Explicit Wave Functions for Any Spin

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(Received 27 April 1965)

A general method is given for calculating the exact form of the helicity eigenstates of a wave function for any spin. Explicit expressions of the wave functions for spin $S=1, \frac{3}{2}, 2,$ and $\frac{5}{2}$ are listed. Examples of helicity amplitudes for the production of $S=\frac{3}{2}^+$ and $S=\frac{5}{2}^+$ isobars in proton-proton scattering are computed.

I. INTRODUCTION

WITH the high-energy beams that experimentalists now have at their disposal, more and more higher-spin resonances are being discovered. Most theoretical calculations pertaining to these resonances require explicit expressions for the higher-spin wave functions, e.g., both in polarization and partial-wave analysis there is need to consider definite helicity states.[1]

In this paper we outline a general method which enables one to compute the full relativistic wave function for any spin S. From the examples given for the calculation of matrix elements, one sees that dealing with higher-order spinors is no more complicated than dealing with spin-$\frac{1}{2}$ particles.

In Sec. II we construct the general spin-wave functions. This is carried out for general momentum by performing a Lorentz transformation on the rest-frame wave functions. For these, we find an explicit expression for the $S^z=S$ state, and by successive application of S^- we find all the others. This leads us to helicity eigenstates for any spin. The Appendix contains explicit expressions for the cases $S=1, \frac{3}{2}, 2,$ and $\frac{5}{2}$. Section III deals with the construction of helicity amplitudes for scattering processes and specific examples are given for the production of the $N_{3/2^+}{}^*(1238)$ and $N_{5/2^+}{}^*(1688)$ isobars.

II. CONSTRUCTION OF GENERAL SPIN-WAVE FUNCTIONS

The Rarita-Schwinger wave functions $\psi_{\mu_1\cdots\mu_k}(p)$ of spin $S=k+\frac{1}{2}$ (k integer) satisfy[2,3]

$$(i\not{p}-m)\psi_{\mu_1\cdots\mu_k}(p)=0\,,\qquad \gamma^\mu\psi_{\mu\mu_2\cdots\mu_k}(p)=0\,,\tag{1}$$

where we use $g_{00}=-1$, $g_{kk}=1$, and $\psi_{\mu_1\cdots\mu_k}$ is symmetric in $\mu_1\cdots\mu_k$.

Equations (1) entail

$$p^\mu\psi_{\mu\mu_2\cdots\mu_k}=0\,,\tag{2}$$

$$\psi^\mu{}_{\mu\mu_3\cdots\mu_k}=0\,.\tag{3}$$

It can be easily verified, using (1), that under a Lorentz transformation the wave functions transform as

$$\psi'_{\mu_1\cdots\mu_k}(p')=\Lambda(L)L_{\mu_1}{}^{\nu_1}\cdots L_{\mu_k}{}^{\nu_k}\psi_{\nu_1\cdots\nu_k}(p)\,,\tag{4}$$

where L is the Lorentz transformation from p to p',

$$p_\mu'=L_\mu{}^\nu p_\nu\tag{5}$$

with

$$g^{\mu\sigma}L_\mu{}^\nu L_\sigma{}^\lambda=g^{\nu\lambda}\,,\tag{6}$$

and $\Lambda(L)$ satisfies

$$\Lambda^{-1}(L)\gamma^\nu\Lambda(L)=L^\nu{}_\lambda\gamma^\lambda\,,\tag{7}$$

which is the same as for the spin-$\frac{1}{2}$ case.

Our aim is to find explicit formulas for the wave functions $\psi_{\mu_1\cdots\mu_k}(p)$. It is more convenient to calculate first the rest-frame wave functions, then perform a Lorentz transformation (4) to a general p. Choosing the direction of $\mathbf{p}$ as the z direction, the required Lorentz transformation is obtained by

$$(L_\nu{}^\lambda)=\begin{array}{c} \begin{matrix} \;0 & \;\;1 & \;2 & \;\;3 \end{matrix} \\ \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \end{array}\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix}\tag{8}$$

$$\Lambda(L)=\cosh(\omega/2)+\begin{pmatrix} 0 & \sigma^z \\ \sigma^z & 0 \end{pmatrix}\sinh(\omega/2)\,,$$

