Infrared Divergence in Quantum Electrodynamics*

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The infrared divergences of quantum electrodynamics are eliminated to all orders of perturbation theory in the matrix elements by an appropriate choice of initial and final soft photon states. The condition for this cancellation restricts these states to representations of the canonical commutation rules which are unitarily inequivalent to the usual Fock representation.

I. INTRODUCTION

THE matrix element in quantum electrodynamics for the scattering from an initial state containing a finite number of electrons and photons into a similar final state contains an integral which diverges logarithmically for small momentum k. The conventional treatment of this "infrared divergence" has been to sum the cross sections over all possible final states consistent with experimental measurements. In particular, when all states with any number of soft photons with momenta below the threshold of observability are considered, the divergences cancel, and the calculated cross sections are consistent with experiment. It is therefore possible to attribute the original divergence in the matrix element to the inappropriate choice of initial and final states to represent the experimental situation. In an actual scattering experiment, an indefinite number of soft photons are emitted, so that in some sense, states which are eigenstates of the number operator are unphysical.

In this paper, we shall show that there exists a representation of the photon states for quantum electrodynamics which appears more appropriate for describing scattering than the usual Fock representation in that the matrix elements do not have infrared divergences. These states are not eigenstates of the number operator, and are parametrized in a manner similar to that used by Glauber,[1] Bargmann,[2] and others. When certain conditions of convergence are imposed, the states can be shown to form irreducible representations of the canonical commutation rules for the "in" and "out" fields which are unitarily inequivalent to the usual Fock representation. Similar results have been obtained by Shroer[3] in certain model field theories.

In the absence of known solutions to the renormalized field equations, we make no pretense to mathematical rigor. In particular, the Feynman-Dyson perturbation techniques are used throughout, and most questions of order in limiting procedures, etc., are treated heuristically.

Section II will summarize the parts of the conventional treatment of infrared divergences which we shall need. This section is based on a more complete discussion made in the article by Yennie, Frautschi, and Suura.[4] The parametrization of the states and its relationship to the usual occupation number parametrization are introduced in Sec. III. We shall make use of the algebra of states developed in Glauber's paper.[1] In Sec. IV, the cancellation of the divergences to second order is demonstrated in order to illustrate the methods used in the succeeding sections. A calculation of the matrix elements for potential scattering in Sec. V shows that the divergences indeed cancel to all orders. In Sec. VI, the structure and the physical meaning of the representations are examined. Then we show that by squaring the matrix elements and summing over the final states results are obtained in low order which agree with those obtained by Yennie *et al.*[4] by the conventional treatment. Some extensions and generalizations of our treatment are carried out in the appendices.

II. SEPARATION OF THE INFRARED FACTORS

The following exposition of the separation of the infrared parts from the matrix element can be found in the review article by Yennie *et al.*[4] We will summarize here what is relevant to our own discussion. For simplicity, we study the example of an electron scattering from a potential, although similar results can be obtained for more general situations.

Consider a process in which there are a fixed number of photons and an electron of momentum p in the initial state and a fixed number of photons with the scattered electron of momentum p' in the final state. The photons may or may not have interacted with the electron line. The complete matrix element for this process is given by

$$M(\mathbf{p},\mathbf{p}')=\sum_{n=0}^{\infty} M_n(\mathbf{p},\mathbf{p}'), \tag{1}$$

where $M_n(\mathbf{p},\mathbf{p}')$ corresponds to the sum of all diagrams in which there are n virtual photons which can be distinguished from the potential interactions in the "basic process" M_0. The real photon variables have been suppressed.

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[3] B. Schroer, Fortschr. Physik **11**, 1 (1963).
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The quantity $\rho_n(k_1,\cdots,k_n)$ is defined by the relation

$$M_n=\frac{1}{n!}\int\cdots\int\prod_{i=1}^{n}\frac{d^4k}{k_i^2-\lambda^2+i\epsilon}\rho_n(k_1,\cdots,k_n), \quad (2)$$

where λ is the photon mass which we allow to approach zero later. It has been shown that ρ_n is of the form

$$\rho_n(k_1,\cdots,k_n)=S(k_n)\rho_{n-1}(k_1,\cdots,k_{n-1}) +\xi^{(1)}(k_1,\cdots,k_{n-1};k_n), \quad (3)$$

where $S(k_n)$ contains the k_n infrared divergence, and can have the form

$$S(k_n)=\frac{1}{2}\left[\frac{ie^2}{(2\pi)^4}\right]\left(\frac{2p_\mu'-k_\mu}{2p'\cdot k-k^2}-\frac{2p_\mu-k_\mu}{2p\cdot k-k^2}\right). \quad (4)$$

The remainder $\xi^{(1)}$ has no infrared divergence in k_n, and its infrared divergence in the other k's has not been made worse by the separation.

By iteration of Eq. (3), $\rho_n(k_1,\cdots,k_n)$ can be expressed as a sum over all permutations of the k's:

$$\rho_n(k_1,\cdots,k_n)=\sum_{\text{perm}}\sum_{r=0}^{n}\frac{1}{r!(n-r)!} \times\prod_{i=1}^{r}S(k_i)\xi_{n-r}(k_{r+1},\cdots,k_n). \quad (5)$$

The functions ξ_r are noninfrared and symmetrical in the k's. If we adopt the definitions

$$\alpha B(\mathbf{p},\mathbf{p}')\equiv\int\frac{d^4k\,S(k)}{k^2-\lambda^2}, \quad (6a)$$

$$m_r(\mathbf{p},\mathbf{p}')\equiv\frac{1}{r!}\int\prod_{i=1}^{r}\frac{d^4k_i}{k_i^2}\xi_r(k_1,\cdots,k_n), \quad (6b)$$

then substitution of (2), (5), (6a), and (6b) into (1) results in the simple expression

$$M=\exp(\alpha B)\sum_{n=0}^{\infty}m_n. \quad (7)$$

In this expression, $m_0=\rho_0=\xi_0=M_0$. The m_n's in (7) are divergence-free, so that the whole infrared divergence has been isolated in the argument αB of the exponential. For future reference, we can write down the form of $\mathrm{Re}(\alpha B)$ which follows from (4) and (6):

$$\mathrm{Re}(\alpha B)=\frac{e^2}{4(2\pi)^3}\int\frac{d^3k}{(k^2+\lambda^2)^{1/2}} \times\left(\frac{2p_\mu'-k_\mu}{2p'\cdot k-\lambda^2}-\frac{2p_\mu-k_\mu}{2p\cdot k-\lambda^2}\right)^2. \quad (8)$$

The extraction of the infrared contribution to the matrix element for the emission of real photons has a form similar to that in Eq. (3). In this case, we let $\tilde{\rho}_n(k_1,\cdots,k_n)$ be the matrix element corresponding to the emission or absorption of n undetectable photons with momenta $k_1,\cdots,k_n$, and for some arbitrary order in the virtual photon corrections. It has been shown that

$$\tilde{\rho}_n(k_1,\cdots,k_n)=\pm\tilde{S}(k_n)\tilde{\rho}_{n-1}(k_1,\cdots,k_n) +\tilde{\xi}^{(1)}(k_1,\cdots,k_{n-1};k_n), \quad (9)$$

where $\tilde{S}(k_n)$ is the factor containing the infrared divergence and has the form

$$\tilde{S}(k)=\frac{e}{[2(2\pi)^3k_0]^{1/2}}\left[\frac{p'\cdot e}{k\cdot p'}-\frac{p\cdot e}{p\cdot k}\right], \quad (10)$$

and the (+) and (−) signs correspond to emission and absorption, respectively. Again the remainder $\tilde{\xi}^{(1)}$ is divergence-free in k_n, and the divergences in the other k's is no worse for the separation.

