-number field theory, it is remarkable that the Heisenberg c field reduces to the massless Dirac field in the two-component limit $\psi\rightarrow\gamma_5\psi$.

The general term "neutrino" may be used for any massless Dirac field. Hereafter, however, we shall focus our attention merely on the neutrino as the two-component limit of a Heisenberg field. Obviously, in that limit, the right-handed state of the neutrino is excluded at every local point of space-time.

III. RESTRICTED CLASS OF NEUTRINOS

In the c-number theory there is a special Heisenberg field satisfying the dynamical restriction[19]

$$\nabla_\mu\psi=\tfrac{1}{2}\epsilon(\bar{\psi}\gamma^\lambda\gamma_5\psi)\gamma_\lambda\gamma_\mu\gamma_5\psi-2\epsilon(\bar{\psi}\gamma_\mu\gamma_5\psi)\gamma_5\psi. \tag{18}$$

This is easily seen by operating on Eq. (18) from the left with γ^μ. The adjoint equation of (18) is

$$\nabla_\mu\bar{\psi}=\tfrac{1}{2}\epsilon(\bar{\psi}\gamma^\lambda\gamma_5\psi)\bar{\psi}\gamma_\mu\gamma_\lambda\gamma_5-2\epsilon(\bar{\psi}\gamma_\mu\gamma_5\psi)\bar{\psi}\gamma_5. \tag{19}$$

Combining Eqs. (18) and (19), one may prove the relations (see Appendix A):

$$\nabla_\mu A_\nu=i\bar{\psi}\gamma_\nu\gamma_5\nabla_\mu\psi+i\nabla_\mu\bar{\psi}\gamma_\nu\gamma_5\psi=0, \tag{20a}$$

$$U_{\mu\nu}=\bar{\psi}\gamma_\nu\nabla_\mu\psi-\nabla_\mu\bar{\psi}\gamma_\nu\psi=2\epsilon(A_\mu A_\nu+\tfrac{1}{2}g_{\mu\nu}A_\lambda A^\lambda). \tag{20b}$$

The integrability condition of Eq. (18) is obtained after calculation of the commutator of two covariant differentiations $(\nabla_\mu\nabla_\nu-\nabla_\nu\nabla_\mu)\psi$ which reads

$$R_{\kappa\lambda\mu\nu}\gamma^{\mu\nu}\psi=\epsilon^2 e_{\alpha\beta\kappa\lambda}e_{\rho\sigma\mu\nu}g^{\beta\sigma}(\bar{\psi}\gamma^\alpha\gamma_5\psi)(\bar{\psi}\gamma^\rho\gamma_5\psi)\gamma^{\mu\nu}\psi, \tag{21}$$

where $R_{\kappa\lambda\mu\nu}$ is the Riemann curvature tensor. Evidently, the integrability condition (21) allows the following solution for $R_{\kappa\lambda\mu\nu}$:

$$R_{\kappa\lambda\mu\nu}=\epsilon^2 e_{\kappa\lambda\alpha\beta}e_{\mu\nu\rho\sigma}g^{\beta\sigma}(\bar{\psi}\gamma^\alpha\gamma_5\psi)(\bar{\psi}\gamma^\rho\gamma_5\psi). \tag{22}$$

Contraction on κ and μ leads to

$$R^\mu{}_{\lambda\mu\nu}=R_{\lambda\nu}=-2\epsilon^2\{(\bar{\psi}\gamma_\lambda\gamma_5\psi)(\bar{\psi}\gamma_\nu\gamma_5\psi)-g_{\lambda\nu}(\bar{\psi}\gamma_\rho\gamma_5\psi)(\bar{\psi}\gamma^\rho\gamma_5\psi)\}. \tag{23}$$

So far as Eq. (18) is completely integrable, a solution exists for an arbitrary initial condition $\psi(x_0)$. Since Eq. (18) is invariant under the chirality transformation (13), any nonvanishing components of $\Lambda_+\psi$ will never turn into a nonzero part of $\Lambda_-\psi$ in the course of dynamical development. Therefore, if $\psi(x_0)$ is limited to $\chi(x_0)$, then the same should be true for $\psi(x)$ at any background metric provided that the integrability condition (21) is fulfilled. Naturally the null property (15) of a two-component field is preserved under the dynamical restriction (18).