where (we use $\hbar=c=1$),

$$\beta=v=\tanh\omega,\quad \gamma=1/(1-v^2)^{1/2}=\cosh\omega\,,\tag{9}$$

and σ^k are the Pauli matrices. We use the following

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[1] S. Mandelstam, J. E. Paton, R. F. Peierls, and A. Q. Sarker, Ann. Phys. (N. Y.) **18**, 198 (1962).
[2] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).
[3] H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Company, Amsterdam, 1956), Chap. IV, paragraph 3; E. M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave-Equations* (Hafner Publishing Company, New York, 1953), pp. 118–121. Further references concerning relativistic wave equations can be found in these two books.

representation for the γ matrices,

$$\gamma^0=-i\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix},\quad \boldsymbol{\gamma}=i\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0\end{pmatrix},$$
$$\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3=-\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}. \tag{10}$$

Let us now find the rest-frame wave functions. Now, in this frame [when we write $\psi_{\mu_1\cdots\mu_k}(\mathbf{p})$, we mean that $p_0=(\mathbf{p}^2+m^2)^{1/2}$],

$$\psi_{0\mu_2\cdots\mu_k}(0)=0\,,$$
$$\psi_{r_1\cdots r_k}(0)=\begin{pmatrix} \chi \\ \eta=0\end{pmatrix}, \tag{11}$$

where χ and η are the two upper and lower components, respectively. Hence, in the rest frame, ψ are two-component spinors. They satisfy

$$\psi^r{}_{rr_3\cdots r_k}(0)=0\,,$$
$$\sigma^r\psi_{rr_2\cdots r_k}(0)=0\,. \tag{12}$$

The spin operator is

$${}^{r_1\cdots r_k}(S^l)^{t_1\cdots t_k}=\tfrac{1}{2}\sigma^l g^{r_1t_1}\cdots g^{r_kt_k} -\sum_{j=1}^{k} i\epsilon^{lr_jt_j}g^{r_1t_1}\cdots \hat{g}^{r_jt_j}\cdots g^{r_kt_k}, \tag{13}$$

where $\hat{g}^{r_jt_j}$ means that $g^{r_jt_j}$ is missing. Equation (13) has to be understood as

$$\langle\psi|S^l|\varphi\rangle=\psi_{r_1\cdots r_k}{}^{\dagger}\,{}^{r_1\cdots r_k}(S^l)^{t_1\cdots t_k}\varphi_{t_1\cdots t_k}. \tag{14}$$

Now, the state with $S^z=S$ satisfies

$${}^{r_1\cdots r_k}(S^+)^{t_1\cdots t_k}\psi_{t_1\cdots t_k}{}^{(S^z=S)}(0)=0\,,$$
$$S^+=S^x+iS^y. \tag{15}$$

It can be easily verified that the desired solution is

$$\psi_{t_1\cdots t_k}{}^{(S^z=S)}(0)=(1/2^{k/2})\begin{pmatrix} i^{n_y} \\ 0\end{pmatrix}\delta_{0,n_z}, \tag{16}$$

where n_x, n_y, n_z are the number of times x, y, z appear among $t_1\cdots t_k$. The solution (16) satisfies Eq. (15) and the conditions (12).

All other $S^z=M$ states can be obtained from (16) by use of the operator S^-, through

$$S^-|SM\rangle=[S(S+1)-M(M+1)]^{1/2}|SM-1\rangle. \tag{17}$$

Our wave function's normalization is

$$\psi^{r_1\cdots r_k\dagger}(0)\psi_{r_1\cdots r_k}(0)=1\,. \tag{18}$$

The eigenvalues of S^z can be most easily calculated using the identity,

$$S\psi^{r_1\cdots r_k\dagger}(0)\boldsymbol{\sigma}\psi_{r_1\cdots r_k}(0) =\psi_{r_1\cdots r_k}{}^{\dagger}(0)^{r_1\cdots r_k}\mathbf{S}^{t_1\cdots t_k}\psi_{t_1\cdots t_k}(0). \tag{19}$$

This identity follows from

$$\psi^{r_1\cdots r_k\dagger}(0)\sigma^l\psi_{r_1\cdots r_k}(0) =-\psi_{r_1r_2\cdots r_k}{}^{\dagger}(0)i\epsilon^{lr_1t_1}\psi_{t_1}{}^{r_2\cdots r_k}(0)\,, \tag{20}$$

which can be obtained from the second of (12).

