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Calculations of Neutron–Deuteron Scattering*

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Exact three-body calculations using separable two-body interactions have been carried out for the neutron-deuteron system at low and intermediate energies. Binding energies, scattering lengths, angular distributions, and total cross sections for *n-d* scattering and the total break-up cross sections for $n+d\rightarrow n+n+p$ have been obtained. These agree well with experimental data where available, indicating that nucleon exchange is the dominant mechanism at these energies. Other theoretical results presented are the partial-wave scattering amplitudes, phase shifts, and angular distributions for the quartet and doublet spin states of the *n-d* system. Diffraction scattering is noted in the doublet state, and its development as a function of energy is traced. Further refinements and future calculations are discussed and outlined.

I. INTRODUCTION

In this paper, we report on calculations of neutron-deuteron scattering at low and intermediate energies and of the triton binding energy. These are exact three-body calculations including the effect of the spin dependence of nuclear forces and may be viewed as an extension of our previous spin-independent calculations.[1] As before, the exact three-body problem is tamed by the introduction of separable interactions. The use of these interactions leads us to equations having only one vector (or after a partial-wave analysis, one scalar) intermediate variable, and therefore equations that are amenable to solution with present computational facilities. The introduction of two interactions gives us two coupled equations, but still in one variable. In fact, it is clear that there will be as many coupled equations as separable interactions, but always all will be one-variable equations.

Our solutions of the equation agree very well with experiment where data are available. Angular distributions, total cross sections, scattering lengths, and binding energies are all very well given, particularly when the effects of the tensor force are approximately taken into account. We have shown that the major features of the low-energy three-nucleon system can be accounted for by simple interaction mechanisms so long as three-body effects are *fully* and *exactly* taken into account. We even show that at intermediate energies our theory is capable of producing genuine diffraction behavior.

In Sec. II, the theory is briefly reviewed, and the equations we are solving are explicitly written down. The method for fixing the two-body parameters in terms of the two-body scattering data is also given. In Sec. III, our results are presented. These include angular distributions, total cross sections, and scattering lengths for $n-d$ scattering and the triton binding energy. Experiment is compared with where available. So far we have not calculated the break-up process $n+d\rightarrow n+n+p$, but are embarking on this. In Sec. IV, further approximation schemes are discussed and future calculations outlined. In the Appendix, the spin algebra is given.

* Supported in part by the National Science Foundation.

[1] R. Aaron, R. D. Amado and Y. Y. Yam, Phys. Rev. **136**, B650 (1964).

II. THEORY

The trouble with the classical three-body problem is the multiplicity of coordinates. Although recent developments have removed certain formal difficulties in the quantum mechanical three-body problem and produced a set of well-defined equations,[2] they have left this problem of the coordinates.

Calculational progress in three-body systems depends on circumventing this multiplicity without sacrificing the three-body nature of the problem. Genuine three-body equations involving few coordinates in intermediate states can be obtained by using separable two-body potentials in the Schrödinger equation[3] or equivalently by writing three-body scattering equations in which pairs interact via a quasiparticle.[4] Recently, it has been shown by a number of authors[5] that using separable two-body potentials in the newly formulated exact three-body Faddeev formalism leads to these same equations. The physical advantage of all these schemes is that it is possible by using separable interactions between pairs to take exact account in the three-body system of certain parts of the two-body interaction and still solve the problem.

We have previously studied the minimal theory of

[2] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960). [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)]; C. A. Lovelace, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Plenum Press, New York, 1964); S. Weinberg, Phys. Rev. **133**, B232 (1964).

[3] A. N. Mitra, Nucl. Phys. **32**, 529 (1962).

[4] R. D. Amado, Phys. Rev. **132**, 485 (1963).

[5] L. Rosenberg, Phys. Rev. **135**, B715 (1964). J. H. Hetherington and L. H. Schick, *ibid.* **137**, B935 (1965).

nucleon-deuteron scattering in this way.[1] Any such theory must have enough of the two-body interaction to give the deuteron. The simplest three-body theory of this system and the one we studied is one which has no more than that. It gives qualitative agreement with the trends of the three-nucleon system. Its most glaring fault for low energies is neglect of the spin dependence of nuclear forces. In this paper, we take this into account. We deal then with real nucleons having spin and isotopic spin and interacting differently in the spin triplet and spin singlet states. To avoid the complication of the Coulomb interaction, we concentrate on neutron-deuteron scattering. The equations for this scattering are given here in the language of the quasiparticle method. One says that each time a spin-triplet pair interacts they form a "d," and each time a spin-singlet pair interacts they form a "ϕ." The renormalized parameters of the interactions in each of these channels are then chosen to fit the physical low-energy triplet and singlet nucleon-nucleon data, so that, for example, we have no "physical" bound ϕ.

Since we are not including tensor forces or spin-orbit forces, it is convenient to study the problem in the $L-S$ representation. There are two possible values of the total spin, $\frac{3}{2}$ and $\frac{1}{2}$, and the amplitudes do not couple between them. We need only study the state of total isotopic spin $\frac{1}{2}$ and z-component $-\frac{1}{2}$. A discussion of the spin and isospin factors is given in the Appendix. The scattering amplitudes in the center-of-mass system are then given by

$(\hbar=2M_n=1)$

$$(\mathbf{k},d|t(E)|\mathbf{k}',d)=\chi_{dd}(\mathbf{k},d|B(E)|\mathbf{k}',d)+\frac{1}{(2\pi)^3}\int d^3p\,\chi_{dd}(\mathbf{k},d|B(E)|\mathbf{p},d)P_d(p^2;E)(\mathbf{p},d|t(E)|\mathbf{k}',d)$$
$$+\frac{1}{(2\pi)^3}\int d^3p\,\chi_{d\phi}(\mathbf{k},d|B(E)|\mathbf{p},\phi)P_\phi(p^2;E)(\mathbf{p},\phi|t(E)|\mathbf{k}',d). \quad (1a)$$

$$(\mathbf{k},\phi|t(E)|\mathbf{k}',d)=\chi_{\phi d}(\mathbf{k},\phi|B(E)|\mathbf{k}',d)+\frac{1}{(2\pi)^3}\int d^3p\,\chi_{\phi\phi}(\mathbf{k},\phi|B(E)|\mathbf{p},\phi)P_\phi(p^2;E)(\mathbf{p},\phi|t(E)|\mathbf{k}',d)$$
$$+\frac{1}{(2\pi)^3}\int d^3p\,\chi_{\phi d}(\mathbf{k},\phi|B(E)|\mathbf{p},d)P_d(p^2;E)(\mathbf{p},d|t(E)|\mathbf{k}',d). \quad (1b)$$

