where $b_{lj}(x)$ in the integrals is defined by Eq. (13b) with r_c replaced by x in the lower limit of integration. The expression Eq. (B7) should be compared with the analogous expression obtained from Eq. (5):

$$\tan\delta_{lj}\approx -a_{lj}(1+b_{lj})+(1+2b_{lj}+a_{lj}c_{lj})\tan\delta_0 - c_{lj}(1+b_{lj})\tan^2\delta_0. \quad (B8)$$

The limitations put on the magnitudes of the parameters of Eq. (10) to obtain Eq. (B7) are probably more restrictive than necessary and can probably be relaxed. The more general restriction appears to be that $\tan\delta(i)$ should not pass through a resonance at any step of the recursion process for solving Eq. (10). Of course, if this more general restriction is violated it is still possible to use Eq. (10) in difference equation form. However, in this instance it is evident that the points r_k where $\tan\delta(k)$ passes through resonance give some difficulty, i.e., that the set $\{r_i\}$ must be dense at r_k. Also it is clear that the accuracy of this method will then depend on the accuracy in locating the largest r_i where a zero for $\tan\delta(i)$ occurs and perhaps the largest r_i where a resonance occurs. Thus, if a zero occurs at r_k [$\tan\delta(k)=0$] and no zeros or resonances occur for $r_i>r_k$, then only the potential for $r>r_k$ contributes to the scattering, but that part nearest to r_k will frequently be the most important. If, however, a resonance occurs for $r_i>r_k$ then the potential beyond r_i as well as the actual location of r_i becomes of greatest importance.

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Calculation of the (π^-, π^0) Reaction on Complex Nuclei*

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The excitation function for the reaction $_ZA(\pi^-,\pi^0)_{Z-1}A$ has been calculated using the Fermi-gas model of the nucleus and the impulse approximation. Experimental data on the differential cross section of the free-particle reaction $\pi^-+p\rightarrow\pi^0+n$ was used, and the effect of the momentum distribution of the nucleons on the kinematics was included. The predicted cross section shows a minimum near 200 MeV and maxima near 100 and 350 MeV. This structure, if confirmed experimentally, would lend support to this simple model, which is frequently used to interpret high-energy nuclear reactions.

INTRODUCTION

MANY features of high-energy nuclear reactions can be successfully predicted by calculations based on a simple model: the nucleus is represented by a degenerate Fermi gas of protons and neutrons with which the incident particle interacts. The impulse approximation is assumed to hold, and free-particle cross sections are used. Extensive Monte Carlo calculations[1,2] have been performed using essentially this model and have given good agreement with experiment, with the notable exception of those for simple nuclear reactions such as the (p,pn) reaction. It is possible to treat such reactions in explicit calculations without resorting to the Monte Carlo method, as was done by Benioff,[3] who used a shell model with harmonic-oscillator wave functions to calculate (p,pn) cross sections. Ericson, Selleri, and Van de Walle[4] have calculated the excitation function for $(p,p\pi^+)$ reactions, using a Fermi-gas model of the nucleus, and Remsberg[5] has improved their calculation and obtained good agreement with experiment.

The present calculation has been done in order to extend this model to pion-induced reactions. One of the simplest of such reactions is charge exchange:

$$\pi^\pm+{}_ZA\rightarrow\pi^0+{}_{Z\pm1}A. \quad (1)$$

According to the model under discussion, reaction (1) occurs by a charge exchange involving a single nucleon[6]

$$\pi^-+p\rightarrow\pi^0+n, \quad (2)$$

in which the neutron remains in the nucleus. In order that the final nucleus not evaporate any particles, its excitation energy must be low, and therefore the momentum transfer to the nucleon must also be low. The incident and outgoing pions are required not to undergo any interactions with other nucleons, since in that case the nucleus would gain too much excitation energy. Using experimental values for the cross section of reaction (2) with the limitations imposed by the low energy transfer to the nucleus, we can calculate the cross section for reaction (1).