It has been shown that the iteration of (9) leads to the form

$$\tilde{\rho}_n(k_1,\cdots,k_n)=\sum_{\text{perm}}\sum_{r=0}^{n}(-1)^m\frac{1}{r!(n-r)!} \times\prod_{i=1}^{r}\tilde{S}(k_i)\tilde{\xi}_{n-r}(k_{r+1},\cdots,k_n), \quad (11)$$

where the functions $\tilde{\xi}$ are noninfrared and symmetrical in the k's and m corresponds to the number of absorbed photons.

III. PARAMETRIZATION OF THE PHOTON STATES

The properties of the states which we will find convenient to use have been discussed by several other authors[1,2] in different contexts from the one in which we intend to use them.

Let $\{f_i(k)\}$ be a complete and orthonormal set of functions defined on some region Ω of momentum space including $k=0$ (perhaps all of momentum space). A typical state "belonging to the ith mode" is defined by

$$|\alpha_i\rangle=\frac{\exp(\alpha_i a_i^\dagger)}{\exp(\frac{1}{2}|\alpha_i|^2)}|0\rangle=\exp(-\tfrac{1}{2}|\alpha_i|^2)\sum_n\frac{(\alpha_i a_i^\dagger)^n}{n!}|0\rangle, \quad (13)$$

where

$$a_i^\dagger=\int d^3k\,f_i(k)a^\dagger(k) \quad (14)$$

is an "in" or "out" creation operator.

In this expression, α_i is a complex number which can take on any value in the complex plane, $a^\dagger(k)$ is the photon creation operator which obeys the commutation rules

$$[a(k),a^\dagger(k')]=\delta(k-k'),$$
$$[a(k),a(k')]=[a^\dagger(k),a^\dagger(k')]=0, \quad (15)$$

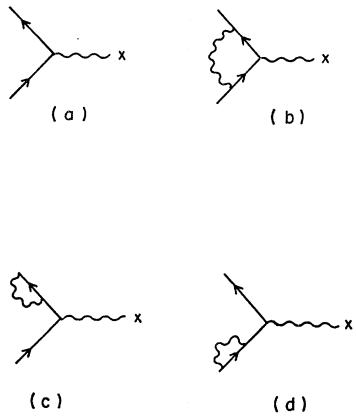

FIG. 1. Contributions to the second-order virtual-photon radiative corrections. Diagram (a) corresponds to the original uncorrected matrix element M_0.

and $|0\rangle$ is the state with no photons; $a_i^\dagger$ obeys the commutation rules

$$[a_i,a_j^\dagger]=\delta_{ij},\quad [a_i,a_j]=[a_i^\dagger,a_j^\dagger]=0. \tag{16}$$

From the commutation rules, it is a trivial matter to show that these states are eigenfunctions of the destruction operator

$$\begin{aligned} &a(k)|\alpha_i\rangle=\alpha_i f_i(k)|\alpha_i\rangle \\ \text{or}\qquad &\\ &a_i|\alpha_i\rangle=\alpha_i|\alpha_i\rangle, \end{aligned} \tag{17}$$

and that the mean number of "photons" is

$$\langle\alpha_i|\hat{N}|\alpha_i\rangle=\int d^3k\langle\alpha_i|a^\dagger(k)a(k)|\alpha_i\rangle=|\alpha_i|^2. \tag{18}$$

It is sometimes useful to note that the state in Eq. (13) can be "created" by a unitary operator

$$D(\alpha_i)=\exp[\alpha_i a_i^\dagger-\alpha_i^* a_i], \tag{19}$$

which has the following "translation" property:

$$D(\alpha_i)D(\beta_i)=\exp[\tfrac{1}{2}(\alpha_i\beta_i^*-\alpha_i^*\beta_i)]D(\alpha_i+\beta_i). \tag{20}$$

The states defined in this manner are nonorthogonal; the overlap between two states $|\alpha_i\rangle$ and $|\beta_i\rangle$ is given by

$$|\langle\alpha_i|\beta_i\rangle|^2=\exp\{-|\alpha_i-\beta_i|^2\}. \tag{21}$$

However, it follows from (21) that the states are normalized, i.e,

$$\langle\alpha_i|\alpha_i\rangle=1. \tag{22}$$

Another property which these states possess is completeness. In fact, it is easy to show that

$$\frac{1}{\pi}\int d^2\alpha_i|\alpha_i\rangle\langle\alpha_i|=\sum_{n_i}|n_i\rangle\langle n_i|=I, \tag{23}$$

where the state denoted by n_i is an eigenstate of the number of photons which have the momentum distribution described by the function $f_i(k)$, and $d^2\alpha=d(\mathrm{Re}\alpha_i)d(\mathrm{Im}\alpha_i)$ is real.

An arbitrary state of the ith mode has an expansion in terms of the n-photon states of the form

$$|\ \rangle=\sum_n c_n|n\rangle=\sum_n c_n\frac{(a_i^\dagger)^n}{(n!)^{1/2}}|0\rangle, \tag{24}$$

where $\sum_n|c_n|^2=1$. We associate with each such state an analytic function,

$$f(z)=\sum_n c_n\frac{z^n}{(n!)^{1/2}}. \tag{25}$$

Equation (24) may then be rewritten as

$$|f\rangle=f(a_i^\dagger)|0\rangle. \tag{26}$$

Using (23), we can expand $|f\rangle$ in terms of the new states:

$$\begin{aligned} |f\rangle&=\frac{1}{\pi}\int d^2\alpha_i|\alpha_i\rangle\langle\alpha_i|f(a_i^\dagger)|0\rangle \\ &=\frac{1}{\pi}\int d^2\alpha_i|\alpha_i\rangle f(\alpha_i^*)\exp(-\tfrac{1}{2}|\alpha_i|^2). \end{aligned} \tag{27}$$