When performing the Lorentz transformation by (8), the eigenstates of S^z are now eigenstates of

$${}^{\mu_1\cdots\mu_k}(S^z)^{\nu_1\cdots\nu_k}=\tfrac{1}{2}\Sigma^z g^{\mu_1\nu_1}\cdots g^{\mu_k\nu_k} -\sum_{j=1}^{k} i\epsilon^{03\mu_j\nu_j}g^{\mu_1\nu_1}\cdots \hat{g}^{\mu_j\nu_j}\cdots g^{\mu_k\nu_k}, \tag{21}$$

$$\Sigma^z=\begin{pmatrix} \sigma^z & 0 \\ 0 & \sigma^z\end{pmatrix}. \tag{22}$$

Since $\mathbf{p}$ is taken to be in the z direction, this procedure gives helicity eigenstates, namely eigenstates of

$$(1/|\mathbf{p}|)\mathbf{S}\cdot\mathbf{p}\,, \tag{23}$$

where now

$$\langle\psi|S|\varphi\rangle=\bar{\psi}_{\mu_1\cdots\mu_k}(\mathbf{p})\,{}^{\mu_1\cdots\mu_k}\mathbf{S}^{\nu_1\cdots\nu_k}\varphi_{\nu_1\cdots\nu_k}(\mathbf{p})\,, \tag{24}$$

$$\bar{\psi}=i\psi^{\dagger}\gamma_0\,, \tag{25}$$

and the invariant normalization is

$$\bar{\psi}_{\mu_1\cdots\mu_k}(\mathbf{p})\psi^{\mu_1\cdots\mu_k}(\mathbf{p})=1\,. \tag{26}$$

The same procedure can be carried out for the integer-spin case. Here, using the tensor wave functions,[3] the equation corresponding to (16) is

$$U_{r_1\cdots r_s}(0)=(1/2^{s/2})\delta_{n_z,0}i^{n_y}\,, \tag{27}$$

S being the spin. The spin operators are now

$${}^{r_1\cdots r_s}(S^l)^{t_1\cdots t_s}=-\sum_{j=1}^{s} i\epsilon^{lr_jt_j}g^{r_1t_1}\cdots \hat{g}^{r_jt_j}\cdots g^{r_st_s}, \tag{28}$$

and the Lorentz transformation is performed by

$$U'_{\mu_1\cdots\mu_s}(p')=L_{\mu_1}{}^{\nu_1}\cdots L_{\mu_s}{}^{\nu_s}U_{\nu_1\cdots\nu_s}(p)\,. \tag{29}$$

The explicit expressions for the helicity states for the spin 1, $\frac{3}{2}$, 2, and $\frac{5}{2}$ cases, are given in the Appendix.

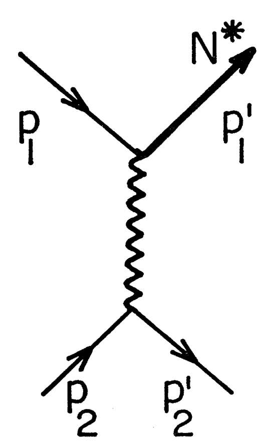

FIG. 1. Feynman diagram for isobar production.

III. HELICITY AMPLITUDES WITH HIGHER-ORDER SPINORS

In this section we calculate matrix elements involving higher-order spinors, and generalize the methods that Goldberger *et al.*[4] have given for spin-$\frac{1}{2}$ particles. As we have made some changes in notation, we will repeat the treatment for the Dirac spinors as well.

As the experimental setup today is such that in most experiments both the incident and target particles have spin $S=\frac{1}{2}$, it is convenient to choose a set of axes so that the momentum ($\mathbf{p}_1'$) of the outgoing isobar is along the positive z axis and confine ourselves to the xz plane. This choice avoids the complication of having to apply the rotation matrices for $S>\frac{1}{2}$. As an example we consider the scattering of two spin-$\frac{1}{2}$ particles, with the production of a single isobar.