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The case of simple pion reactions is particularly interesting because of the resonances in pion-nucleon scattering. The $(\frac{3}{2},\frac{3}{2})$ resonance at 200-MeV pion kinetic energy has been shown to manifest itself as a peak in the cross section of the $C^{12}(\pi^-, \pi^- n)C^{11}$ reaction.[7] Reaction (2) shows peaks at 200 and 900 MeV; corresponding structure in the calculated cross section of reaction (1) would be especially suitable for comparison with experiment, because it is the shape, rather than the absolute value, which is characteristic.

DETAILS OF THE CALCULATION

Let the momentum of the struck proton be p_1, that of the recoil neutron be p_2, and the Fermi momentum be p_F. We have the following restrictions:

$$\begin{aligned} &p_1 \leq p_F, \\ &p_2 \geq p_F, \\ &p_2^2 - p_1^2 \leq 2mE_0. \end{aligned} \tag{3}$$

The last relation requires the residual excitation to be less than E_0, the energy above which the residual nucleus is particle-unstable. As pointed out by Grover,[8] the probability of neutron evaporation may not become comparable to that of gamma emission until the excitation energy exceeds the neutron binding energy by an appreciable margin. However, we are interested primarily in the shape of the excitation function, and the parameter E_0 affects only its absolute magnitude. We therefore use a value of $E_0 = 8$ MeV in these calculations.

To calculate the cross section using this model we have the expression used by Ericson, Selleri, and Van de Walle,[4]

$$\sigma = Z(\tfrac{4}{3}\pi p_F^3)^{-1} \int d^3p_1 \int (d\sigma/d\Omega) R(\phi) d\Omega, \tag{4}$$

where $d\sigma/d\Omega$ is the differential cross section of reaction (2), $d\Omega = 2\pi d\cos\theta$, θ is the c.m. scattering angle of the pion, and $R(\phi)$ is the reduction factor, expressing the probability that neither the incident pion nor the outgoing pion makes any interactions. The angle ϕ is the laboratory scattering angle of the pion. Integrating over the azimuthal angle of p_1 and writing the limits in, we have

$$\sigma = Z\frac{3\pi}{p_F^3} \int_{(p_F^2 - 2mE_0)^{1/2}}^{p_F} p_1^2 dp_1 \int_{-1}^{1} d\cos\alpha \times \int_{\cos\theta_1}^{\cos\theta_2} \frac{d\sigma}{d\Omega} R(\phi)\, d\cos\theta, \tag{5}$$

where α is the angle of p_1 with respect to the incident pion, and the lower and upper limits on $\cos\theta$ are the scattering angles for which the final neutron momentum $p_2 = (p_1^2 + 2mE_0)^{1/2}$, and $p_2 = p_F$, respectively. These angles are a function of p_1, $\cos\alpha$, and the incident pion energy, and are calculated using relativistic kinematics.[9]

The differential cross section for reaction (2) is evaluated at the c.m. energy W,

$$W^2 = M^2 + \mu^2 + 2E_1E_\pi - 2p_1p_\pi \cos\alpha, \tag{6}$$

where E_π and p_π are the total energy and momentum of the incident pion, M is the nucleon mass, and μ the pion mass. The differential cross section was expressed as a power series in $\cos\theta$,

$$d\sigma/d\Omega = \sum_n a_n \cos^n\theta, \tag{7}$$

with the coefficients a_n determined experimentally. Below the $(\frac{3}{2},\frac{3}{2})$ resonance the coefficients a_0, a_1, and a_2 were calculated from the phase shifts[10]; above that energy the available experimental values[11–13] were plotted against W, and smooth curves used for interpolation.