In (27) we have used the fact that the states $|\alpha_i\rangle$ are eigenstates of the destruction operator a_i:

$$a_i|\alpha_i\rangle=\alpha_i|\alpha_i\rangle. \tag{28}$$

In a similar fashion, the adjoint state vectors $\langle g|$ can be shown to possess an analogous expansion,

$$\langle g|=\frac{1}{\pi}\int[g(\beta_i^*)]^*\langle\beta_i|\exp(-\tfrac{1}{2}|\beta_i|^2)d^2\beta_i. \tag{29}$$

A basis for the whole electromagnetic field is a direct product of the states $|\alpha_i\rangle$ of the individual modes

$$|\ \rangle=\prod_i|\alpha_i\rangle\equiv|\{\alpha_i\}\rangle, \tag{30}$$

and the mean number of photons in such a state is

$$\sum_i|\alpha_i|^2. \tag{31}$$

Equations (25), (27), and (30) ensure that states containing a finite number of photons (the usual Fock representation) can be expanded in terms of the states $|\{\alpha_i\}\rangle$ which satisfy $\sum_i|\alpha_i|^2<\infty$. This will be shown in Sec. VI. However, this restriction will not be imposed in the discussion that follows, i.e., we shall allow for the possibility that there exist states in which the average number of photons is not bounded.

IV. CANCELLATION OF THE INFRARED DIVERGENCES TO SECOND ORDER

In order to illustrate the general procedure, we shall choose two particular photon states parametrized in the manner just discussed in Sec. III and calculate in lowest order the matrix element for potential scattering from one to the other. The photon is assumed to have

a finite mass λ which is allowed to approach zero at the end of the calculation.

Let the initial momentum of the electron be p_i and the final momentum of the scattered electron be p_f. For the initial state, we choose

$$\begin{aligned}|\ \rangle_i &= \exp\left\{-\tfrac{1}{2}\sum_{l=1}^{2}\int|\tilde{S}_i^{(l)}(k)|^2 d^3k\right\}\\ &\times\exp\left\{\sum_{l=1}^{2}\int d^3k\,\tilde{S}_i^{(l)}(k)e^{(l)}(k)a^{(l)\dagger}(k)\right\}|\Psi(p_i)\rangle\\ &=\exp\{-\tfrac{1}{2}\sum_{l,a}|\beta_{ia}{}^l|^2\}\\ &\times\exp\left\{\sum_{l,a}\beta_{ia}{}^l\int d^3k\,f_a(k)e^{(l)}(k)a^{(l)\dagger}(k)\right\}|\Psi(p_i)\rangle,\end{aligned} \tag{32}$$

where

$$\tilde{S}_i^{(l)}(k)=\frac{e}{[2(2\pi)^3k_0]^{1/2}}\frac{p_i\cdot e^{(l)}}{k\cdot p_i} \tag{33}$$

is a function which depends on the momentum of the initial electron. $|\Psi(p_i)\rangle$ is the wave function for the electron, and $e^{(l)}(k)$ are the polarization vectors. The superscript (l) is the polarization index. Since Eq. (33) is meant to define the momentum distribution only as $|k|\to 0$, the function $\tilde{S}_i(k)$ for $k\neq 0$ can be chosen in any manner which makes the integrals in (32) converge as $k\to\infty$. The second form given above exhibits the relation to the discussion in Sec. III. The coefficients $\beta_{ia}{}^l$ are the coefficients obtained in the expansion of $\tilde{S}_i{}^l(k)$ in terms of the chosen orthonormal set.

The initial state can then be expanded in lowest order to give

$$\begin{aligned}|\ \rangle_i &\cong\left(1-\tfrac{1}{2}\sum_{l=1}^{2}\int|\tilde{S}_i^{(l)}(k)|^2d^3k\right)\\ &\times\left(1+\sum_{l=1}^{2}\int d^3k\,\tilde{S}_i^{(l)}(k)e^{(l)}(k)a^{(l)\dagger}(k)\right)|\Psi(p_i)\rangle.\end{aligned} \tag{34}$$

Similarly, the final state can be expanded to give

$$\begin{aligned}|\ \rangle_f &\cong\left(1-\tfrac{1}{2}\sum_{l=1}^{2}\int|\tilde{S}_f^{(l)}(k)|^2d^3k\right)\\ &\times\left(1+\sum_{l=1}^{2}\int d^3k\,\tilde{S}_f^{(l)}(k)e^{(l)}(k)a^{(l)\dagger}(k)\right)|\Psi(p_f)\rangle,\end{aligned} \tag{35}$$

where

$$\tilde{S}_f^{(l)}(k)=\frac{e}{[2(2\pi)^3k_0]^{1/2}}\frac{p_f\cdot e^{(l)}}{k\cdot p_f}.$$

Let the basic interaction given by the matrix element M_0 be the single-potential interaction shown in Fig. 1(a). In order to calculate the S matrix to order e^2, the contributions from all the diagrams in Fig. 1 and Fig. 2 must be summed. Diagrams (b), (c), and (d) of Fig. 1

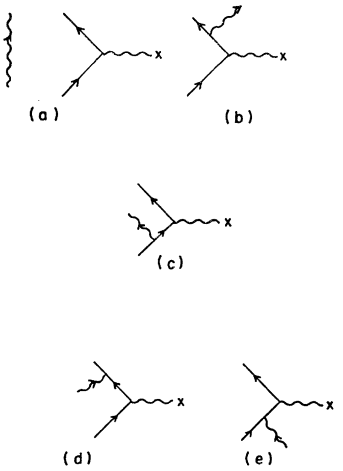

FIG. 2. Contributions to the second-order corrections due to emission or absorption of real soft photons. Diagram (a) accounts for the possibility that the photon does not interact with the electrons at all.

correspond to the virtual photon radiative corrections to M_0. Diagrams (b), (c), (d), and (e) of Fig. 2 account for the possibility of emission or absorption of single real photons, while diagram (a) of Fig. 2 accounts for the possibility that the photon does not interact with the electrons at all.