In the center-of-mass system we use the notation

$$p_1=(\mathbf{K},E_1),\quad p_1'=(\mathbf{q},E_1'),\quad p_2=(-\mathbf{K},E_2),\quad p_2'=(-\mathbf{q},E_2'),$$

where $\mathbf{K}$ and $\mathbf{q}$ denote the three-momenta of the incoming and outgoing particles, respectively, and λ denotes the helicity state of the particle. (See Figs. 1 and 2.) From Fig. 2 we see that the helicities of particles with index 2 are in the negative z direction and thus must always be taken with a minus sign prefixed. Taking p_1, p_2, and p_2' to be the spin-$\frac{1}{2}$ particles, we can represent these Dirac spinors as

$$\begin{aligned}\psi_{\lambda_1}(p_1)&=N(p_1)\begin{pmatrix}E_1+m_1\\ 2\lambda_1 K\end{pmatrix}e^{i\sigma_y\theta/2}\chi_{\lambda_1},\\ \psi_{\lambda_2}(p_2)&=N(p_2)\begin{pmatrix}E_2+m_2\\ 2\lambda_2 K\end{pmatrix}e^{i\sigma_y\theta/2}\chi_{-\lambda_2},\\ \bar{\psi}_{\lambda_2'}(p_2')&=N(p_2')((E_2'+m_2'),\,-2\lambda_2'q)\chi^{\dagger}{}_{-\lambda_2'},\end{aligned}\tag{30}$$

where $N(p)=1/[2m(E+m)]^{1/2}$ is a normalization factor

$$\chi_+=\begin{pmatrix}1\\0\end{pmatrix}\quad\text{and}\quad\chi_-=\begin{pmatrix}0\\1\end{pmatrix}.$$

Following the notation of GGMW[4] we will denote our helicity amplitude by $\langle\lambda_1'\lambda_2'|\Phi|\lambda_1\lambda_2\rangle$.

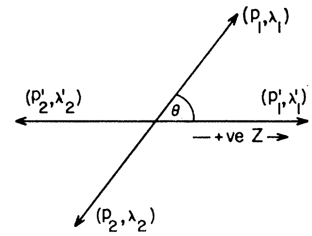

FIG. 2. Four-momenta and helicities in the c.m. system.

(a) Production of Spin-$\frac{3}{2}$ Isobar

We consider the process of the production of a $N_{3/2}{}^{+*}(1238)$ isobar in proton-proton scattering, with the exchanged particle (in Fig. 1) taken to be a pion. Since the pion is a spinless particle we can treat the upper and lower vertices separately.

(i) *Lower Vertex* $\langle\lambda_2'|\lambda_2\rangle$

The interaction is of the form

$$L_I=g\bar{\psi}_{\lambda_2'}(p_2')\gamma_5\psi_{\lambda_2}(p_2)\Phi(p_2-p_2').$$

Spin factors which have to be evaluated are of the type

$$\Sigma^{\pm\pm}=\chi_{\pm}{}^{\dagger}e^{i\sigma_y\theta/2}\chi_{\pm},$$

and have the values

$$\begin{aligned}\Sigma^{++}&=\Sigma^{--}=\cos(\theta/2),\\ \Sigma^{+-}&=-\Sigma^{-+}=\sin(\theta/2).\end{aligned}\tag{31}$$

From Eq. (30) we have

$$\begin{aligned}\langle\tfrac{1}{2}|\pm\tfrac{1}{2}\rangle&=N(p_2)N(p_2')[\mp KA+qB]\Sigma^{-\mp},\\ \langle-\tfrac{1}{2}|\pm\tfrac{1}{2}\rangle&=N(p_2)N(p_2')[\mp KA-qB]\Sigma^{+\mp},\end{aligned}\tag{32}$$

where $A=(E_2'+m)$, $B=(E_2+m)$, and m denotes the proton mass.