Because of the restrictions [Eqs. (3)] on the momentum transfer the allowed range of $\cos\theta$ is rather small. Likewise, the range of laboratory scattering angle ϕ is also small, and the reduction factor was calculated using an average value of ϕ. As shown below, there is only a weak dependence of $R(\phi)$ on ϕ, so this proves to be a good approximation. For low incident energies and $\cos\alpha < 0$ ("head-on" collisions) it is kinematically impossible to satisfy the restriction $p_2 \geq p_F$. This situation was taken into account by not allowing $\cos\alpha$ to be less than the allowed value.

The reduction factor is the integral over the nucleus of the probability that the incident pion, traveling in the z direction with impact parameter r (cylindrical coordinates) will reach the point (r,z) without interacting, and that the final pion will leave at angle ϕ without interacting. If $P(r,z,\phi)$ is this joint probability, then

$$R(\phi) = \int\int \rho(r,z) P(r,z,\phi) r\, dr\, dz \Big/ \int\int \rho(r,z) r\, dr\, dz, \tag{8}$$

where $\rho(r,z)$ is the nucleon density. $P(r,z,\phi)$ is obtained by averaging over the azimuthal angle ψ of the outgoing pion:

$$P(r,z,\phi) = \exp\left[-\sigma_1 \int_{-\infty}^{z} \rho(r,z) dz\right] \times \frac{1}{\pi} \int_0^{\pi} \exp\left[-\sigma_2 \int_z^{\infty} \rho(r,z) ds\right] d\psi. \tag{9}$$

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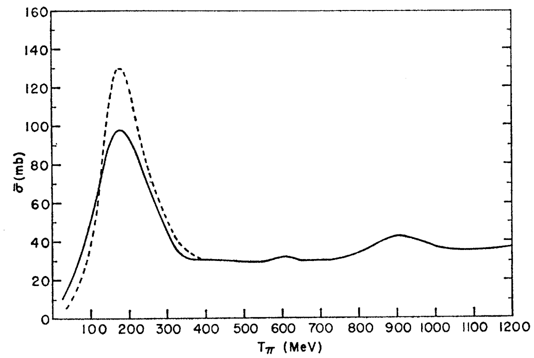

FIG. 1. Average pion-nucleon cross sections. (——): $\bar{\sigma}$ (- - -): free-particle cross section.

The path length of the outgoing pion ds is a function of r, z, ϕ, and ψ. For a constant-density nucleus with radius R_0 and density ρ_0, the integral in the second exponential is

$$\int_z^\infty \rho(r,z)ds=\rho_0\Big\{-(r\sin\phi\cos\psi+z\cos\phi)+[(r\sin\phi\cos\psi+z\cos\phi)^2+R_0^2-r^2-z^2]^{1/2}\Big\}. \quad (10)$$

σ_1 and σ_2 are the effective pion-nucleon cross sections of the incident and exiting pions, respectively. Because of the momentum restrictions (3) the pion energies are very nearly equal, and hence for a nucleus with equal numbers of neutrons and protons,

$$\sigma_1=\sigma_2=\bar{\sigma}=\tfrac{1}{2}[\sigma(\pi^-p)+\sigma(\pi^+p)]. \quad (11)$$

At sufficiently high energies one can use the free-particle cross sections in (11), but below about 300 MeV the Pauli principle reduces the cross section significantly.[14] The pion-capture process by a pair of nucleons,[15] which is not possible with a free nucleon, must be included in the effective cross section. We used the calculations of Frank *et al.*[16] for $\bar{\sigma}$, replacing their expression for the capture cross section, which is valid only at low energies, with a more recent one.[1] Above 350 MeV the free-particle cross sections[17] were used. The curves of $\bar{\sigma}$ and the average free-particle cross section are given in Fig. 1.