From the discussion in Sec. II of this paper, one knows that the diagrams in Fig. 1 will contribute

$$M_0+(\alpha B+\eta)M_0, \tag{36}$$

where η is a quantity which is not infrared divergent as $\lambda\to 0$. The contribution from diagrams (b) and (c) of Fig. 2 gives a term with a factor

$$\sum_l\int d^3k\,\tilde{S}_f^{(l)}(k)[\tilde{S}^{(l)}(k)M_0+\xi(k)], \tag{37}$$

where $\tilde{S}^{(l)}(k)$ was defined in Eq. (10). A similar contribution from diagrams (d) and (e) of Fig. 2 is

$$-\sum_l\int d^3k\,\tilde{S}_i^{(l)}(k)[\tilde{S}^{(l)}(k)M_0+\xi(k)]. \tag{38}$$

The disconnected diagram (a) of Fig. 2 is given by

$$M_0\sum_l\int d^3k\,\tilde{S}_i^{(l)}(k)\tilde{S}_f^{(l)}(k). \tag{39}$$

Summing Eqs. (36) to (39) with the proper normalization given by Eq. (54), one finds the result

$$\begin{aligned}M_{i\to f}&\cong(1+\alpha B+\eta)M_0\left(1-\tfrac{1}{2}\sum_l\int|\tilde{S}_i^{(l)}(k)|^2d^3k\right)\\ &\times\left(1-\tfrac{1}{2}\sum_l\int|\tilde{S}_f^{(l)}(k)|^2d^3k\right)\\ &+\left[\sum_l\int(\tilde{S}_f^{(l)}-\tilde{S}_i^{(l)})\tilde{S}^{(l)}(k)d^3k+\tilde{\eta}\right]M_0\\ &+M_0\sum_l\int d^3k\,\tilde{S}_i^{(l)}(k)\tilde{S}_f^{(l)}(k),\end{aligned} \tag{40}$$

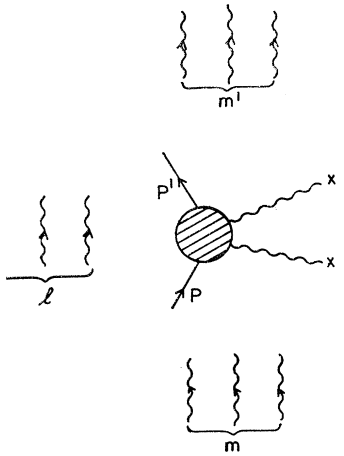

FIG. 3. Representation of l noninteracting real soft photons, m real soft photons absorbed by the electron line, and m' real soft photons emitted by the electron line.

where $\tilde{\eta}$ is noninfrared divergent. Thus

$$M_{i\to f}=(1+\alpha B)M_0+(\eta+\tilde{\eta})M_0+M_0\sum_l\int|\tilde{S}^{(l)}(k)|^2d^3k$$
$$-M_0\sum_l\int d^3k\{\tfrac{1}{2}|\tilde{S}_i{}^{(l)}(k)|^2+\tfrac{1}{2}|\tilde{S}_f{}^{(l)}(k)|^2$$
$$-\tilde{S}_f{}^{(l)}(k)\tilde{S}_i{}^{(l)}(k)\}+O(e^3)$$
$$=\left(1+\alpha B+\tfrac{1}{2}\sum_l\int|\tilde{S}^{(l)}(k)|^2d^3k\right)M_0$$
$$+(\eta+\tilde{\eta})M_0+O(e^3), \quad (41)$$

where we have used

$$\tilde{S}^{(l)}(k)\equiv\tilde{S}_f{}^{(l)}(k)-\tilde{S}_i{}^{(l)}(k). \quad (42)$$

From (10) we have the relation

$$\tfrac{1}{2}\sum_{l=1}^{2}\int|\tilde{S}^{(l)}(k)|^2d^3k$$
$$=\sum_{l=0}^{3}\frac{e^2}{4(2\pi)^3k_0}\int d^3k\left[\frac{p_\mu'\cdot e^{(l)}}{p'\cdot k}-\frac{p\cdot e^{(l)}}{p\cdot k}\right]^2(-g^{ll})$$
$$=-\frac{e^2}{4(2\pi)^3k_0}\int d^3k\left[\frac{p_\mu'}{p\cdot k}-\frac{p_\mu}{p\cdot k}\right]^2\equiv\alpha\tilde{B}. \quad (43)$$

Thus

$$M_{i\to f}=(1+\alpha B+\alpha\tilde{B})M_0+(\eta+\tilde{\eta})M_0+O(e^3). \quad (44)$$

By comparing Eq. (43) with Eq. (8), one can see that the infrared divergence which occurs when $\lambda\to 0$ has been canceled in Eq. (44).

Note that it is the matrix element which is finite. Thus in the calculation of the cross section, there will be no need to deal with an infinite sum of divergent integrals, as must be done in the conventional treatment of infrared divergences.

V. CANCELLATION OF INFRARED DIVERGENCES TO ALL ORDERS

It is now a matter of algebra to calculate the matrix element for the transition from a state of electron momentum $\mathbf{p}$ and photon "quantum numbers" $\{\alpha_a{}^\lambda\}$ to a state of electron momentum $\mathbf{p}'$ and photon "quantum numbers" $\{\gamma_c{}^\lambda\}$, where

$$|\{\alpha_a{}^\lambda\}\rangle=\prod_a\frac{\exp\sum_\lambda\left[\alpha_a{}^\lambda\int_\Omega d^3kf_a(k)e_\mu{}^{(\lambda)}(k)a^{(\lambda)\dagger}(k)\right]}{\exp[\sum_\lambda\tfrac{1}{2}|\alpha_a{}^\lambda|^2]}|0\rangle. \quad (45)$$

The superscripts (λ) refer to the polarization indices.

Consider all the diagrams represented by Fig. 3, in which there are m real photons absorbed by the electron line, m' real photons emitted by the electron line, and l photons which do not interact with the electron at all.

The matrix element for the process $|\{\alpha_a{}^\lambda\},p_i\rangle\to|\{\gamma_c{}^{\lambda'}\},p_f\rangle$ is then a sum over all diagrams of the kind shown in Fig. 3 for all values of m, m', and l, and with the proper factors determined by Eqs. (45).

Considerations which enter the calculation of this matrix element are expalined below:

(a) There is an infrared divergent factor $e^{\alpha B}$ due to the virtual photon corrections. [See Eq. (7)].

(b) The overlap of the l initial-state noninteracting photons with the l final-state noninteracting photons contributes a factor

$$l!\left[\sum_{\substack{\mu,a,c,\\ \lambda,\lambda'}}\alpha_a{}^\lambda\gamma_c{}^{*\lambda'}\int d^3kf_c{}^*(k)f_a(k)e_\mu{}^{(\lambda)}(k)e_\mu{}^{(\lambda')}(k)\right]^l$$
$$=l![\sum_{\substack{\lambda,\lambda',\\ a,c}}\alpha_a{}^\lambda\gamma_c{}^{*\lambda'}\delta_{\lambda\lambda'}\delta_{ac}]^l. \quad (46)$$

(c) Equation (11) gives the contribution due to the interaction of m initial-state photons and m' final-state photons with the electron line

$$\tilde{\rho}_{m+m'}{}^{\{\lambda\}}(k_1,\cdots,k_{m+m'})=\sum_{\substack{\text{perm}\\ k's}}\sum_{t=0}^{m+m'}(-1)^m(\prod_{i=1}^{t}\tilde{S}^{(\lambda)}(k_i))$$
$$\times\xi_{m+m'-t}{}^{\{\lambda\}}(k_{t+1},\cdots,k_{m+m'})\frac{1}{t!(m+m'-t)!}. \quad (47)$$