In the two-component limit, any Heisenberg c field tends to the neutrino field. In the same limit, Eq. (18) can be written as

$$\nabla_\mu\chi=-\epsilon(\bar{\chi}\gamma_\mu\chi)\chi, \tag{24}$$

which we shall refer to as the *restricted* neutrino equation, with the field χ describing a restricted class of neutrinos. For the limiting field χ the relations (20a) and (20b) are also simplified as

$$\nabla_\mu\varphi_\nu=i\bar{\chi}\gamma_\nu\nabla_\mu\chi+i\nabla_\mu\bar{\chi}\gamma_\nu\chi=0, \tag{25a}$$

$$U_{\mu\nu}=\bar{\chi}\gamma_\nu\nabla_\mu\chi-\nabla_\mu\bar{\chi}\gamma_\nu\chi=2\epsilon\varphi_\mu\varphi_\nu, \tag{25b}$$

where $\varphi_\mu=i\bar{\chi}\gamma_\mu\chi$. From (23), therefore, there follows an allowed geometry in which the restricted field equation (24) is integrable,[20]

$$R_{\mu\nu}=-2\epsilon^2(\bar{\chi}\gamma_\mu\chi)(\bar{\chi}\gamma_\nu\chi). \tag{26}$$

Positive-definiteness of R_{00} in Eq. (26) results from the property (11) of the vector current for any metric value. Apparently the Ricci tensor (26) fulfills all Rainich null-field conditions (3). We may thus conclude that the restricted neutrino equation is integrable in a Rainich-Riemannian null space. In addition, the curl

[17] This is due to the following properties of $\Lambda_\pm$: $\Lambda_\mp Q\Lambda_\pm=0$ for scalar, tensor, or pseudoscalar, and $\Lambda_\mp Q\Lambda_\pm=\pm\Lambda_\mp\gamma_\mu\Lambda_\pm$ for vector or axial vector. See R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[18] W. Heisenberg, Rev. Mod. Phys. **29**, 269 (1957); H. P. Dürr, W. Heisenberg, H. Mitter, S. Schlieder, and K. Yamazaki, Z. Naturforsch. **14a**, 441 (1959). For the covariant form, see T. Kimura, Progr. Theoret. Phys. (Kyoto) **24**, 386 (1960).

[19] This field is also self-consistent in the sense that the stress-energy tensor serves, if $\kappa=2\epsilon$, as the source to the geometry in which the field equation is integrable; A. Inomata, Bull. Am. Phys. Soc. **10**, 577 (1965).

[20] The integrability condition of Eq. (24) is $R_{\kappa\lambda\mu\nu}\gamma^{\mu\nu}\chi=0$ (because the right-hand side of Eq. (21) vanishes for χ), from which Morinaga derived the equation $R_{\kappa\lambda\mu\nu}={}^*R_{\kappa\lambda\mu\nu}$ and obtained the pure gravitational space $R_{\mu\nu}=0$ as a necessary consequence; K. Morinaga, J. Sci. Hiroshima Univ. Ser. A **5**, 151 (1935). The present solution (22) is obviously another possible geometry which reduces to the pure gravitational case in the limit $\epsilon\rightarrow0$. Emphasis, however, must be placed on the point that the restricted neutrino field χ is a limiting case of the Heisenberg field. Otherwise the geometry (23) will be subject to ambiguity by a constant multiple.

of the null current $\varphi_\mu = i\bar{\chi}\gamma_\mu\chi$ vanishes because of the relation (25a)

$$\partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu = 0. \tag{27}$$

Hence the null space belongs to Peres' exceptional case.

Before closing this section, let us obtain a plane-wave-like solution of the restricted neutrino equation (24). The Peres equation (27) holds true if and only if φ_μ is the gradient of a scalar; say $\varphi_\mu = \partial_\mu \varphi$. On this substitution, Eq. (24) becomes

$$\nabla_\mu \chi = i\epsilon \partial_\mu \varphi \chi \tag{28}$$

and is readily solved with the field

$$\chi = e^{i\epsilon\varphi}\chi_0, \tag{29}$$

where χ_0 is a covariantly constant two-component spinor, i.e.,

$$\nabla_\mu \chi_0 = 0; \quad \chi_0 = \gamma_5 \chi_0. \tag{30}$$

On each local Minkowskian background, the solution (29) describes a plane wave

$$\chi = e^{ip_\mu x^\mu}\chi_0,$$

with momentum-energy $p_\mu = i\epsilon\bar{\chi}_0\gamma_\mu\chi_0$. In general, the *eikonal*-like function[21] φ can be found by a line integration,

$$\varphi(x) = \int \varphi_\mu dx^\mu + \varphi_0. \tag{31}$$

As the integral is independent of the path, it is completely determined up to an additive constant. The undertermined constant merely gives rise to a phase gauge of the field which is physically irrelevant.