The reduction factor was evaluated for two density distributions: a uniform distribution with radius $R_0=1.40\,A^{1/3}$F and the Fermi distribution,

$$\rho(r,z)=\rho_0\{1+\exp([(r^2+z^2)^{1/2}-c]/a)\}^{-1}, \quad (12)$$

with parameters[18] $c=1.07A^{1/3}$ F, $a=0.55$ F. The reduction factor, $R(\phi)$, is shown as a function of $\cos\phi$ for two values of $\bar{\sigma}$ for the two density distributions in Fig. 2. The uniform density results in an $R(\phi)$ which is smaller and more strongly dependent on ϕ than the Fermi density distribution for large $\bar{\sigma}$ and forward scattering angles. We emphasize here that we assume complete independence of momentum and density distributions; the momentum distribution is taken to be the same at every location in the nucleus. This allows us to separate the calculation of $R(\phi)$ from the rest of the calculation.

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RESULTS AND DISCUSSION

We first present the results of a numerical integration of Eqs. (5) and (8), for particular values of the parameters. The dependence of the cross section on these parameters is discussed next, to indicate how the results will change on changing the particular nucleus. Finally we mention some of the omissions and approximations involved and estimate their effect on the results.

We calculated the cross section for the $Cu^{65}(\pi^-,\pi^0)Ni^{65}$ reaction, which is a favorable case for measurement. The binding energy of a neutron in Ni^{65} is 6.13 MeV; we use $E_0=8.0$ MeV for the maximum excitation energy. Two values of the Fermi momentum were used: 280 and 217 MeV/c; these correspond to Fermi energies of 41 and 25 MeV, respectively. The reduction factor was calculated for both a uniform and the Fermi density distributions. The results are shown in Fig. 3. There is a pronounced minimum in the cross section near 200 MeV, and peaks near 100 and 350 MeV. This structure results from the balance between the reduction factor and the effect of the momentum distribution on the cross section. Figure 4 shows the results obtained if one neglects the variation of $\bar{\sigma}$ with pion energy (dashed line) or takes a constant reduction factor, independent of angle and pion energy (solid line). The single peak is at 140 MeV, shifted downward from the free-particle peak energy. This is due to the effects of the nucleon momentum and the angular distribution for reaction (2). Below the $(\frac{3}{2},\frac{3}{2})$ resonance the latter is peaked backward in the center-of-mass system; and for pion energies

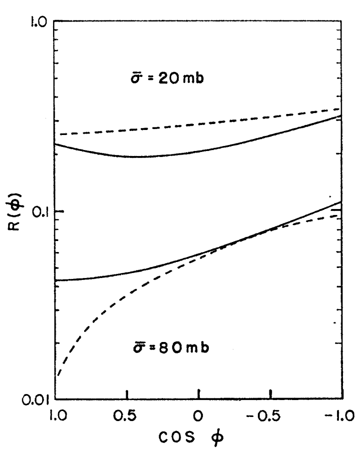

FIG. 2. Reduction factor, $R(\phi)$. (——): Fermi density distribution; (- - -): uniform density, $R_0=1.40\,A^{1/3}$.

below about 150 MeV it is just the backward c.m. scatterings which mainly result in the allowed momentum transfers to the nucleon, due to the restriction $p_2 \geq p_F$. On the other hand, at high pion energies the forward scatterings are favored, because of the upper limit on p_2, and it is there that the angular distribution shifts to forward peaking. The result is to shift the peak in Fig. 4 to a lower energy than the peak in the free-particle cross section, and to lessen the rate of decrease on the high-energy side. When the energy variation of the reduction factor is included, its minimum near 200 MeV, together with the shifted peak, causes the cross-section minimum seen in Fig. 3. The second peak in Fig. 3 is due to the reduction factor increasing to a plateau value as the cross section decreases more slowly.

The variation of $R(\phi)$ with the angle ϕ is different for the uniform and the Fermi density distributions, but the over-all variation of R with pion energy is similar, because of the strong dependence of R on the mean pion-nucleon cross section $\bar{\sigma}$. When $\bar{\sigma}$ is large, near 200 MeV, the uniform density distribution results in R significantly smaller than does the Fermi distribution for forward scattering angles. The result is to make the minimum near 200 MeV deeper for the uniform distribution.