The momentum labels the nucleon momentum and ϕ or d labels which pair state the nucleon is incident on. The χ's are spin and isospin factors. In the quartet state, only the $n-d$ channel contributes, and we have $\chi_{dd}=-1.0$ and $\chi_{d\phi}=\chi_{\phi d}=\chi_{\phi\phi}=0$. In the doublet state, $\chi_{dd}=\chi_{\phi\phi}=0.5$ and $\chi_{d\phi}=\chi_{\phi d}=-1.5$. Here B is the Born approximation representing the exchange of a nucleon between the pairs and P the full propagator in the intermediate state. The explicit forms are given below. E is the total energy variable. These equations,

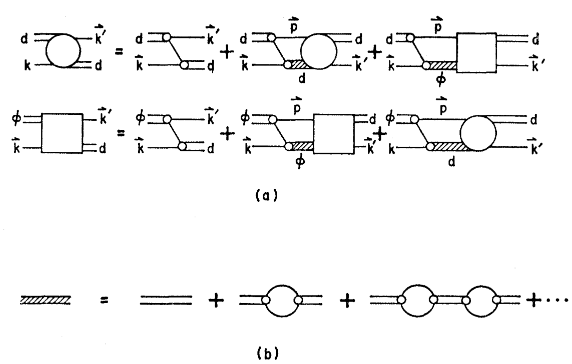

FIG. 1. (a) Coupled integral equations for neutron-deuteron scattering. (b) Perturbative representation of the full propagator P. The single line represents a nucleon; the double lines, a two-nucleon pair—ϕ for the singlet and d for the triplet. The small circle is the nucleon-nucleon vertex; the large circle and rectangle are the three-body amplitudes.

like most linear scattering equations, involve going off the energy shell. The equations are represented diagrammatically in Fig. 1. They are the two-channel analogs of the equations derived in Ref. 4. The simplification introduced by the separable interactions is clear, since the intermediate states of the equation are labeled by only one vector variable. After a partial-wave analysis, this goes to one scalar variable. Thus we need only solve a one-dimensional Fredholm integral equation. All this in spite of the fact that three-body effects—such as the possibility of $n+d\to n+n+p$—are exactly taken into account. The introduction of two interactions has given us two coupled integral equations to solve. In fact, it is clear that more and more separable interactions or quasiparticles can be introduced, and thus the two-body force approximated more and more closely, and one will just get more and more coupled one-variable equations. The scheme for solving the three-body problem is thus to replace, in each partial wave, the exact single multivariable integral equation with local potentials by a set (infinite in general) of coupled one-dimensional equations, and then approximate this set by truncating it where money, computer time, and physical honesty will allow. Returning to Eq. (1), we give the explicit form of the terms. As before, we choose Hulthén forms[6]

[6] L. Hulthén and M. Sugawara, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.

for the vertex functions. We then get ($\hbar = 2M_n = 1$)

$$(\mathbf{k},d|B(E)|\mathbf{k}',d) = \frac{\gamma_d^2}{[(\mathbf{k}'+\frac{1}{2}\mathbf{k})^2+\beta_d^2][E-k^2-k'^2-(\mathbf{k}+\mathbf{k}')^2+i\eta][(\mathbf{k}+\frac{1}{2}\mathbf{k}')^2+\beta_d^2]}, \tag{2a}$$

$$(\mathbf{k},\phi|B(E)|\mathbf{k}',\phi) = \frac{\gamma_\phi^2}{[(\mathbf{k}'+\frac{1}{2}\mathbf{k})^2+\beta_\phi^2][E-k^2-k'^2-(\mathbf{k}+\mathbf{k}')^2+i\eta][(\mathbf{k}+\frac{1}{2}\mathbf{k}')^2+\beta_\phi^2]}, \tag{2b}$$

$$(\mathbf{k},\phi|B(E)|\mathbf{k}',d) = (\mathbf{k}',d|B(E)|\mathbf{k},\phi) = \frac{\gamma_\phi\gamma_d}{[(\mathbf{k}'+\frac{1}{2}\mathbf{k})^2+\beta_\phi^2][E-k^2-k'^2-(\mathbf{k}+\mathbf{k}')^2+i\eta][(\mathbf{k}+\frac{1}{2}\mathbf{k}')^2+\beta_d^2]}, \tag{2c}$$

$$P_d(p^2;E) = \left\{[\sigma+\epsilon]\left[Z-\frac{\gamma_d^2}{(2\pi)^3}\int\frac{d^3n}{(\beta_d^2+n^2)^2(\epsilon+2n^2)^2(\sigma-2n^2+i\eta)}\right]\right\}^{-1}, \tag{2d}$$

$$P_\phi(p^2;E) = -\left[1+\frac{\gamma_\phi^2}{(2\pi)^3}\int\frac{d^3n}{(n^2+\beta_\phi^2)^2(\sigma-2n^2+i\eta)}\right]^{-1}, \tag{2e}$$

and

$$\sigma = E-\tfrac{3}{2}p^2, \tag{2f}$$

where γ_ϕ and γ_d are the coupling constants of the nucleon to the ϕ and d quasiparticles. β_ϕ and β_d are the vertex ranges in momentum space, and ϵ is the deuteron binding energy. Z is the wave function renormalization constant of the deuteron. It should strictly be zero to represent a composite deuteron.[7] Making it greater than zero weakens the nucleon-nucleon force in the spin triplet state, but leaves the binding energy of the deuteron fixed. We shall have recourse to this later.

We may express the parameters of Eq. (2) in terms of the low-energy parameters of the nucleon-nucleon system. In terms of the singlet scattering length a_s and effective range r_s, we have

$$\beta_\phi = \frac{3}{2r_s}\left[1+\left(1-\frac{16r_s}{9a_s}\right)^{1/2}\right], \tag{3a}$$

$$\gamma_\phi^2 = \frac{16\pi\beta_\phi^4 a_s}{a_s\beta_\phi-2}; \tag{3b}$$

and in terms of the deuteron binding energy ϵ, triplet scattering length a_t, and Z, we have

$$\frac{1}{a_t} = \frac{16\pi\alpha_d^2\beta_d^4 Z}{\gamma_d^2}+\frac{\alpha_d\beta_d(2\beta_d+\alpha_d)}{2(\alpha_d+\beta_d)^2}, \tag{4a}$$

$$\gamma_d^2 = 32\pi\alpha_d\beta_d(\alpha_d+\beta_d)^3(1-Z), \tag{4b}$$

where

$$\alpha_d^2 = \epsilon/2. \tag{4c}$$

The values we use for the physical parameters[8] are given in Table I. We have not investigated in detail the effects of varying them in the three-body system, but would expect similar variations as in our previous work.[1]