IV. NEUTRINO STRESS-ENERGY TENSOR

In the general theory of relativity, the physical field serves as a source to the geometrized gravitational field through the Einstein equation

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}. \tag{32}$$

By the variational principle, the Einstein equation (32) is obtainable from an appropriate Lagrangian which consists of the geometrical and physical parts. The stress-energy tensor can be defined as the variational derivative of the physical part with respect to the metric. In order for the Newtonian correspondent to exist in the weak-field limit, the coupling constant κ in Eq. (32) must be universal for all massive sources ($\kappa = 1$ in natural units). There is, however, no particular experimental reason for applying the same argument to massless fields which do not have the rigorous Newtonian limit. For the moment, we leave the value of κ unspecified for convenience.

Using the Lagrangian from which the Dirac equation (17) is derived, one finds the stress-energy tensor for neutrinos,[22]

$$T_{\mu\nu} = \tfrac{1}{4}\{\bar{\psi}\gamma_\mu\nabla_\nu\psi - \nabla_\nu\bar{\psi}\gamma_\mu\psi + \bar{\psi}\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar{\psi}\gamma_\nu\psi\}. \tag{33}$$

On account of Eq. (17), the trace of this tensor vanishes

$$T^\mu{}_\mu = 0, \tag{34}$$

just as the trace of the Maxwell tensor does. Nevertheless, the fact that the field equation is needed in proving the vanishing trace contrasts with the situation of electromagnetism for which the trace disappears algebraically. The expression (34) is a dynamical consequence rather than a pure algebraic relation. The Rainich algebraic conditions, whether null or non-null, are not satisfied in general by the stress-energy tensor of the form (33), even though the field equation is taken into consideration.

In particular, for the restricted neutrino field χ, the stress-energy tensor (33) reduces to the simple form

$$T_{\mu\nu} = -\epsilon(\bar{\chi}\gamma_\mu\chi)(\bar{\chi}\gamma_\nu\chi), \tag{35}$$

which has the algebraic properties

$$T^\mu{}_\mu = 0; \quad T_{\mu\lambda}T_\nu{}^\lambda = 0. \tag{36}$$

The restricted field satisfies the relations (25a) and (25b), and the second relation has been used in reducing the tensor (33) to the form (35). Applying the first relation as well to the reduced tensor (35), we obtain the differential restriction on the neutrino tensor,

$$\nabla_\lambda T_{\mu\nu} = 0. \tag{37}$$

In the expression (35), positive-definiteness of T_{00} is not assured as it depends on the sign of ϵ, a quantity which is somehow related to the energy value of the field. For instance, in a local Lorentz frame, $T_{00} = \epsilon(\chi^\dagger\chi)^2$. In fact, the lack of positive-definiteness of the energy in the c-number theory is a feature of all fields with half-integral spin. As is well known, anticommutation rules are necessary for the spinor field in order to resolve the negative-energy difficulty. Unfortunately, such anticommutation relations are not compatible with the null property (15). The recent work of Klauder on the c-number quantization[23] could be of importance in this context.

For a source with vanishing trace, the Einstein equation becomes

$$R_{\mu\nu} = \kappa T_{\mu\nu}. \tag{38}$$

According to this equation, geometry must be endowed with the same properties as those of the physical source. Now one may raise the question: Would the c-number neutrinos demand a negative geometry ($R_{00} < 0$) as well as a positive geometry ($R_{00} > 0$) depending on the value of ϵ? Before answering this question, we recall that the

[21] For example, see L. Landau and E. Lifshitz, *The Classical Theory of Fields*, translated by M. Hamermesh (Addison-Wesley Publishing Company, Reading, Massachusetts, 1951), p. 136. The null relation (15′) corresponds to the eikonal equation.