It is seen from Fig. 3 that a lower assumed Fermi momentum leads to a higher cross section, but with nearly the same shape. The dependence on Fermi momentum is complicated by the change in the kinematics of the collision, but to a fair approximation we can account for it by considering only the limits on the integrals in Eq. (5). For the small range of allowed values of p_2, the quantity $\cos\theta$ is approximately linear in p_2. The range of the last integral in Eq. (5) is therefore

$$(\cos\theta_2 - \cos\theta_1) \propto [(p_1^2 + 2mE_0)^{1/2} - p_F].$$

If we neglect the changes in the integrand due to small

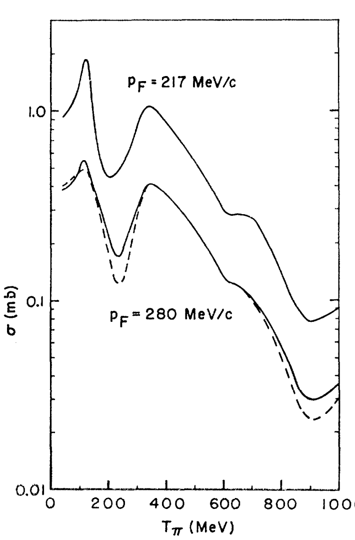

FIG. 3. Calculated excitation function for $Cu^{65}(\pi^-, \pi^0)Ni^{65}$, for two values of Fermi momentum. (——): Fermi density distribution; (– – –): uniform density, $R_0 = 1.40\, A^{1/3}$.

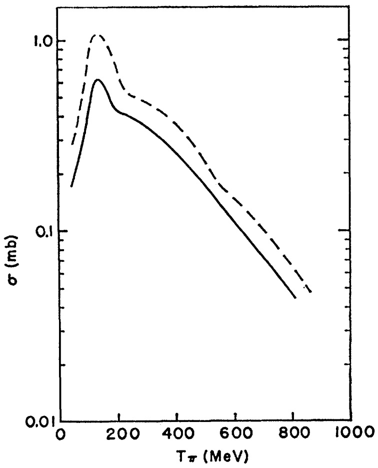

FIG. 4. Calculated excitation function for $Cu^{65}(\pi^-, \pi^0)Ni^{65}$; (——): constant reduction factor ($R = 0.1$); (– – –): Fermi density distribution, $\bar{\sigma} = 30$ mb. $p_F = 280$ MeV/c.

changes in p_F or E_0, the value of the integral is proportional to the above expression. The cross section is thus

$$\sigma \propto \frac{\sigma_0}{p_F^3} \int_{(p_F^2 - 2mE_0)^{1/2}}^{p_F} p_1^2 ((p_1^2 + 2mE_0)^{1/2} - p_F) dp_1,$$

where σ_0 is the average value of $d\sigma/d\Omega$ over the range of $\cos\theta$ and $\cos\alpha$. Evaluating this integral, and using the expansion

$$(p_F^2 \pm 2mE_0)^{1/2} = p_F[1 \pm mE_0/p_F^2 - \tfrac{1}{2}(mE_0/p_F^2)^2 \pm \cdots],$$

we find

$$\sigma \propto \sigma_0 (mE_0)^2 / p_F^3. \tag{13}$$

The approximate dependence of the cross section on p_F^{-3} is borne out by the curves in Fig. 3; the dependence on E_0^2 is confirmed by calculation and is in agreement with the calculation of Ericson *et al.*,[4] who also used the Fermi-gas model of the nucleus.

As pointed out by Ericson *et al.*,[4] the reduction factor is approximately inversely proportional to the mass of the nucleus, and so the cross section is proportional to ZE_0^2/A. The variation from nucleus to nucleus is primarily due to E_0.