It remains only to solve the equations. This is done by making a partial wave decomposition and then writing the integrals as sums and turning the equations into matrix equations. The coupling of the channels doubles the size of the matrix for a given integration mesh. We are interested in two general problems, the there-body bound state and the scattering problem. For the former, E is negative, but all the matrix elements are real. For the scattering, we put k' on the energy shell as $E=\frac{3}{4}k'-\epsilon$, but now the elements are complex so the matrix size again doubles. Furthermore, the presence of singularities in the matrix elements makes them rapidly varying, and considerable care must be exercised in the choice of grid, etc. This is particularly true above the threshold for deuteron breakup. Even when using Gaussian quadratures, we find that we need 35 mesh points to give 5% stability against mesh variation. This means that our matrices are 140×140, which comes very close to filling the memory of a large computer and places a limit on including more channels. Having chosen a mesh, the equations can be inverted by the IBM Share subroutine MATINV. Threshold singularities and propagator poles are treated by the same prescriptions as those in the one-channel case.[1] For the higher energies and higher partial waves, it is not necessary to invert the equations,

TABLE I. Low-energy parameters of the two-nucleon system. a_t and a_s are the triplet and singlet scattering lengths, respectively. r_s is the singlet effective range and ϵ, the deuteron binding energy. See Ref. 8.

$a_t=$	5.41 F
$a_s=$	−23.78 F
$r_s=$	2.67 F
$\epsilon=$	2.226 MeV

[7] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

[8] M. J. Moravcsik, *The Two-Nucleon Interaction* (Oxford University Press, New York, 1963).

TABLE II. Summary of experimental and theoretical results for the low-energy neutron-deuteron system. The triton binding energy is in MeV. The quartet and doublet scattering lengths ($a_{3/2}$ and $a_{1/2}$) are in fermis. See Refs. 11 and 12 for references to the theoretical and experimental results, respectively.

	S.K.	Mitra *et al.* Tensor force	Mitra *et al.* Central force	A.A.Y. $Z=0$	A.A.Y. $Z=0.0496$	Experiment Set I	Experiment Set II
$a_{3/2}$	6.28	···	6.19 → 6.38[b]	6.32	6.20	6.38±0.06	2.6 ±0.2
$a_{1/2}$	−2.76	···	−2.62 → −1.31[b]	−1.04	0.7	0.7 ±0.3	8.26±0.12
E_T	12.5	8.85 → 10.4[a]	11.8 → 12.2[a]	11.01	8.53	8.49	

[a] B. S. Bhakar and A. N. Mitra, see Ref. 11.
[b] V. S. Bhasin, G. L. Shrenk, and A. N. Mitra, see Ref. 11.

but rather they can be solved by iteration. The Neumann series for the equations converges at sufficiently high energy.[9] The crossover point from inversion to iteraction is chosen empirically by seeing to it that both methods agree.

III. RESULTS

In this section we present some of the results obtained from solving Eq. (1). We begin with the bound state and zero-energy scattering data. Some of these results have been presented before,[10] and are very similar to some previous results of other groups.[11] Experimentally, the results we are aiming at are a triton-binding energy of 8.49 MeV and the doublet and quartet neutron-deuteron scattering lengths. Because of the way these are determined,[12] two possible sets agree with experiment and they are shown in Table II.

With the physical parameters chosen as in Table I, and with the deuteron taken as a composite particle with zero wave function renormalization, our calculation gives the result labeled $Z=0$ in Table II. In the quartet state the s-wave interaction is repulsive, there is no bound state, and the scattering length is large and positive. In the doublet state, putting Z equal to zero leads to too much binding for the triton and correspondingly too small a doublet scattering length, when viewed from Set I. That this is true about the scattering length can be seen if one recalls that increasing the strength of an interaction so that the bound state goes from just barely bound to there being nearly a second bound state corresponds to the scattering length from $+\infty$ to $-\infty$ and passing through zero in between. Our values are near this zero. Thus $a_{1/2}=-1.04$ F goes with *more* binding than $a_{1/2}=0.7$ F. That one gets too much binding with our forces is not a surprise. We have left out a number of repulsive effects—among these are hard cores and the effects of the tensor force. The tensor force is relatively more repulsive in the three-body system because the triton, being a much more symmetric object then the deuteron, has less tensor force contribution to its binding. Thus, if we fit the deuteron with a purely central force, we overestimate the effect of this central triplet force in the triton. This can be rectified by making $Z\neq0$. One can think of Z not as a measure of the elementarity of the deuteron, but rather as a measure of the fraction of the time that the deuteron *is not* an s-wave triplet pair. The noncentral force responsible for this fraction should not be very effective in the triton. Only in a field-theoretic formalism can we introduce this flexibility without changing the physical energy eigenvalue of the deuteron. To implement this idea we recalculate the doublet scattering length with Z as a parameter and vary Z to give the value of set I. This gives the results shown in Table II marked $Z=0.0496$. We see that the triton binding energy is also well given and the value of Z is a reasonable number for the fraction of noncentral contribution to the deuteron. Hence we believe that the picture emerges with considerable logical consistency as well as agreement with experiment.

A calculation also fitting low-energy nucleon-nucleon parameters with separable potentials and then looking for three-body bound states has recently been reported by Bander.[13] He finds too much binding for the triton and finds a second bound state. However, his separable potentials do not cut off as quickly as ours at large momentum and therefore are more attractive than ours or than are warranted by nucleon-nucleon data above the effective range region. We have already seen[1] that a little attraction can give a second bound state and presumably that is what happens in Bander's case. He also points out that one of the bound states shows up in the scattering amplitude as a pole with the "wrong sign" for the residue. Following S-matrix theory usage, he calls this a "ghost." It is certainly not a ghost in the field theory sense of a state vector with negative norm. In an exact solution of Schrödinger's equation with a bounded, Hermitian Hamiltonian, no such states can exist, and Bander's solution is such an exact solution, as is ours. It is true that in local field theory the sign of the residue is related to the norm, but we are certainly not studying local field theory. No doubt the use of nonlocal potentials does some damage to the analytic properties of the amplitude. Many of the potentials used in nuclear physics (harmonic oscillators, square well) do as well, but they are thought to represent the more correct potential in some limited region of energy and momentum. The hope is that separable potentials do as well, even if in Bander's

[9] Y. Y. Yam, thesis, University of Pennsylvania, 1965 (unpublished).
[10] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. Letters 13, 574 (1964).
[11] A. N. Mitra and V. S. Bhasin, Phys. Rev. 131, 1265 (1963); A. G. Sitenko and V. F. Kharchenko, Nucl. Phys. 49, 15 (1963); V. S. Bhasin, G. L. Schrenk, and A. N. Mitra, Phys. Rev. 137, B398 (1965); B. S. Bhakar and A. N. Mitra, Phys. Rev. Letters 14, 143 (1965).
[12] D. Hurst and N. Alcock, Can. J. Phys. 29, 36 (1951).
[13] M. Bander, Phys. Rev. 138, B322 (1965).