(d) Contribution (c) must be integrated over the momentum distribution that is obtained from the formal expansion of the initial and final states [see Eq. (45)]:

$$\left[\left(\prod_{r=1}^{m}\sum_{\lambda,a}a_a{}^\lambda\int d^3k_rf_a(k_r)\right)\right.$$
$$\times\left(\prod_{r'=m+1}^{m+m'}\sum_{\lambda',c}\gamma_c{}^{*\lambda'}\int d^3k_{r'}\,f_c{}^*(k_{r'})\right)$$
$$\left.\times\tilde{\rho}_{m+m'}{}^{\{\lambda,\lambda'\}}(k_1,\cdots,k_{m+m'})\right]. \quad (48)$$

(e) The formal expansion of Eq. (45) also leads to the factors

$$\frac{1}{(m+l)!}\frac{1}{(m'+l)!}\exp(-\tfrac{1}{2}\sum_{a,\lambda}|\alpha_a{}^{\lambda}|^2)\times\exp(-\tfrac{1}{2}\sum_{c,\lambda'}|\gamma_c{}^{\lambda'}|^2). \quad (49)$$

(f) In addition to the above, there is a combinatorial factor which accounts for the number of ways that $(m+l)$ initial-state photons and $(m'+l)$ final-state photons can be distributed among m initial-state interacting photons, m' final-state interacting photons, and l noninteracting photons:

$$\frac{(m+l)!}{m!l!}\frac{(m'+l)!}{m'!l!}. \quad (50)$$

After summing over all numbers m, m', and l, we arrive at the following expression for the matrix element $\tilde{M}$:

$$\tilde{M}=e^{\alpha B}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty}\left[\frac{1}{(m+l)!}\frac{1}{(m'+l)!}\right]\left[\frac{(m+l)!(m'+l)!}{m!m'!l!l!}\right][\exp(-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^{\lambda}|^2)\exp(-\tfrac{1}{2}\sum_{\lambda,a}|\gamma_a{}^{\lambda}|^2)][l!][\sum_{\lambda,a}\alpha_a{}^{\lambda}\gamma_a{}^{*\lambda}]^l$$
$$\times\left[\left(\prod_{r=1}^{m}\sum_{\lambda,a}\alpha_a{}^{\lambda}\int d^3k_r\, f_a(k_r)\right)\left(\prod_{r'=m+1}^{m+m'}\sum_{\lambda',c}\gamma_c{}^{*\lambda'}\int d^3k_{r'}\, f_c{}^{*}(k_{r'})\right)\tilde{\rho}_{m+m'}{}^{\{\lambda,\lambda'\}}(k_1,\cdots,k_{m+m'})\right]. \quad (51)$$

Another factor corresponding to the contribution from the scattering of photons by photons could have been included explicitly, but since this term does not contribute to the cancellation of infrared divergences, nor does add to the divergences, it has not been considered in this analysis.

Making the appropriate cancellations, and combining the terms with a little bit of careful counting, we arrive at the expression

$$M=e^{\alpha B}\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{m'=0}^{\infty}\frac{1}{m'!}[\sum_{a,\lambda}\alpha_a{}^{\lambda}\gamma_a{}^{*\lambda}]^l\left\{\sum_{j=0}^{m}\sum_{j'=0}^{m'}\frac{m!}{j!(m-j)!}\frac{m'!}{j'!(m'-j')!}\right.$$
$$\left.\times[-\sum_{a,\lambda}\alpha_a{}^{\lambda}(\tilde{S}^{\lambda},f_a)_{\Omega}]^j[\sum_{c,\lambda'}\gamma_c{}^{*\lambda'}(f_c{}^{*},\tilde{S}^{\lambda'})_{\Omega}]^{j'}P_{m-j,m'-j'}(p_i,p_f)\right\}[\exp(-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^{\lambda}|^2)\exp(-\tfrac{1}{2}\sum_{\lambda,a}|\gamma_a{}^{\lambda}|^2)], \quad (52)$$

where

$$(\tilde{S}^{\lambda},f_a)_{\Omega}\equiv\int_{\Omega}d^3k\,\tilde{S}^{(\lambda)}(k)f_a(k),\quad (f_c{}^{*},\tilde{S}^{\lambda'})_{\Omega}\equiv\int_{\Omega}d^3k\,\tilde{S}^{(\lambda')}(k)f_c{}^{*}(k),$$

and

$$P_{j,j'}\equiv(-1)^j\left(\prod_{r=1}^{j}\int d^3k_r\sum_{\lambda,a}\alpha_a{}^{\lambda}f_a(k_r)\right)\left(\prod_{r'=1}^{j'}\int d^3k_{r'}\sum_{\lambda',c}\gamma_c{}^{*\lambda'}f_c{}^{*}(k_{r'})\right)\tilde{\xi}_{j+j'}{}^{\{\lambda\}}(k_1,\cdots,k_{j+j'}). \quad (53)$$

Defining the "residuals" $m_{j,j'}$ by

$$m_{j,j'}(p_i,p_f)=\frac{P_{j,j'}}{j!j'!}, \quad (54)$$

and reordering the sums in (52), we can write

$$\tilde{M}=e^{\alpha B}\exp[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^{\lambda}|^2]\exp[-\tfrac{1}{2}\sum_{\lambda,a}|\gamma_a{}^{\lambda}|^2]\exp[\sum_{\lambda,a}\alpha_a{}^{\lambda}\gamma_a{}^{*\lambda}]$$
$$\times\exp[-\sum_{\lambda,a}\alpha_a{}^{\lambda}(\tilde{S}^{\lambda},f_a)_{\Omega}]\exp[\sum_{\lambda,c}\gamma_c{}^{*\lambda}(f_c{}^{*},\tilde{S}^{\lambda})_{\Omega}]\{\sum_{m,m'=0}^{\infty}m_{m,m'}\}. \quad (55)$$

To simplify the notation further, we define the coefficients $\beta_a{}^{\lambda}$ by

$$\beta_a{}^{\lambda}\equiv(f_a{}^{*},\tilde{S}^{\lambda})_{\Omega}. \quad (56)$$

Since the function $\tilde{S}$ is real, Eq. (55) becomes

$$\tilde{M}=e^{\alpha B}\exp[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^{\lambda}|^2]\exp[-\tfrac{1}{2}\sum_{\lambda,a}|\gamma_a{}^{\lambda}|^2]\exp[\sum_{\lambda,a}\alpha_a{}^{\lambda}\gamma_a{}^{*\lambda}]$$
$$\times\exp[-\sum_{\lambda,a}\alpha_a{}^{\lambda}\beta_a{}^{*\lambda}]\exp[\sum_{\lambda,a}\beta_a{}^{\lambda}\gamma_a{}^{*\lambda}]\{\sum_{m,m'=0}^{\infty}m_{m,m'}(p_i,p_f)\}. \quad (57)$$