(ii) *Upper Vertex* $\langle\lambda_1'|\lambda_1\rangle$

The interaction is

$$L_I=(G^*/m_\pi)\bar{\psi}_\mu(p_1')\psi_{\lambda_1}(p_1)(p_1'-p_1)^\mu\Phi(p_1'-p_1).$$

Applying Eq. (2), this simplifies to

$$L_I=-(G^*/m_\pi)\bar{\psi}_\mu(p_1')\psi_{\lambda_1}(p_1)p_1{}^\mu\Phi(p_1'-p_1).$$

We can immediately write down the matrix elements for the various helicity states using the expressions appearing in Eqs. (A3) and (30):

$$\begin{aligned}\langle\tfrac{3}{2}|\pm\tfrac{1}{2}\rangle&=-(G^*/m_\pi)[N(p_1)N(p_1')/\sqrt{2}][(C\mp Kq)K\sin\theta]\Sigma^{+\pm},\\ \langle\tfrac{1}{2}|\pm\tfrac{1}{2}\rangle&=-(G^*/m_\pi)[N(p_1)N(p_1')/\sqrt{6}]\\ &\quad\times[(C\pm Kq)(K\sin\theta)\Sigma^{-\pm}-(2E_1'/M)(C\mp Kq)(K\cos\theta)\Sigma^{+\pm}+2q(E_1/M)(C\mp Kq)\Sigma^{+\pm}],\\ \langle-\tfrac{1}{2}|\pm\tfrac{1}{2}\rangle&=-(G^*/m_\pi)[N(p_1)N(p_1')/\sqrt{6}]\\ &\quad\times[-(C\mp Kq)(K\sin\theta)\Sigma^{+\pm}-(2E_1'/M)(C\pm Kq)(K\cos\theta)\Sigma^{-\pm}+2E_1(q/M)(C\pm Kq)\Sigma^{-\pm}],\\ \langle-\tfrac{3}{2}|\pm\tfrac{1}{2}\rangle&=-(G^*/m_\pi)[N(p_1)N(p_1')/\sqrt{2}][-(C\pm Kq)(K\sin\theta)\Sigma^{-\pm}],\end{aligned}\tag{33}$$

[4] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

where $C=(E_1'+M)(E_1+m)$ and M denotes the isobar mass. These amplitudes are related by

$$\langle\lambda|\lambda'\rangle=\langle-\lambda|-\lambda'\rangle(-)^{\lambda+\lambda'+s-\frac{1}{2}}.$$

This relation follows from parity invariance.

The amplitude for the required helicity state can be constructed by forming the product of the proper terms from the two vertices, and introducing the propagator, e.g.,

$$\langle\tfrac{1}{2}\tfrac{1}{2}|\Phi|\tfrac{1}{2}\tfrac{1}{2}\rangle=(-g)(G^*/m_\pi)N(p_1)N(p_1')N(p_2)N(p_2')(1/\sqrt{6})$$
$$\times[\{(qB-KA)\}[1/(m_\pi^2-t)]\{[(qE_1-KE_1'\cos\theta)/M](C-Kq)(1+\cos\theta)-\tfrac{1}{2}(C+Kq)K\sin^2\theta\}],$$

and $t=2(m^2-E_2E_2'+Kq\cos\theta)$.

(b) Production of Spin-$\frac{5}{2}$ Isobar

For the production of the $N_{5/2^+}{}^*(1688)$ in proton-proton scattering with one-pion exchange we take the following interaction at the upper vertex:

$$L_I=(G'/m_\pi^2)\bar{\psi}_{\alpha\beta}(p_1')\gamma_5\psi(p_1)p_1{}^\beta p_1{}^\alpha\Phi(p_1'-p_1),$$

where the spin-$\frac{5}{2}$ wave function is symmetric, i.e., $\psi_{\alpha\beta}=\psi_{\beta\alpha}$, and we have used Eq. (2). The lower vertex is as in the previous case.

Writing down the helicity amplitudes of the upper vertex in the form $\langle\lambda_1'|\lambda_1\rangle$, we have

$$\langle\tfrac{5}{2}|\pm\tfrac{1}{2}\rangle=(G'/m_\pi^2)[N(p_1)N(p_1')/2][FK^2(\sin^2\theta)\Sigma^{+\pm}],$$

$$\langle\tfrac{3}{2}|\pm\tfrac{1}{2}\rangle=(G'/m_\pi^2)[N(p_1)N(p_1')/2\sqrt{5}]$$
$$\times[-HK^2(\sin^2\theta)\Sigma^{-\pm}-4F(E_1'/M)K^2(\cos\theta\sin\theta)\Sigma^{+\pm}+4FE_1(q/M)K(\sin\theta)\Sigma^{+\pm}],$$