TABLE III. Scattering amplitudes for neutron-deuteron scattering in the two-spin states (quartet and doublet) for various values of the orbital angular momentum (l) and laboratory energy of the incident neutron (E_{lab}). Energies are in units of MeV; phase shifts are in units of π radians; the real part ($\mathrm{Re}t_l$) and imaginary part ($\mathrm{Im}t_l$) of the scattering amplitude are in units such that $\hbar=2M_n=1$ and $\epsilon=1.5$. Its normalization is given in the text.

		Quartet			Doublet		
E_{lab}	l	$\mathrm{Re}t_l$	$\mathrm{Im}t_l$	$\mathrm{Re}\delta_l$	$\mathrm{Re}t_l$	$\mathrm{Im}t_l$	$\mathrm{Re}\delta_l$
2.45	0	+8.4745	−18.020	−0.3601	+6.2457	−1.9445	+0.9039
	1	−7.7758	−3.2184	+0.1249	+2.3423	−0.2519	−0.0341
	2	+1.3048	−0.07765	−0.0189	−0.7220	−0.02372	+0.0105
	3	−0.2696	−0.00330	+0.0039	+0.1324	−0.000796	−0.0019
	4	+0.5541	−0.000139	−0.0008	−0.02769	−0.000035	+0.0004
3.27	0	+5.7422	−17.122	−0.3970	+6.4998	−2.5628	+0.8805
	1	−7.2540	−3.3528	+0.1378	+1.9542	−0.2027	−0.0329
	2	+1.4845	−0.1164	−0.0249	−0.8376	−0.03690	+0.0140
	3	−0.3495	−0.00641	+0.0058	+0.1708	−0.00153	−0.0029
	4	+0.08235	−0.000356	−0.0014	−0.04118	−0.000089	+0.0007
5.5	0	+0.5500	−14.650	−0.4880	+5.560	−4.030	+0.8355
	1	−6.0040	−3.190	+0.1537	+0.4500	−0.463	−0.0104
	2	+1.6500	−0.192	−0.0361	−1.0030	−0.0970	+0.0219
	3	−0.4650	−0.0149	+0.0101	+0.2210	−0.00542	−0.0048
	4	+0.1347	−0.00125	−0.0029	−0.06733	−0.000438	+0.0015
	5	−0.01188	−0.000108	+0.0009	+0.01985	−0.000036	−0.0004
9.0	0	−1.880	−11.110	+0.4464	+3.890	−4.960	+0.7815
	1	−4.709	−2.836	+0.1620	−0.770	−1.290	+0.0273
	2	+1.564	−0.2550	−0.0442	−1.037	−0.224	+0.0296
	3	−0.502	−0.0259	+0.0139	+0.226	−0.0231	−0.0063
	4	+0.1684	−0.002924	−0.0047	−0.08423	−0.002634	+0.0023
	5	−0.05713	−0.000340	+0.0016	+0.02856	−0.000326	−0.0008
	6	+0.01967	−0.000041	−0.0005	−0.009832	−0.000042	+0.0003
14.1	0	−3.060	−7.30	+0.3655	+2.140	−5.30	+0.6988
	1	−3.590	−2.398	+0.1629	−1.390	−1.82	+0.0741
	2	+1.313	−0.303	−0.0473	−0.962	−0.336	+0.0354
	3	−0.469	−0.0388	+0.0164	+0.194	−0.049	−0.0068
	4	+0.1734	−0.00569	−0.0060	−0.0867	−0.00763	+0.0030
	5	−0.0645	−0.000865	+0.0022	+0.0323	−0.00127	−0.0011
	6	+0.02438	−0.000137	−0.0008	−0.0122	−0.000219	+0.0004
	7	−0.00931	−0.000023	+0.0003	+0.00465	−0.000039	−0.0002

case they may lead to three-body bound state wave functions with unusual asymptotic properties in configuration space (again as do harmonic oscillators). It is an unfortunate doctrine of S-matrix theory that the scattering amplitude at the bound state pole has more physical meaning "on-the-energy-shell" when momenta are pure imaginary, than at that energy pole with the momenta real.

We turn now to the calculations of scattering. The results of the calculation are the real and imaginary parts of the amplitudes in each partial wave in the doublet and quartet state. These are given in Table III. Also given is the phase shift. This is only a complete description below the deuteron breakup threshold; above it, the real part of the phase-shift is defined by

$$\tan[\mathrm{Re}\delta_l]=\frac{\mathrm{Im}t_l}{\mathrm{Re}t_l}-\frac{3\pi}{k\,\mathrm{Re}t_l}\times\left\{1-\left[1+\frac{2k}{3\pi}\left(\mathrm{Im}t_l+\frac{k}{6\pi}|t_l|^2\right)\right]^{1/2}\right\}. \quad (5)$$

The normalization of the amplitudes is such that the elastic unitarity relation is of the form

$$\mathrm{Im}t_l=-(k/6\pi)|t_l|^2. \quad (6)$$

In the table, we have adopted a convention for the phase shifts according to Levinson's theorem,[14] so that for example the doublet s-wave phase shift is π for $k=0$. The proper branch of the arctan is chosen to give a continuous curve across the break-up threshold.

To compare with experiment, the results of Table III are assembled into angular distributions, etc. Good angular-distribution measurements of $n-d$ scattering exist at 2.45,[15] 3.27,[15] 14.1,[16] and 5.5 and 9.0 MeV,[17] lab neutron energies. Our theory is compared with each of these in Figs. 2–6. The agreement is generally very good. The backward peaking is due to the basic exchange nature of the interaction. This is brought out in Fig. 2, where the Born approximation is also plotted and shows both the clear backward peaking and the clear inadequacy of this approximation. In the forward direction, unitarity raises the amplitude. This is

[14] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **25**, 9 (1949).

[15] J. D. Seagrave and L. Cranberg, Phys. Rev. **105**, 1816 (1957).

[16] J. C. Allard, A. H. Armstrong, and L. Rosen, Phys. Rev. **91**, 90 (1953); J. D. Seagrave, *ibid.* **97**, 757 (1954).

[17] The 5.5-MeV *n-d* scattering data are those of E. Wantuch [Phys. Rev. **84**, 169 (1951)]. The theoretical curves are calculated for 5.5 and 9.0 MeV. The 5.64- and 9.0-MeV data are those of Bonner *et al.* [B. E. Bonner, thesis, Rice University, 1965 (unpublished)]. We are grateful to Dr. Bonner for making these data available to us.