As in Sec. IV, it proves convenient to split the function $\tilde{S}^{(\lambda)}$ into two parts:

$$\tilde{S}_i^{(\lambda)}(k)=\frac{e}{[2(2\pi)^3k_0]^{1/2}}\frac{p_i\cdot e^{(\lambda)}}{k\cdot p_i},\quad \tilde{S}_f^{(\lambda)}(k)=\frac{e}{[2(2\pi)^3k_0]^{1/2}}\frac{p_f\cdot e^{(\lambda)}}{k\cdot p_f}. \tag{58}$$

We can then define the coefficients $\beta_{ia}{}^{\lambda}$ and $\beta_{fa}{}^{\lambda}$ by

$$\beta_{ia}{}^{\lambda}=(f_a{}^*,\tilde{S}_i^{(\lambda)})_\Omega,\quad \beta_{fa}{}^{\lambda}=(f_a{}^*,\tilde{S}_f^{(\lambda)})_\Omega, \tag{59}$$

so that by Eqs. (10) and (56),

$$\beta_a{}^{\lambda}=\beta_{fa}{}^{\lambda}-\beta_{ia}{}^{\lambda}. \tag{60}$$

The complex coefficients $\{\alpha_a{}^{\lambda}\}$ and $\{\gamma_c{}^{\lambda'}\}$ which specify the initial and final states of the photons may be regarded as a set of coordinates in some complex infinite-dimensional space X. It is then possible to simplify Eq. (57) by making a translation of the coordinate system in X by the amounts defined in Eq. (59):

$$\gamma_a{}^{\lambda}=\beta_{fa}{}^{\lambda}+\epsilon_{fa}{}^{\lambda},\quad \alpha_a{}^{\lambda}=\beta_{ia}{}^{\lambda}+\epsilon_{ia}{}^{\lambda}. \tag{61}$$

Thus

$$\begin{aligned}\tilde{M}=e^{\alpha B}\{\exp\sum_{\lambda,a}[&-\tfrac{1}{2}|\beta_{ia}{}^{\lambda}+\epsilon_{ia}{}^{\lambda}|^2-\tfrac{1}{2}|\beta_{fa}{}^{\lambda}+\epsilon_{fa}{}^{\lambda}|^2+(\beta_{ia}{}^{\lambda}+\epsilon_{ia}{}^{\lambda})(\beta_{fa}{}^{\lambda}+\epsilon_{fa}{}^{\lambda})^*\\&-(\beta_{ia}{}^{\lambda}+\epsilon_{ia}{}^{\lambda})\beta_a{}^{*\lambda}+\beta_a{}^{\lambda}(\beta_{fa}{}^{\lambda}+\epsilon_{fa}{}^{\lambda})^*]\}\{\sum_{m,m'=0}^{\infty}m_{m,m'}\}\\=e^{\alpha B}\{\exp\sum_{a,\lambda}[&+\tfrac{1}{2}|\beta_f-\beta_i|^2-\tfrac{1}{2}|\epsilon_f-\epsilon_i|^2+i\,\mathrm{Im}(\beta_i{}^*\epsilon_i+\beta^{\alpha}\epsilon_f{}^*-\beta_i\beta_f{}^*+\epsilon_i\epsilon_f{}^*)]\}\{\sum_{m,m'=0}^{\infty}m_{m,m'}\},\end{aligned} \tag{62}$$

where the mode and polarization indices have been suppressed for convenience.

By Eqs. (43), (53), (56), and (60), we have

$$\begin{aligned}\sum_{\lambda,a}\tfrac{1}{2}|\beta_{fa}{}^{\lambda}-\beta_{ia}{}^{\lambda}|^2=\sum_{\lambda,a}\tfrac{1}{2}|\beta_a{}^{\lambda}|^2=\sum_{\lambda,a}\Big[\int d^3k\,\tilde{S}^{(\lambda)}(k)f_a(k)\Big]\Big[\int d^3k'\,\tilde{S}^{(\lambda)}(k')f_a{}^*(k')\Big]\\=\sum_{\lambda}\frac{1}{2}\int d^3k\,\tilde{S}^{(\lambda)}(k)\tilde{S}^{(\lambda)}(k)=\alpha\tilde{B}.\end{aligned} \tag{63}$$

Substituting Eq. (63) into (62), we arrive at the important result

$$\tilde{M}=\exp(\alpha B+\alpha\tilde{B})\exp(-\tfrac{1}{2}\sum_{\lambda,a}|\epsilon_{fa}{}^{\lambda}-\epsilon_{ia}{}^{\lambda}|^2)\times e^{i\phi}\{\sum_{m,m'=0}^{\infty}m_{m,m'}\}, \tag{64}$$

where ϕ is real.

The argument of the first exponential was shown in Sec. IV to be infrared divergenceless in the limit of zero photon mass. The third exponential has modulus unity, and the last sum is term by term divergence-free. If the possible states of the system are restricted by the condition

$$\sum_{\lambda,a}|\epsilon_{fa}{}^{\lambda}-\epsilon_{ia}{}^{\lambda}|^2<\infty, \tag{65}$$

the second exponential is nonzero, but less than or equal to unity. With this condition satisfied the infrared divergences have been eliminated. The interpretation of this restriction is discussed in the next section.

VI. INTERPRETATION OF THE PHOTON STATES

In the beginning of this section, we will show that Eq. (65) defines a separable Hilbert space. To do this, we study a related space $\mathfrak{F}$ which will turn out to be identical to the ordinary Fock space $\mathfrak{F}_\infty$. Translations like Eq. (61) will not change the intrinsic properties of this space. Finally, a calculation of the total cross section will relate this whole discussion to experiment.

Much of the mathematical material here will be treated heuristically, but a more rigorous formulation of the statements can be found in the papers by Bargmann.[2,5]

We will define a separable Hilbert space $\mathfrak{F}$ in the following manner: Let $\{\theta_i\}$ be an infinite sequence of complex numbers. A set of "principal vectors" $|\{\theta_i\}\rangle$ is

[5] V. Bargmann, Comm. Pure Appl. Math. 14, 187 (1961).

then defined by the equation

$$|\{\theta_i\}\rangle=\prod_i|\theta_i\rangle$$
$$=\prod_i \exp[-\tfrac{1}{2}|\theta_i|^2]\exp[\theta_i a_i^\dagger]|0\rangle$$
$$=\prod_i \exp[-\tfrac{1}{2}|\theta_i|^2]\exp\left[\theta_i\int f_i(k)a^\dagger(k)\right]|0\rangle \quad (66)$$

and the condition

$$\sum_i|\theta_i|^2<\infty\,. \quad (67)$$

The elements of $\mathfrak{F}$ are taken to be the closure of all finite linear combinations of the principal vectors.