[22] For a detailed account, see Ref. 15.

[23] J. R. Klauder, Ann. Phys. (N. Y.) **11**, 123 (1960).

restricted neutrino as a limit of the special Heisenberg field is meaningless in the negative null space. On comparison of the Ricci tensor (26) with the neutrino stress-energy tensor (35), it is seen that the theory is self-consistent only if

$$\kappa = 2\epsilon. \tag{39}$$

As a result, the geometry turns out to be positive-definite regardless of the sign of ϵ. If the energy concept at a local point could be carried over to curved space, we would say that the positive and negative states of neutrinos are degenerate in the geometry with the restricted source (35). At any rate, it is clear that the restricted class of neutrinos is a solution of the Rainich null-field problem.

V. NEUTRINO GEOMETRY

In the preceding section, we have seen that the restricted neutrino field may form a Rainich null geometry. Conversely, can a geometry suitable for the neutrino physics be singled out from general Rainich null spaces? In the following, we shall show that the null geometry of a spinor field in the two-component limit has a physical correspondent which obeys the neutrino equation in the circumstance that the null eigenvector of the curvature is the gradient of the chirality complexion of the field. This situation is quite analogous to the case of the Einstein-Maxwell geometry of non-null fields where the Rainich vector (2) composed of the curvature is taken as the gradient of the duality complexion of the electromagnetic field.

As the Maxwell stress-energy tensor remains unchanged under the duality transformation with a real function α (the α rotation),[12] the reduced neutrino tensor (35) is invariant under the chirality transformation with a real gauge β (the β rotation),[12]

$$\chi \rightarrow \chi' = e^{i\beta\gamma_5}\chi. \tag{40}$$

If the values of β are different at different points of space-time, $\nabla_\mu\chi$ transforms according to

$$\nabla_\mu\chi \rightarrow \nabla_\mu\chi + i\partial_\mu\beta\chi, \tag{41}$$

and hence the Dirac equation is transformed into

$$\gamma^\mu\nabla_\mu\chi + i\partial_\mu\beta\gamma^\mu\chi = 0, \tag{42}$$

which is not generally a free-neutrino equation. This implies that the stress-energy tensor of the form (35) does not necessarily represent the neutrino physics.

With a choice of the chirality function β such that $\partial_\mu\beta\gamma^\mu\chi = 0$, the transformed field again satisfies the neutrino equation. In this case, the gauge can have any value provided that its gradient is either orthogonal or proportional to φ_μ. For the restricted field, this degeneracy does not exist. Therefore, in a way analogous to the Einstein-Maxwell case, specifying an extremal field as standard of reference, one would be able to determine the restricted neutrino field by the chirality transformation with a very special gauge.

To see this, we choose the field χ_0 appearing in Eq. (30) as the extremal field. Now the chirality transformation generates on the basis of the extremal field a new field

$$\chi = e^{i\beta}\chi_0. \tag{43}$$

If the gauge function is taken as

$$\partial_\mu\beta = \epsilon\varphi_\mu, \tag{44}$$

then the field χ obeys the restricted neutrino equation (24) and coincides with the plane-wave-like solution (29). Following Misner and Wheeler's terminology of *duality complexion* in the case of electromagnetism, we call the particular gauge β in Eq. (44) the *chirality complexion* of neutrinos. Thus, by assuming the existence of the extremal field χ_0, and by imposing the condition that the null vector φ_μ is the gradient of the chirality complexion, we may read the restricted neutrino physics out of a Rainich null geometry.

From Eq. (25a), the chirality-invariant null current is covariantly constant or uniform. This uniform condition is, as is seen in Appendix B, sufficient as well as necessary for the null current to be the gradient of the chirality complexion of the restricted neutrino field. In summary, neutrino geometry is characterized by a uniform null curvature:

$$R=0;\ R_{00}\geqslant 0;\ \ R_{\mu\lambda}R_\nu{}^\lambda=0;\ \nabla_\lambda R_{\mu\nu}=0. \tag{45}$$

Since the uniform condition on the current (25a) is obviously stronger than the Peres condition (6), neutrino geometry is more restrictive than the Rainich-Peres null geometry. The uniform condition is the counterpart, in gravito-neutrinodynamics, of the Rainich-Misner-Wheeler equation in gravito-electrodynamics.