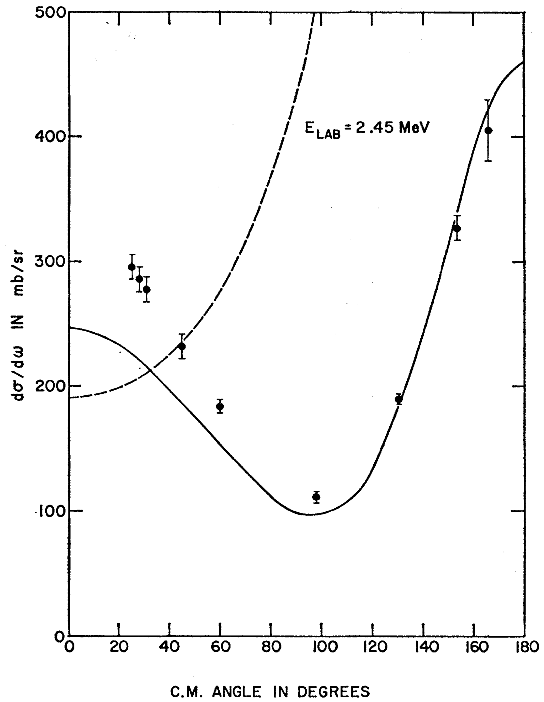

FIG. 2. Experimental and theoretical angular distributions for neutron-deuteron scattering at 2.45 MeV. The dashed line is the Born approximation which rises monotonically to 1560 mb/sr at 180°. The experimental data are those of Seagrave and Cranberg. See Ref. 15.

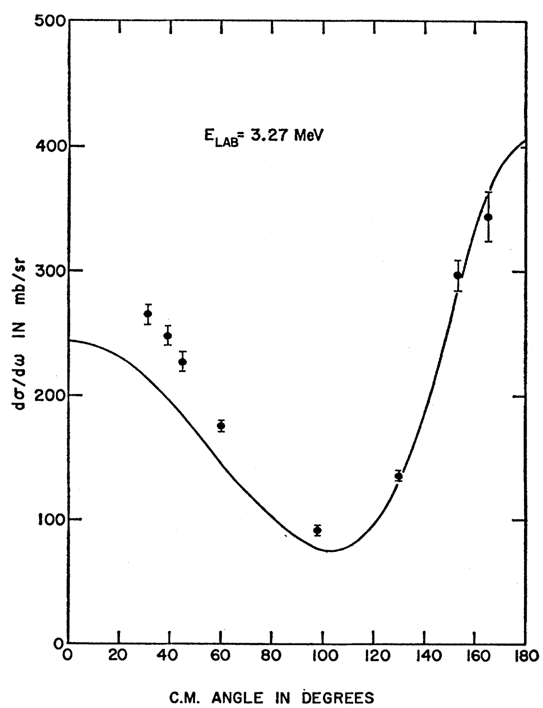

FIG. 3. Experimental and theoretical angular distributions for neutron-deuteron scattering at 3.27 MeV. The experimental data are those of Seagrave and Cranberg. See Ref. 15.

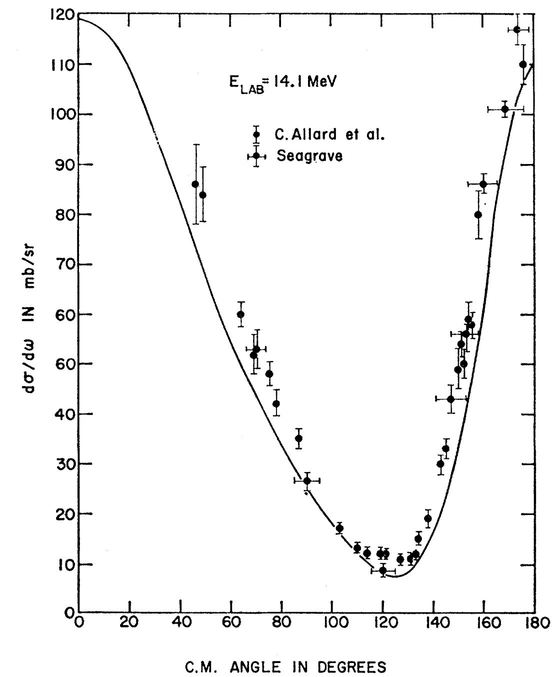

FIG. 4. Experimental and theoretical angular distributions for neutron-deuteron scattering at 14.1 MeV. See Ref. 16.

particularly true at 14.1 MeV, where the forward peak is largely the shadow of the break-up process. The theory is a bit low in the forward direction, probably because we have left out the high momentum components of

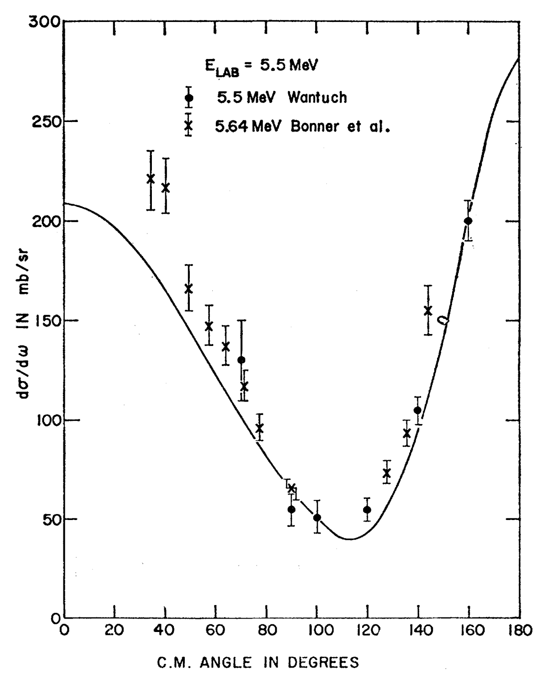

FIG. 5. Experimental and theoretical angular distributions for neutron-deuteron scattering at 5.5 MeV. See footnote 17.

the two-body force. These could be included in an impulse approximation. In fact, impulse approximation calculations at higher energies have given good results in the forward direction but not at backward angles.[18] We hope to study the combination of these two. It should be noted that except at the lowest energies where the effect of the triton pole is important, the difference between $Z=0$ and $Z=0.0496$ is very small and we have set $Z=0$.