$$\langle\tfrac{1}{2}|\pm\tfrac{1}{2}\rangle=(G'/m_\pi^2)[N(p_1)N(p_1')/\sqrt{10}][-FK^2(\sin^2\theta)\Sigma^{+\pm}+2H(E_1'/M)K^2(\sin\theta\cos\theta)\Sigma^{-\pm}$$
$$-2HE_1(q/M)K(\sin\theta)\Sigma^{-\pm}+2F(E_1'^2/M^2)K^2(\cos^2\theta)\Sigma^{+\pm}-4F(E_1'/M^2)E_1qK(\cos\theta)\Sigma^{+\pm}+2FE_1^2(q^2/M^2)\Sigma^{+\pm}],$$

$$\langle-\tfrac{1}{2}|\pm\tfrac{1}{2}\rangle=(G'/m_\pi^2)[N(p_1)N(p_1')/\sqrt{10}][+HK^2(\sin^2\theta)\Sigma^{-\pm}+2F(E_1'/M)K^2(\sin\theta\cos\theta)\Sigma^{+\pm}$$
$$-2F(E_1/M)qK(\sin\theta)\Sigma^{+\pm}-2H(E_1'^2/M^2)K^2(\cos^2\theta)\Sigma^{-\pm}$$
$$+4H(E_1'/M^2)E_1qK(\cos\theta)\Sigma^{-\pm}-2HE_1^2(q^2/M^2)\Sigma^{-\pm}],$$

$$\langle-\tfrac{3}{2}|\pm\tfrac{1}{2}\rangle=(G'/m_\pi^2)[N(p_1)N(p_1')/2\sqrt{5}]$$
$$\times[FK^2(\sin^2\theta)\Sigma^{+\pm}-4H(E_1'/M)K^2(\cos\theta\sin\theta)\Sigma^{-\pm}+4HE_1(q/M)K(\sin\theta)\Sigma^{-\pm}],$$

$$\langle-\tfrac{5}{2}|\pm\tfrac{1}{2}\rangle=(G'/m_\pi^2)[N(p_1)N(p_1')/2][-HK^2(\sin^2\theta)\Sigma^{-\pm}],$$

where

$$F=[\mp K(E_1'+M)+q(E_1+m)],$$
$$H=[\pm K(E_1'+M)+q(E_1+m)],$$

and hence we have the helicity amplitude

$$\langle\tfrac{5}{2}\tfrac{1}{2}|\Phi|\tfrac{1}{2}\tfrac{1}{2}\rangle=(G'g/m_\pi^2)[N(p_1)N(p_1')N(p_2)N(p_2')/4][F(qB-KA)K^2(\sin^2\theta)(1+\cos\theta)]\times1/(m_\pi^2-t).$$

All other amplitudes can be constructed in the same way. The relation $\langle\lambda|\lambda'\rangle=\langle-\lambda|-\lambda'\rangle(-)^{\lambda+\lambda'+s-\frac{1}{2}}$ relates the various amplitudes, as before.

The calculation of matrix elements for the production of isobars via vector-meson exchange can be tackled as in the above. The one difference is that for this case we cannot treat the upper and lower vertices separately. The appearance of Dirac gamma matrices between the two-component spinors χ is treated as for the spin-$\frac{1}{2}$ case. Useful results are contained in GGMW,[4] Eqs. (4.11) to (4.16).

IV. APPENDIX

Below are listed the explicit forms of the wave functions for different helicity states:

S=1

$$\begin{array}{lllll} S^z=1: & U_1=1/\sqrt{2}, & U_2=i/\sqrt{2}, & U_3=0, & U_0=0. \\ S^z=0: & U_1=0, & U_2=0, & U_3=-E/m, & U_0=-p/m. \\ S^z=-1: & U_1=-1/\sqrt{2}, & U_2=i/\sqrt{2}, & U_3=0, & U_0=0. \end{array} \quad \text{(A1)}$$

$S=2$

The wave functions $U_{\mu\nu}$ are symmetric in μ and ν, i.e., $U_{\mu\nu}=U_{\nu\mu}$.