VI. CONCLUSIONS

As is shown in Table I, there is striking parallelism between the null neutrino situation and the non-null electromagnetic situation. The neutrino field is presented as a *square root* of the null curvature vector, in

TABLE I. Comparison between the non-null and null cases.

	Non-null case[a]	Null case
Geometry	$R_{\mu\nu}=\bar{\omega}_{\mu\lambda}\omega_\nu{}^\lambda$ $\bar{\omega}_{\mu\lambda}\omega^{\mu\lambda}=0;\ \bar{\omega}_{0\lambda}\omega_0{}^\lambda\geqslant 0$ $\partial_\mu\alpha_\nu-\partial_\nu\alpha_\mu=0$	$R_{\mu\nu}=2\epsilon^2\varphi_\mu\varphi_\nu$ $\varphi_\mu\varphi^\mu=0;\ \varphi_0\varphi_0\geqslant 0$ $\nabla_\mu\varphi_\nu=0$
Invariance groups	$\omega_{\mu\nu}\rightarrow e^{i\alpha_0}\omega_{\mu\nu}$	$\chi\rightarrow e^{i\beta_0}\chi$
Extremal fields	$\omega'_{\mu\lambda}\omega'_\nu{}^\lambda=\bar{\omega}'_{\mu\lambda}\bar{\omega}'_\nu{}^\lambda$	$\nabla_\mu\chi_0=0$
Complexions	$\alpha=\int\alpha_\mu dx^\mu+\alpha_0$	$\beta=\epsilon\int\varphi_\mu dx^\mu+\beta_0$
Physical fields	$\omega_{\mu\nu}=e^{i\alpha}\omega'_{\mu\nu}$	$\chi=e^{i\beta}\chi_0$
Field equations	$\nabla^\mu\omega_{\mu\nu}=0$	$\nabla_\mu\chi=i\epsilon\varphi_\mu\chi$
Physics	Gravito-electrodynamics	Gravito-neutrinodynamics

[a] $\omega_{\mu\nu}=f_{\mu\nu}+i(^*f_{\mu\nu})$ and $\bar{\omega}_{\mu\nu}=f_{\mu\nu}-i(^*f_{\mu\nu})$ are complex electromagnetic fields; see Ref. 13.

much the same way that the non-null electromagnetic field is described as a *square root* of the non-null curvature tensor. The chirality transformation defines the complexion of neutrinos, while the duality rotation designates the complexion of electromagnetism. The uniform condition bestows neutrinodynamics on a null geometry and the Rainich-Misner-Wheeler condition endows a non-null geometry with electrodynamics.

It is common to both the null and non-null cases that there is a free scale factor independent of all geometric conditions. In other words, if a curvature tensor $R_{\mu\nu}$ satisfies the algebraic and the differential conditions, so does another tensor which differs from the $R_{\mu\nu}$ by a constant multiple. On the other hand, the Einstein equation (38) shows that different multiples in geometry give rise to different field strengths in physics. Neither the duality rotation nor the chirality transformation can remove or generate such a multiple. In order that the theory be self-consistent, any dynamical restriction imposed on the physical field such as Eq. (24) must be completely integrable in the geometry formed with the restricted field itself. This requirement works to determine the multiple in principle. For a field which is two-component *ab initio*, the restricted Eq. (24) is integrable in a null geometry with an arbitrary multiple, and the ambiguity does still remain unresolved. To get rid of this difficulty, we have adopted the view that the neutrino is the two-component limit of a Heisenberg field rather than an essentially two-component field. The limiting procedure demands the gravitational coupling κ to be equated to the self-interaction 2ϵ, ensuring positive-definiteness of neutrino geometry. The Einstein equation is then put into the form

$$R_{\mu\nu}=2\epsilon T_{\mu\nu}(\epsilon)\,, \tag{46}$$

which signifies that the gravito-neutrino coupling is directly related to the internal structure of neutrinos.