The total cross section and the break-up cross section are shown together in Fig. 7 and compared with experiment.[19] At zero energy our theory is adjusted to give the correct answer, since it gives the correct scattering lengths. Away from zero, the fact that our angular distributions are a little low at forward angles

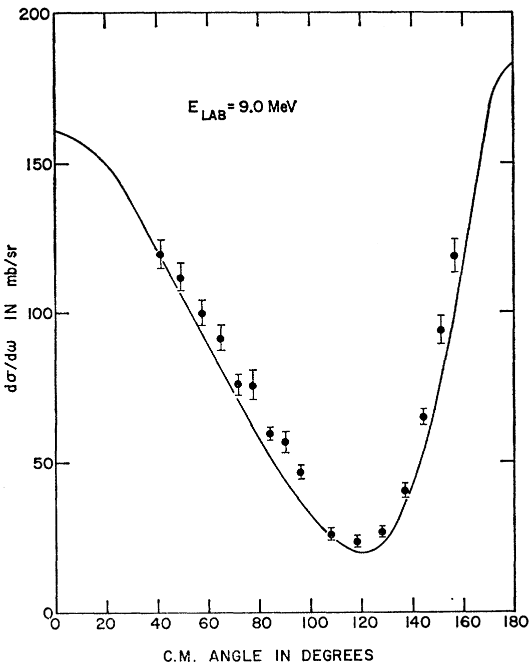

FIG. 6. Experimental and theoretical angular distributions for neutron-deuteron scattering at 9.0 MeV. See footnote 17.

gives a total cross section a little too small; but still the general agreement is gratifying and indicates that the bulk of the reaction is being properly described.

It is interesting to compare separately the contribution of the quartet and doublet states to the angular distributions. These add, of course, incoherently. From Table III we see that the doublet s-wave amplitude is small. We should expect this since the scattering length is minus the slope of the phase shift at threshold

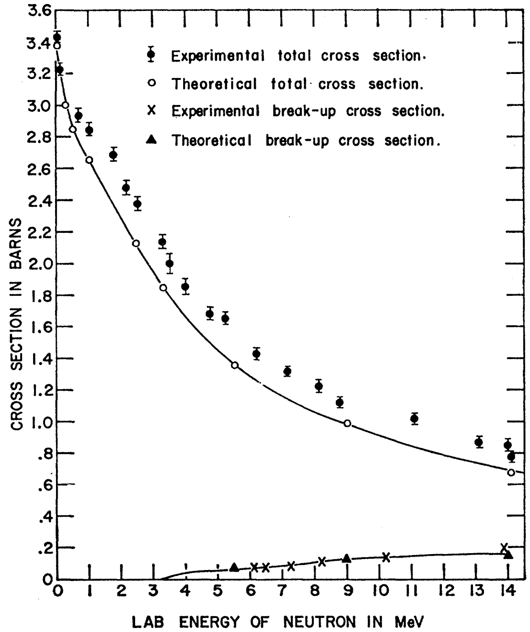

FIG. 7. Total and break-up cross sections for neutron-deuteron scattering. For references to experimental data, see Ref. 19. Note that for convenience, smooth curves have been drawn through the points calculated from the theory.

and the small doublet scattering length keeps the phase shift near π. The doublet p-wave phase shift is also small since it starts out repulsive and turns attractive above the breakup threshold due to the

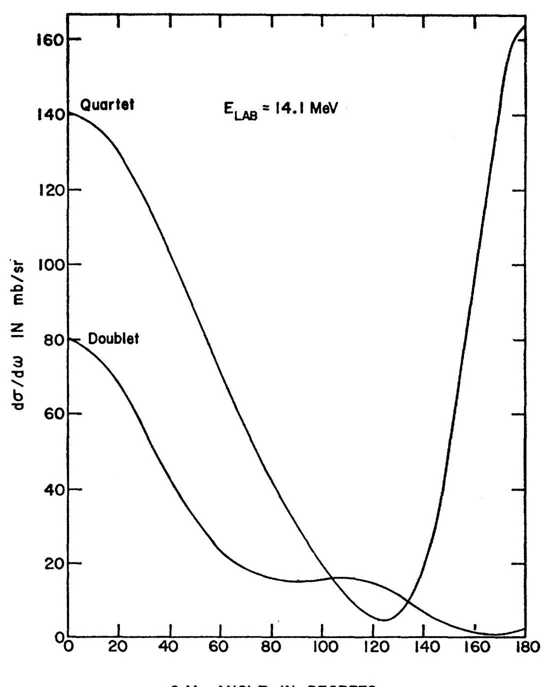

FIG. 8. Theoretical quartet and doublet angular distributions for neutron-deuteron scattering at 14.1 MeV.

[18] H. Kottler and K. L. Kowalski, Phys. Rev. **138**, B619 (1965).

[19] The total n-d cross section data are from *Tabulated Neutron Cross Sections*, R. J. Howerton, University of California Radiation Laboratory Report, UCRL 5573, 1961 (unpublished). The break-up cross section data are those of H. C. Catron *et al.* [Phys. Rev. **123**, 218 (1961)]. Very different results on the break-up cross section have been reported by Bonner *et al.* (see Ref. 17).

virtual effects of the breakup. The amplitudes are also strongly imaginary in this regime. All these facts plus the 2 to 1 statistical weight contribute to making the doublet contribution very much smaller than the quartet. For example, the doublet and quartet angular distributions at 14.1 MeV are shown separately in Fig. 8.

At least as striking as the large quartet to doublet ratio in Fig. 8 is the clear diffraction structure of the doublet amplitude. This is traced at other energies in Figs. 9–12. It is not surprising that there should be diffraction scattering in the scattering of a neutron from a diffuse object like the deuteron, but it is reassuring that a theory as simple as ours is capable of revealing this structure.

IV. DISCUSSION

The results presented in Sec. III are some of the first *exact* results for scattering in physical three-body systems. They are exact in the sense that three-body effects and asymptotic states are taken fully into account. Of course, the interaction mechanism itself is greatly simplified. The amazing agreement of our results with experiment indicates that our simple exchange mechanism is the dominant one, and that putting in the three-body effects exactly with this mechanism is more important than using more sophisticated interactions and less sophisticated three-body dynamics. It is perhaps surprising that so much can emerge from such a simple mechanism—for example, diffraction effects—but that is apparently the case.

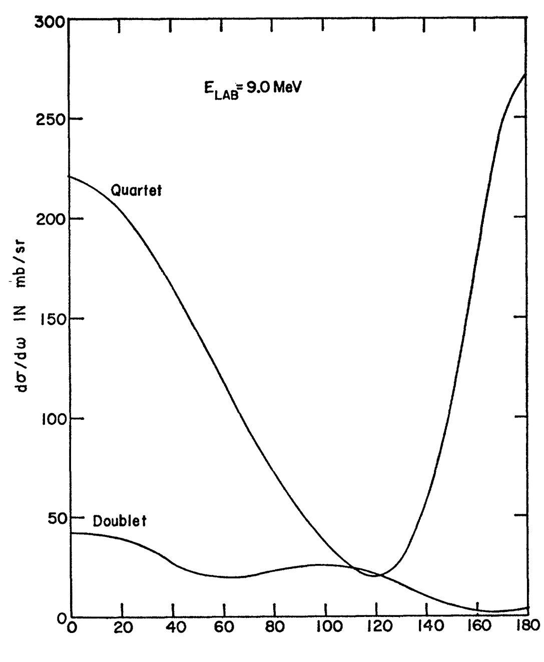

FIG. 9. Theoretical quartet and doublet angular distributions for neutron-deuteron scattering at 9.0 MeV.