From (66) and the commutation rules for $a^\dagger(k)$, the inner product of two elements, $|f\rangle=\sum_{j=1}^{p}\lambda_j|\{\theta_i^{(j)}\}\rangle$ and $|f'\rangle=\sum_{k=1}^{q}\mu_k|\{\theta_i^{(k)}\}\rangle$, is given by

$$\langle f|f'\rangle=\sum_{j,k}\lambda_j^*\mu_k\{\exp[\sum_i\theta_i^{*(j)}\theta_i^{(k)}]$$
$$\times\exp[-\tfrac{1}{2}\sum_i|\theta_i^{(j)}|^2]\exp[-\tfrac{1}{2}\sum_i|\theta_i^{(k)}|^2]\}\,. \quad (68)$$

In particular, the inner product of two principal vectors $|\{\theta_i^{(j)}\}\rangle$ and $|\{\theta_i^{(k)}\}\rangle$ has the property

$$|\langle\{\theta_i^{(j)}\}|\{\theta_i^{(k)}\}\rangle|^2$$
$$=|\exp[\textstyle\sum_i\theta_i^{*(j)}\theta_i^{(k)}]\exp[-\frac{1}{2}\sum_i|\theta_i^{(j)}|^2]$$
$$\times\exp[-\tfrac{1}{2}\textstyle\sum_i|\theta_i^{(k)}|^2]|^2$$
$$=\exp\{-\textstyle\sum_i|\theta_i^{(j)}-\theta_i^{(k)}|^2\}\,, \quad (69)$$

so that the principal vectors are all normalized to unit length. Moreover, by Eq. (67), no two principal vectors are normal to each other.

The properties of the space could in fact have been derived by using Eq. (69) instead of Eq. (66), but we wish to retain the connection with the previous sections of this paper.

The separability of $\mathfrak{F}$ follows from the existence of a countable sequence of vectors which is dense in $\mathfrak{F}$. Let $|\{\theta_i\}\rangle$ be any principal vector. From Eq. (67), it is known that for any $\delta_N>0$ there exists an integer N such that $\sum_{i>N}|\theta_i|^2<\delta_N$. Moreover, for any $i<N$ and $\delta>0$ it is always possible to find rational numbers $\{R_i\}$ such that $|\theta_i-R_i|^2<\delta$.

Consider a principal vector $|\{\theta_i'\}\rangle$ such that $\theta_i'=0$ for $i>N$, and $\theta_i'=R_i$ for $i\leq N$. Let $\sum_i|\theta_i|^2=A^2$. Then

$$\sum_i|\theta_i'-\theta_i|^2\leqslant N\delta+\delta_N$$

and

$$|\operatorname{Im}\textstyle\sum_i\theta_i^*\theta_i'|=|\operatorname{Im}\sum_i\theta_i^*\theta_i+\operatorname{Im}\sum_i\theta_i^*(\theta_i'-\theta_i)|\leqslant|\sum_i\theta_i^*(\theta_i'-\theta_i)|\leqslant(\textstyle\sum_i|\theta_i|^2\sum_i|\theta_i'-\theta_i|^2)^{1/2}\leqslant A(N\delta+\delta_N)^{1/2}.$$

Thus,

$$\begin{aligned}||\{\theta_i\}\rangle-|\{\theta_i'\}\rangle|^2&=2-\langle\{\theta_i\}|\{\theta_i'\}\rangle-\langle\{\theta_i'\}|\{\theta_i\}\rangle\\&=2-\exp[-\tfrac{1}{2}\textstyle\sum_i|\theta_i|^2]\exp[-\frac{1}{2}\sum_i|\theta_i'|^2]\{\exp[\sum_i\theta_i^*\theta_i']+\exp[\sum_i\theta_i^{*\prime}\theta_i]\}\\&=2-\exp[-\tfrac{1}{2}\textstyle\sum_i|\theta_i'-\theta_i|^2]\{\exp[i\operatorname{Im}\sum_i\theta_i^*\theta_i']+\exp[i\operatorname{Im}\sum_i\theta_i^{*\prime}\theta_i]\}\\&=2\{1-(\exp[-\tfrac{1}{2}\textstyle\sum_i|\theta_i'-\theta_i|^2])(\cos[\operatorname{Im}\sum_i\theta_i^*\theta_i'])\}\leqslant(A^2+1)(N\delta+\delta_N)\,.\end{aligned}$$

Since $|\{\theta_i\}\rangle$, δ_N, and δ were arbitrary, we have shown that any principal vector can be approximated by another principal vector belonging to a denumerable set. The denumerable set which consists of all finite sums of principal vectors like $|\{\theta_i\}\rangle$ is dense in $\mathfrak{F}$.

In the case of massless soft photons, there is no reason to restrict the photon states by Eq. (67). Let $\{\theta_i^{(0)}\}$ be a sequence of complex numbers which are not square-summable, i.e.,

$$\sum_i|\theta_i^{(0)}|^2\not<\infty\,. \quad (70)$$

Then the states defined by the complex numbers $\{\theta_i\}$, and which satisfy the condition

$$\sum_i|\theta_i-\theta_i^{(0)}|^2<\infty\,, \quad (71)$$

form a separable Hilbert space $\mathfrak{F}^{(0)}$ with all the properties of $\mathfrak{F}$, except Eq. (67). $\mathfrak{F}^{(0)}$ is unitarily inequivalent to $\mathfrak{F}$, i.e., it forms a unitarily inequivalent representation of the canonical commutation rules.

In Sec. III, we discussed the connection between the Fock states and the principal vectors for a single mode. We will now briefly study the relationship between the occupation number parametrization and the principal vector parametrization.

The states in the Fock space $\mathfrak{F}_\infty$ are specified by a set M of infinite sequences of nonnegative integers $\{m_i\}$, or "occupation numbers" of which a finite number are different from zero. An orthonormal basis of Fock space is given by

$$|u_{\{m\}}\rangle=\prod_i\frac{(a_i^\dagger)^{m_i}}{(m_i!)^{1/2}}|0\rangle. \quad (72)$$

An arbitrary state $|f\rangle$ of $\mathfrak{F}_\infty$ is given by

$$|f\rangle=\sum_{\{m\}\in M}\gamma_{\{m\}}|u_{\{m\}}\rangle, \quad (73)$$

where the complex coefficients $\gamma_{\{m\}}$ satisfy

$$\sum_{\{m\}\in M}\gamma_{\{m\}}{}^2<\infty\,. \quad (74)$$

At this point, it should be apparent that $\mathfrak{F}\subset\mathfrak{F}_\infty$, since

$$|\{\theta_i\}\rangle=\sum_{\{m\}\in M}\prod_i\frac{(\theta_i a_i^\dagger)^{m_i}}{m_i!}|0\rangle$$
$$=\sum_{\{m\}\in M}\prod_i\frac{(\theta_i)^{m_i}}{(m_i!)^{1/2}}|u_{\{m\}}\rangle \quad (75)$$

and

$$\sum_{\{m\}\in M}\left|\prod_i \frac{(\theta_i)^{m_i}}{(m_i!)^{1/2}}\right|^2=\prod_i\left(\sum_{m=0}^{\infty}\frac{|\theta_i|^{2m_i}}{m_i!}\right)$$
$$=\exp\left(\sum_i|\theta_i|^2\right)<\infty\,. \quad (76)$$