$$
\begin{aligned}
S^z=2:\quad & U_{11}=\tfrac{1}{2},\quad U_{12}=i/2,\quad U_{22}=-\tfrac{1}{2},\\
& U_{13}=U_{23}=U_{33}=U_{00}=U_{01}=U_{02}=U_{03}=0.\\
S^z=1:\quad & U_{13}=-E/2m,\quad U_{23}=-iE/2m,\quad U_{10}=-p/2m,\quad U_{20}=-ip/2m,\\
& U_{11}=U_{22}=U_{12}=U_{33}=U_{30}=U_{00}=0.\\
S^z=0:\quad & U_{11}=-1/\sqrt{6},\quad U_{22}=-1/\sqrt{6},\quad U_{33}=(2/\sqrt{6})E^2/m^2,\\
& U_{30}=(2/\sqrt{6})pE/m^2,\quad U_{00}=(2/\sqrt{6})p^2/m^2,\\
& U_{12}=U_{13}=U_{23}=U_{10}=U_{20}=0.\\
S^z=-1:\quad & U_{13}=E/2m,\quad U_{23}=-iE/2m,\quad U_{10}=p/2m,\quad U_{20}=-ip/2m,\\
& U_{11}=U_{22}=U_{12}=U_{33}=U_{30}=U_{00}=0.\\
S^z=-2:\quad & U_{11}=\tfrac{1}{2},\quad U_{12}=-i/2,\quad U_{22}=-\tfrac{1}{2},\\
& U_{13}=U_{23}=U_{33}=U_{00}=U_{01}=U_{02}=U_{03}=0.
\end{aligned}
\tag{A2}
$$

$S=\tfrac{3}{2}$

Using the notation

$$\psi_+=N\begin{pmatrix}E+m\\ p\end{pmatrix}\chi_+,\quad \psi_-=N\begin{pmatrix}E+m\\ -p\end{pmatrix}\chi_-,$$

where

$$N=1/[2m(E+m)]^{1/2},\quad \chi_+=\begin{pmatrix}1\\0\end{pmatrix},\quad\text{and}\quad \chi_-=\begin{pmatrix}0\\1\end{pmatrix},$$

we have

$$
\begin{aligned}
S^z=\tfrac{3}{2}:\quad & \psi_1=\frac{1}{\sqrt{2}}\psi_+,\quad \psi_2=\frac{i}{\sqrt{2}}\psi_+,\quad \psi_3=\psi_0=0;\\
S^z=\tfrac{1}{2}:\quad & \psi_1=\frac{1}{\sqrt{6}}\psi_-,\quad \psi_2=\frac{i}{\sqrt{6}}\psi_-,\quad \psi_3=-\frac{2}{\sqrt{6}}\frac{E}{m}\psi_+,\quad \psi_0=-\frac{2}{\sqrt{6}}\frac{p}{m}\psi_+;\\
S^z=-\tfrac{1}{2}:\quad & \psi_1=-\frac{1}{\sqrt{6}}\psi_+,\quad \psi_2=\frac{i}{\sqrt{6}}\psi_+,\quad \psi_3=-\frac{2}{\sqrt{6}}\frac{E}{m}\psi_-,\quad \psi_0=-\frac{2}{\sqrt{6}}\frac{p}{m}\psi_-;\\
S^z=-\tfrac{3}{2}:\quad & \psi_1=-\frac{1}{\sqrt{2}}\psi_-,\quad \psi_2=\frac{i}{\sqrt{2}}\psi_-,\quad \psi_3=\psi_0=0.
\end{aligned}
\tag{A3}
$$

$S=\tfrac{5}{2}$

(where $\psi_{\mu\nu}=\psi_{\nu\mu}$)

$$
\begin{aligned}
S^z=\tfrac{5}{2}:\quad & \psi_{11}=\frac{1}{2}\psi_+,\quad \psi_{12}=\frac{i}{2}\psi_+,\quad \psi_{22}=-\frac{1}{2}\psi_+,\\
& \psi_{13}=\psi_{23}=\psi_{33}=\psi_{00}=\psi_{01}=\psi_{02}=\psi_{03}=0.\\
S^z=\tfrac{3}{2}:\quad & \psi_{11}=\frac{1}{2\sqrt{5}}\psi_-,\quad \psi_{12}=\frac{i}{2\sqrt{5}}\psi_-,\quad \psi_{22}=-\frac{1}{2\sqrt{5}}\psi_-,\quad \psi_{13}=-\frac{1}{\sqrt{5}}\frac{E}{m}\psi_+,\\
& \psi_{23}=-\frac{i}{\sqrt{5}}\frac{E}{m}\psi_+,\quad \psi_{10}=-\frac{1}{\sqrt{5}}\frac{p}{m}\psi_+,\quad \psi_{20}=-\frac{i}{\sqrt{5}}\frac{p}{m}\psi_+,\\
& \psi_{33}=\psi_{30}=\psi_{00}=0.
\end{aligned}
$$