The success of the calculation is no doubt also due to the relatively diffuse nature of the deuteron and even of the triton. The high-momentum parts of the nuclear force and even the higher partial waves in that force have been left out. This would certainly be more serious in dealing with more compact objects.

The next step in improving on the calculation would be the inclusion of some of these missing parts of the two-body potential. Some can, no doubt, be included perturbatively. For example, much of the effects of the short-range forces can be incorporated into an impulse approximation. Calculations of this type give fairly good answers for the forward angles in $n-d$ scattering.[18] Combining this with our results would

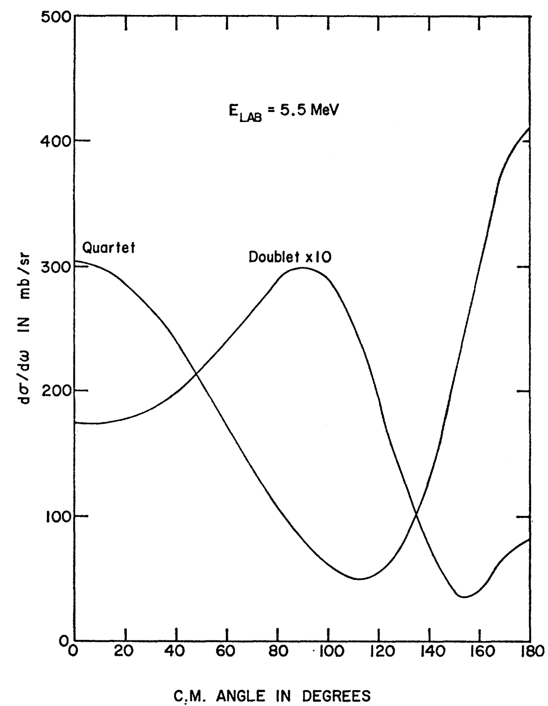

FIG. 10. Theoretical quartet and doublet angular distributions for neutron-deuteron scattering at 5.5 MeV.

doubtless go some way to rectifying our small-angle disagreement. The introduction of these missing terms in the bound state, particularly the effect of tensor forces, would no doubt also remove the need of making Z not zero. Care must be taken in this since it does not seem to be consistent with the two-body data to introduce the tensor force only into the two-body system without also introducing some central d wave and the hard core.[20] It is clear from the calculation of Bhakar and Mitra[11] that the tensor force goes in the right direction in the triton, but its exact effect is ambiguous. To introduce all the above-named terms is difficult, if not prohibitive, from the point of view of computer space. Perturbation or approximate

[20] F. Tabakin, Ann. Phys. (N. Y.) 30, 51 (1964).

schemes may be possible and we are currently investigating the question.

A possible approximation scheme is that of introducing high-energy effects, like hard cores, into the two-body propagators, but not introducing a new vertex for them. In multiple scattering language, this is to say that at low momentum, pairs mostly interact first and last via the long-range attractive part of the interaction, but that after it pulls them together, the short-range repulsive part is also effective. This approximation would not add to the number of channels needed for the problem, but would change the form of the propagators. At present the approximation is being compared against a calculation involving hard cores explicitly.[21]

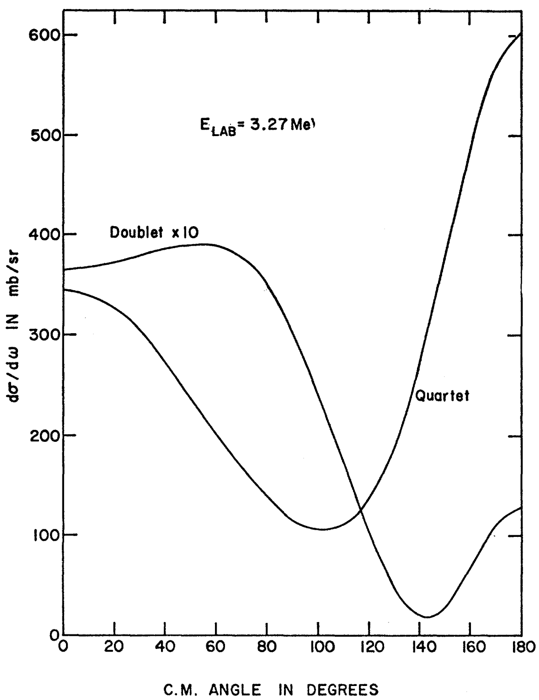

FIG. 11. Theoretical quartet and doublet angular distributions for neutron-deuteron scattering at 3.27 MeV.

Beyond improving the interactions for the bound states or $n-d$ scattering, there are a number of areas of considerable interest into which our calculation naturally leads. We plan, for example, to study the breakup reaction $n+d\rightarrow n+n+p$. In our formalism the amplitude for this is an integral over the off-energy-shell $n-d$ amplitude itself. There are no new equations to solve. A calculation of the break-up process including final state interactions exactly, that is between all three particles, would be of considerable importance in attempts to extract the $n-n$ scattering length. This calculation is the next item in our program.

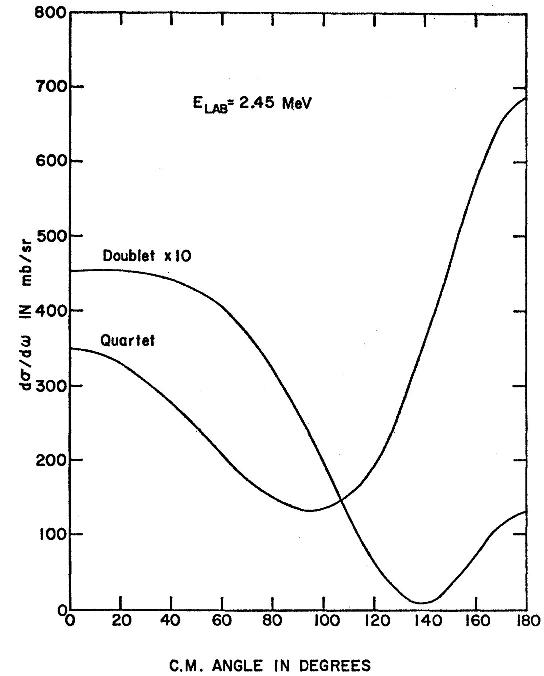

FIG. 12. Theoretical quartet and doublet angular distributions for neutron-deuteron scattering at 2.45 MeV.