Suppose the gravitational constant κ has a universal value as it does for all massive sources. Presumably, the most natural generalization of the idea of the universal gravitational interaction is to choose the value of κ as unity $(=8\pi G)$ in natural units. By Eq. (46), the coupling should be positive for the positive energy state $(T_{00}>0)$ and negative for the negative state $(T_{00}<0)$. As has been shown in Sec. III, the restricted neutrino equation (24) has a plane-wave-like solution. In a local Lorentz frame, the energy density of the field is given by $\pm\frac{1}{2}(\chi^\dagger\chi)^2$, the form of which is to be compared with that of the electromagnetic energy density $\frac{1}{2}(E^2+H^2)$. The field density itself can be identified with the energy density. Comparing, at the same local point, such a solution for the neutrino field and a plane-wave solution of the full Dirac equation (17), one may regard the restricted field as standing for a monochromatic wave carrying the energy $\epsilon(\chi^\dagger\chi)$. A general solution of the local Dirac equation is to be found by superposition. In general, however, the covariant Dirac equation is nonlinear in over-all character, so that the simple superposition principle is not applicable. Whether the set of restricted neutrino fields may cover all possible solutions of the Dirac equation (17) is yet an unanswered question.[24] A speculation has been made that the Eq. (24) would govern the geometrical behavior of two-component neutrinos in curved space.[25]

It is also important to see how the neutrino field relates to the Maxwell null field in the special case of the Rainich-Peres geometry. Could both neutrino and electromagnetic fields coexist in the null geometry? Could the Maxwell field perhaps be composed of neutrino fields? Or is there any way to discriminate between the Maxwell null geometry and the neutrino null geometry? At the present stage, however, none of these issues can be settled.

Finally, one may ask: If the two-component limiting process be essential, would the neutrino field be no more fundamental than the Heisenberg field, contrary to our earlier expectation? The Heisenberg nonlinear physics is not at all reproducible in the framework of Rainich geometry. Only in the two-component limit does the Heisenberg physics become significant in the Rainich scheme. As a matter of course, geometrodynamics by no means claims that Rainich geometry is the only way to describe all physical laws. A geometry other than Riemannian geometry might be adequate for the Heisenberg field. It is already known that the nonlinear terms can be understood geometrically as torsion of space-time.[26] For instance, a Finkelstein space[27] with uniform torsion is equivalent to the Rainich uniform null space.[28] Even for understanding the obvious coexistence of neutrinos and electromagnetism in reality, it seems inevitable to introduce a new degree of freedom such as torsion into geometry. Geometrodynamics asks one ultimately to fashion mass, field, and perhaps spin out of geometry itself. In an effort to speak of elementary particles in the geometric language, it is almost meaningless to deal from the beginning with the Dirac field with nonvanishing mass. If there exists a purely geometrical description for the Heisenberg physics, the field is apparently more general than that for neutrinos, whichever field is more fundamental.

ACKNOWLEDGMENTS

One of the authors (A. I.) would like to thank Professor G. Y. Rainich for stimulating conversations and Professor C. W. Misner for valuable suggestions.

[24] It is noteworthy that there is a similar solvable problem for neutrinos in the non-null case if four-component solutions are acceptable: A. Inomata, Bull. Am. Phys. Soc. **10**, 88 (1965). Unfortunately, such a solution loses its meaning in the two-component limit. Details of this non-null Rainich problem will be given elsewhere.

[25] A. Inomata and W. A. McKinley, Bull. Am. Phys. Soc. **9**, 450 (1964).

[26] T. W. B. Kibble, J. Math. Phys. **2**, 212 (1961); D. W. Sciama, an article in *Recent Developments in General Relativity*, dedicated to Leopold Infeld (Pergamon Press, Inc., New York, 1962), p. 415.

[27] R. Finkelstein, Ann. Phys. (N. Y.) **15**, 223 (1961).

[28] A. Inomata (to be published).