$$S^z=\tfrac{1}{2}:\quad \psi_{11}=-\frac{1}{\sqrt{10}}\psi_+,\quad \psi_{22}=-\frac{1}{\sqrt{10}}\psi_+,\quad \psi_{13}=-\frac{1}{\sqrt{10}}\frac{E}{m}\psi_-,\quad \psi_{23}=-\frac{i}{\sqrt{10}}\frac{E}{m}\psi_-,$$
$$\psi_{33}=\frac{2}{\sqrt{10}}\frac{E^2}{m^2}\psi_+,\quad \psi_{10}=-\frac{1}{\sqrt{10}}\frac{p}{m}\psi_-,\quad \psi_{20}=-\frac{i}{\sqrt{10}}\frac{p}{m}\psi_-,\quad \psi_{30}=\frac{2}{\sqrt{10}}\frac{pE}{m^2}\psi_+,$$
$$\psi_{00}=\frac{2}{\sqrt{10}}\frac{p^2}{m^2}\psi_+,\quad \psi_{12}=0.$$

$$S^z=-\tfrac{1}{2}:\quad \psi_{11}=-\frac{1}{\sqrt{10}}\psi_-,\quad \psi_{22}=-\frac{1}{\sqrt{10}}\psi_-,\quad \psi_{13}=\frac{1}{\sqrt{10}}\frac{E}{m}\psi_+,\quad \psi_{23}=-\frac{i}{\sqrt{10}}\frac{E}{m}\psi_+,$$
$$\psi_{33}=\frac{2}{\sqrt{10}}\frac{E^2}{m^2}\psi_-,\quad \psi_{10}=\frac{1}{\sqrt{10}}\frac{p}{m}\psi_+,\quad \psi_{20}=-\frac{i}{\sqrt{10}}\frac{p}{m}\psi_+,\quad \psi_{30}=\frac{2}{\sqrt{10}}\frac{pE}{m^2}\psi_-,$$
$$\psi_{00}=\frac{2}{\sqrt{10}}\frac{p^2}{m^2}\psi_-,\quad \psi_{12}=0. \tag{A4}$$

$$S^z=-\tfrac{3}{2}:\quad \psi_{11}=\frac{1}{2\sqrt{5}}\psi_+,\quad \psi_{12}=-\frac{i}{2\sqrt{5}}\psi_+,\quad \psi_{22}=-\frac{1}{2\sqrt{5}}\psi_+,\quad \psi_{13}=\frac{1}{\sqrt{5}}\frac{E}{m}\psi_-,$$
$$\psi_{23}=-\frac{i}{\sqrt{5}}\frac{E}{m}\psi_-,\quad \psi_{10}=\frac{1}{\sqrt{5}}\frac{p}{m}\psi_-,\quad \psi_{20}=-\frac{i}{\sqrt{5}}\frac{p}{m}\psi_-,$$
$$\psi_{33}=\psi_{30}=\psi_{00}=0.$$

$$S^z=-\tfrac{5}{2}:\quad \psi_{11}=\frac{1}{2}\psi_-,\quad \psi_{12}=-\frac{i}{2}\psi_-,\quad \psi_{22}=-\frac{1}{2}\psi_-,$$
$$\psi_{13}=\psi_{23}=\psi_{33}=\psi_{00}=\psi_{01}=\psi_{02}=\psi_{03}=0.$$

ACKNOWLEDGMENTS

We would like to thank Professor G. Yekutieli and other members of the Experimental High Energy group at the Weizmann Institute for helpful discussions, and for suggesting the problem which arose from their recent experimental findings. Thanks are also due to Dr. G. Alexander and B. Harber for drawing our attention to an error in the Appendix.