Another aspect of the three-body system of current interest is the triton and He^3 form factors. Our model is presently being investigated in this context by one of us (RDA).

ACKNOWLEDGMENTS

We would like to thank the computer center at the Massachusetts Institute of Technology and at the University of Pennsylvania for making their extensive facilities available to us.

APPENDIX: SPIN PROJECTION FACTORS

To obtain the spin projection factors χ, we need to calculate the Born terms in a representation in which the total spin angular momentum and total isospin are diagonal. With the Born terms in this representation, the scattering amplitudes obtained from Eq. (1) will be in it as well. To do this, it is convenient to write out an explicit form for the interaction Hamiltonian in the quasiparticle formalism. Since we shall use only the rotation group properties of this Hamiltonian, our results are of course independent of the detailed interaction form. We write the interaction term in second quantization. This insures that antisymmetry takes care of itself. We are interested in an interaction which describes the pair state (represented by its own field) going into two nucleons and back. In order to get the correct linear combination of spin and isospin, we use Clebsch-Gordon coefficients explicitly. We have for the

[21] F. Tabakin (private communication).

interaction

$$H_{\text{int}}=\sum_{\Sigma T}\frac{\gamma_{\Sigma T}}{\sqrt{2}}\sum_{\mathbf{qk}} f_{\Sigma T}(q^2)\langle T\tau|\tfrac{1}{2}i_1\tfrac{1}{2}i_2\rangle\langle\Sigma\sigma|\tfrac{1}{2}\mu_1\tfrac{1}{2}\mu_2\rangle \times[\Phi_{\Sigma T;\sigma\tau}(\mathbf{k})\Psi^\dagger{}_{\mu_1 i_1}(\tfrac{1}{2}\mathbf{k}-\mathbf{q})\Psi^\dagger{}_{\mu_2 i_2}(\tfrac{1}{2}\mathbf{k}+\mathbf{q}) -\Phi_{\Sigma T;\sigma\tau}{}^+(\mathbf{k})\Psi_{\mu_1 i_1}(\tfrac{1}{2}\mathbf{k}-\mathbf{q})\Psi_{\mu_2 i_2}(\tfrac{1}{2}\mathbf{k}+\mathbf{q})], \quad \text{(A1)}$$

where $\Psi_{\mu i}(\mathbf{k})$ is the field operator for a nucleon of momentum $\mathbf{k}$, spin component μ and isospin projection i. The nucleon operators obey anticommutation relations, and this accounts for the minus in the Hermitian conjugate. Combined with the symmetry of the Clebsch-Gordan coefficients, the anticommutation relations also restrict the set (Σ,T) to (1,0)— the d pair state, and (0,1)—the ϕ pair state. $\Phi_{\Sigma T;\mu\tau}$ is the renormalized field operator for the pair quasiparticle with appropriate spin, isospin, and projections of these. $\gamma_{\Sigma T}$ and $f_{\Sigma T}$ are the renormalized coupling constant and vertex function; as indicated, they depend on the pair state involved. The brackets $\langle jm|j_1m_1j_2m_2\rangle$ are Clebsch-Gordon coefficients for adding $\mathbf{j}_1+\mathbf{j}_2=\mathbf{j}$ and $m_1+m_2=m$.

We can use this expression for H_{int} to construct the Born approximation. This can be done, for example, using contraction rules,[22] or diagrammatically. The Born terms are most easily expressed in a representation in which the third components of spin and isospin are diagonal. For a nucleon of moment $\mathbf{k}$ (in the center-of-mass system) and components (μ,i) incident on a pair state (Σ,T) with components (σ,τ) going to a state of nucleon of momenta $\mathbf{p}$ and components (μ',i') and a pair state $(\Sigma',T';\sigma'\tau')$, we get

$$\langle\Sigma T,\sigma\tau;\mu i;\mathbf{k}|B(E)|\Sigma'T',\sigma'\tau';\mu'i';\mathbf{p}\rangle =\sum_{i_1\mu_1} 2\langle T\tau|\tfrac{1}{2}i_1\tfrac{1}{2}i'\rangle\langle\Sigma\sigma|\tfrac{1}{2}\mu_1\tfrac{1}{2}\mu'\rangle\langle T'\tau'|\tfrac{1}{2}i_1\tfrac{1}{2}i'\rangle \times\langle\Sigma'\sigma'|\tfrac{1}{2}\mu_1\tfrac{1}{2}\mu\rangle\langle\Sigma T;\mathbf{k}|B(E)|\Sigma'T';\mathbf{p}\rangle, \quad \text{(A2)}$$

where the remaining matrix element is one of Eq. (2). μ_1 and i_1 are the spin and isospin components of the exchanged nucleon. We want the Born term not in this representation, but rather in a representation in which the total spin S, z component S_z, total isospin I, z component I_z, are specified. This is given by

$$\langle\Sigma T;SI,S_zI_z;\mathbf{k}|B(E)|\Sigma'T';SI,S_zI_z;\mathbf{p}\rangle =\sum\langle SS_z|\Sigma\sigma\tfrac{1}{2}\mu\rangle\langle II_z|T\tau\tfrac{1}{2}i\rangle \times\langle SS_z|\Sigma'\sigma'\tfrac{1}{2}\mu'\rangle\langle II_z|T'\tau'\tfrac{1}{2}i\rangle \times\langle\Sigma T;\sigma\tau;\mu i;\mathbf{k}|B(E)|\Sigma'T';\sigma'\tau';\mu'i';\mathbf{p}\rangle. \quad \text{(A3)}$$

Substituting (A2) into (A3), we see that we need only do a sum over four Clebsch-Gordan coefficients, which gives a Racah coefficient. We get[23]

$$\langle\Sigma T;SI,S_zI_z;\mathbf{k}|B(E)|\Sigma'T';SI,S_zI_z;\mathbf{p}\rangle =2[(2\Sigma+1)(2\Sigma'+1)(2T+1) \times(2T'+1)]^{1/2}(-1)^{1+\Sigma'+T'-S-I}W(\tfrac{1}{2}\tfrac{1}{2}S\tfrac{1}{2};\Sigma\Sigma') \times W(\tfrac{1}{2}\tfrac{1}{2}I\tfrac{1}{2};TT')\langle\Sigma T;\mathbf{k}|B(E)|\Sigma'T';\mathbf{p}\rangle, \quad \text{(A4)}$$

where the W are Racah coefficients. Putting in the appropriate values for everything and recalling that we are studying the case $I=\frac{1}{2}$, we get the coefficients χ of the text.

[22] See for example, R. D. Amado, Phys. Rev. **127**, 261 (1962).

[23] See, for example, M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957). Note that Table I.4 in this edition should have an additional minus sign for the Racah coefficient $W(bbdd;1F)$.