APPENDIX A

Here we prove Eqs. (20a) and (20b). The proofs are direct if we use the following identities:

$$\gamma_\mu\gamma_\lambda\gamma_\nu-\gamma_\nu\gamma_\lambda\gamma_\mu=2ie_{\mu\lambda\nu\kappa}\gamma^\kappa\gamma_5\,, \tag{A1}$$

$$\gamma_\mu\gamma_\lambda\gamma_\nu+\gamma_\nu\gamma_\lambda\gamma_\mu=2(g_{\lambda\nu}\gamma_\mu-g_{\nu\mu}\gamma_\lambda+g_{\lambda\mu}\gamma_\nu)\,. \tag{A2}$$

On substitution of Eqs. (18) and (19), the left-hand side of Eq. (20a) becomes

$$\nabla_\mu A_\nu=-\tfrac{1}{2}i\epsilon(\bar\psi\gamma^\lambda\gamma_5\psi)\{\bar\psi(\gamma_\mu\gamma_\lambda\gamma_\nu-\gamma_\nu\gamma_\lambda\gamma_\mu)\psi\}\,.$$

Use of (A1) readily leads to

$$\nabla_\mu A_\nu=-\epsilon e_{\mu\lambda\nu\kappa}(\bar\psi\gamma^\kappa\gamma_5\psi)(\bar\psi\gamma^\lambda\gamma_5\psi)=0\,. \tag{A3}$$

The two-component current φ_μ is also necessarily uniform;

$$\nabla_\mu\varphi_\nu=0\,. \tag{A4}$$

Next, substituting Eqs. (18) and (19) into the left-hand side of Eq. (20b) yields

$$U_{\mu\nu}=+\tfrac{1}{2}\epsilon(\bar\psi\gamma^\lambda\gamma_5\psi)\{\bar\psi(\gamma_\mu\gamma_\lambda\gamma_\nu+\gamma_\nu\gamma_\lambda\gamma_\mu)\gamma_5\psi\}-4\epsilon(\bar\psi\gamma_\mu\gamma_5\psi)(\bar\psi\gamma_\nu\gamma_5\psi)\,.$$

Making use of (A2), we find

$$U_{\mu\nu}=-2\epsilon\{(\bar\psi\gamma_\mu\gamma_5\psi)(\bar\psi\gamma_\nu\gamma_5\psi)+\tfrac{1}{2}g_{\mu\nu}(\bar\psi\gamma_\lambda\gamma_5\psi)(\bar\psi\gamma^\lambda\gamma_5\psi)\}$$

or

$$U_{\mu\nu}=U_{\nu\mu}=2\epsilon(A_\mu A_\nu+\tfrac{1}{2}g_{\mu\nu}A_\lambda A^\lambda)\,. \tag{A5}$$

In the two-component limit,

$$U_{\mu\nu}=2\epsilon\varphi_\mu\varphi_\nu\,. \tag{A6}$$

APPENDIX B

In the following we wish to show that the uniform condition (A4) is sufficient for the two-component spinor field χ to be the restricted neutrino field.

We assume that the covariant derivative of a spinor field can be expressed as a linear combination of 16 independent products of the γ matrices, and write

$$\nabla_\mu\chi=\Sigma_\mu\chi\,. \tag{B1}$$

In this case, the field χ is subject to the constraint $\chi=\gamma_5\chi$ and hence Σ_μ takes the simple form

$$\Sigma_\mu=iA_\mu+C_{\mu\alpha\beta}\gamma^{\alpha\beta}\,. \tag{B2}$$

Now the uniform condition requires that A_μ and $C_{\mu\alpha\beta}$ be real. Furthermore, $C_{\mu\alpha\beta}$ must satisfy the condition

$$C_{\mu\alpha\beta}\varphi^\beta=0\,. \tag{B3}$$

Since the Fock-Ivanenko connection involves an arbitrary vector field, the field A_μ may be adjusted so as to vanish from the right-hand side of Eq. (B1). Since the field $C_{\mu\alpha\beta}$ under the condition (B3) has only eight independent components, we can express it in terms of the two real vector fields C_μ and D_μ which are orthogonal and proportional to φ_μ, respectively,

$$C_{\mu\alpha\beta}=C_\mu C_\alpha D_\beta-C_\mu C_\beta D_\alpha+ie_{\mu\alpha\beta\sigma}D^\sigma\,. \tag{B4}$$

On insertion of (B4) into (B1), the first and second terms disappear because of the null property of φ_μ. The third term alone contributes to (B1). Again making use of the null property of φ_μ, we obtain

$$\nabla_\mu\chi=i\epsilon\varphi_\mu\chi\,, \tag{B5}$$

where we have put $D_\mu=\epsilon\varphi_\mu$. Although the proportionality constant seems arbitrary, once the extremal field has been specified, the chirality transformation determines the restricted neutrino field uniquely up to a constant-phase gauge.