The contrast of this calculation with the experimental $C^{12}(\pi^-, \pi^- n)C^{11}$ cross section[7,19] is striking. This is the only pion-induced reaction whose excitation function is known, and the $(\frac{3}{2},\frac{3}{2})$ resonance manifests itself as a single broad peak. We conclude that the reduction factor for this reaction does not suppress the cross section near 200 MeV. Reeder and Markowitz[7] conclude on the basis of a Fermi-gas calculation that the mechanism is a direct knockout (one-step mechanism). In order for the neutron to gain enough kinetic energy to have a good probability of escaping the nucleus, the pion must lose a considerable fraction of its energy. For an incident pion at the resonance, the

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outgoing pion will have a much smaller cross section, hence the reduction factor will not be excessively small. At higher energies, the outgoing pion will have an appreciable probability of being near the resonance and having a large cross section, but the incident pion has a small cross section. In addition, because the outgoing pion energy can have a wide range of values, unlike the (π^-, π^0) reaction, the effect of the resonance is washed out. A two-step mechanism (inelastic scattering followed by neutron evaporation) for the $(\pi^-, \pi^- n)$ reaction would require the outgoing pion kinetic energy to be 20–30 MeV below that of the incident pion, and would result in a minimum in the cross section near resonance, an effect which is seen in the calculation of Reeder and Markowitz.[7]

We now consider the approximations which were used in the calculation. The degenerate Fermi-gas model is admittedly crude, particularly in its handling of the availability of final states, which is all in the parameter E_0. Shell structure, which may cause variations in cross section from nucleus to nucleus not correlated with E_0, is ignored. However, these primarily affect the magnitude, not the shape, of the excitation function. The magnitude is also sensitive to the choice of a Fermi momentum and to the estimated values of $\bar{\sigma}$ as a function of pion energy. The minimum near 200 MeV and the peaks near 100 and 350 MeV are a result of the general features of pion-nucleon cross sections and the presence of a momentum distribution inside the nucleus, not the details of the latter.

The calculation should be most valid at high energies, where the impulse approximation is most applicable. Near 200 MeV the mean free path of the pion is of the same order of magnitude as the inter-nucleon distance and it is not valid to consider only two-body interactions. At lower energies the pion wavelength becomes too large to make the classical trajectory a good approximation. The peak near 100 MeV is therefore less reliable than the other features.

We have separated the calculation of the reduction factor from the rest of the problem. This is not valid unless the momentum distribution is independent of position inside the nucleus. Since the Fermi momentum depends on density one might expect p_F to decrease near the nuclear surface. However, the (π^-, π^0) reaction is probably concentrated near the nuclear surface because of the absorptive effects which are accounted for by $R(\phi)$, and is similar in that regard to the (p,pn),[3] $(\pi^-, \pi^- n)$,[7] and $(p, p\pi^+)$[5] reactions. In that case it is probably a good assumption to use a single value of p_F as an effective Fermi momentum for the region in which the reaction occurs.

The nuclear potential felt by the pion has not been included. Its effect is to shift the energy scale by $-V$, where V is the real part of the potential. Below 150 MeV the experimental evidence[20] is that $V \approx -30$ MeV. At higher energies V appears to be positive,[21,22] and this is borne out by the forward pion-nucleon scattering amplitudes.[23] The energy at which the potential crosses zero is about 200 MeV. It may be possible to determine the nuclear potential for π mesons by comparison of experimental cross sections with the calculation.

We have calculated only the elastic charge-exchange contribution to reaction (1). Inelastic processes become significant above ca. 400 MeV, and probably would represent the major contribution to the nuclear reaction above ca. 1 GeV. It has been suggested[4] that measurement of radiochemical $(\pi,2\pi)$ cross sections could yield information about the (π,π) elementary cross section. The present calculation may be useful in subtracting out the elastic contribution from the observed cross section to obtain the inelastic $(\pi,2\pi)$ cross section.

In conclusion, it appears that a measurement of the (π^-, π^0) excitation function below 500 MeV offers the possibility of confirming the usefulness of the commonly used Fermi-gas model to describe simple high-energy nuclear reactions. In addition, it may be possible to determine indirectly the effective pion-nucleon cross section inside the nucleus and the nuclear potential of pions.

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