The scalar product of two vectors $|f\rangle$ and $|f'\rangle$ in $\mathfrak{F}_\infty$ can be obtained from (72) and (73):

$$\langle f|f'\rangle=\sum_{\{m\}\in M}\gamma_{\{m\}}{}^*\gamma_{\{m\}}{}'. \quad (77)$$

Let $Q_n\{m_i\}$ be the truncated sequence

$$Q_n\{m_i\}=(m_1,m_2,\cdots,m_n,0,0,\cdots)\,, \quad (78)$$

and define a projection on $\mathfrak{F}_\infty$ by

$$E_n|f\rangle=E_n\sum_{\{m\}\in M}\gamma_{\{m\}}|u_{\{m\}}\rangle=\sum_{\{m\}\in M_n}\gamma_{\{m\}}|u_{\{m\}}\rangle, \quad (79)$$

where M_n is the set of all sequences of the form given by Eq. (78). Then it follows from Eq. (74) that $E_n|f\rangle$ converges strongly to $|f\rangle$ as $n\to\infty$. We will show that $E_n|f\rangle$ is contained in $\mathfrak{F}$, which implies that $\mathfrak{F}_\infty\equiv\mathfrak{F}$.

The expansion of $E_n|f\rangle$ in terms of the principal vectors follows directly from Eq. (27):

$$E_n|f\rangle=\sum_{\{m\}\in M_n}\gamma_{\{m\}}\prod_{i=1}^{n}\int d\mu_i$$
$$\times|(\theta_1,\theta_2,\cdots,\theta_n,0,0,\cdots)\rangle\frac{(\theta_i{}^*)^{m_i}}{(m_i!)^{1/2}}\,, \quad (80)$$

where

$$d\mu_i=\pi^{-1}\exp[-\tfrac{1}{2}|\theta_i|^2]d(\mathrm{Re}\theta_i)d(\mathrm{Im}\theta_i)\,. \quad (81)$$

From Eq. (69), it is clear that principal vectors $|\{\theta_i\}\rangle$ which do not satisfy Eq. (67) are orthogonal to $E_n|f\rangle$. Therefore $E_n|f\rangle\in\mathfrak{F}$ and the result $\mathfrak{F}_\infty=\mathfrak{F}$ follows, i.e., the Fock space built from states with a finite number of photons, and the space of principal vectors satisfying Eq. (67), are the same space.

In order to satisfy the requirement Eq. (65) for finite matrix elements, it will be necessary not to restrict the scattering states to $\mathfrak{F}$. For if the initial state were in $\mathfrak{F}$, i.e., the $\{\alpha_a{}^\lambda\}$ of Eq. (45) satisfied the condition

$$\sum_{\lambda,a}|\alpha_a{}^\lambda|^2<\infty\,, \quad (82)$$

then the final state parametrized by the sequence of complex numbers $\{\gamma_a{}^\lambda\}$ would be given by

$$\gamma_a{}^\lambda=\alpha_a{}^\lambda+(\beta_{fa}{}^\lambda-\beta_{ia}{}^\lambda)+(\epsilon_{fa}{}^\lambda-\epsilon_{ia}{}^\lambda)\,, \quad (83)$$

where we have used Eq. (61), and the ϵ's would satisfy

$$\sum_{\lambda,a}|\epsilon_{fa}{}^\lambda-\epsilon_{ia}{}^\lambda|^2<\infty\,. \quad (84)$$

But we know from Sec. IV and Eq. (63) that

$$\sum_{\lambda,a}|\beta_{fa}{}^\lambda-\beta_{ia}{}^\lambda|^2=\tfrac{1}{2}\sum_{l=1}^{2}\int|\tilde{S}^{(l)}(k)|^2d^3k=\alpha\tilde{B}\,, \quad (85)$$

and $\tilde{B}\to\infty$ as the photon mass approaches zero. Therefore

$$\sum_{\lambda,a}|\gamma_a{}^\lambda|^2\to\infty\,, \quad (86)$$

as the photon mass approaches zero. Thus the final state cannot belong to $\mathfrak{F}$.

Neverthesless, the coefficients $\{\gamma_a{}^\lambda\}$ define a final state. It must be that the final state belongs to an inequivalent representation of the canonical commutation rules whose most outstanding feature is that the average number of photons is infinite. Not any final state will do, however, for the boundaries of this new space $\mathfrak{F}'$ are restricted by the condition Eq. (65).

One of many ways to satisfy Eq. (65) which preserves symmetry between the initial and final states is to write these states as

$$\alpha_a{}^\lambda=\epsilon_{ia}{}^\lambda+\beta_{ia}{}^\lambda+\epsilon_{0a}{}^\lambda\,,$$
$$\gamma_a{}^\lambda=\epsilon_{fa}{}^\lambda+\beta_{fa}{}^\lambda+\epsilon_{0a}{}^\lambda\,,$$

and to restrict the states by

$$\sum_{\lambda,a}|\epsilon_{ia}{}^\lambda|^2<\infty\,,\quad \sum_{\lambda,a}|\epsilon_{fa}{}^\lambda|^2<\infty\,.$$

Then we would get different theories by different choices of the sequence $\{\epsilon_{0a}{}^\lambda\}$. With such a choice, the photon states would have a dependence upon the momenta of the participating electrons.

So far we have not spoken at all about Ω, the region of momentum space on which the single photon states $\{f_a\}$ were defined. In an experimental situation, there is always a threshold below which a single photon cannot be detected. We identify Ω with what we shall call the "resolution region," i.e., all photons with momentum $k\in\Omega$ are not detectable, while those which satisfy $k\in\Omega$ are detectable. In what follows, the nondetectable photons will be spoken of as "soft" photons, while the others will be called "hard." Furthermore, we shall indicate the resolution of the momentum space by a subscript, e.g., X_Ω.

In a practical calculation where one wants to treat, for example, the scattering of an electron with the emission of hard photons, the hard photons can be dealt with by the conventional occupation-number parametrization, while the soft photons are described in terms of the translated principal vectors. More specifically, consider the calculation of the cross section for an electron of momentum p_i scattering into a state with an electron of momentum p_f plus several hard photons. The incoming electron is associated with a photon field described by a sequence $\{\alpha_a{}^\lambda\}$ and the outgoing electron has a photon field $\{\gamma_a{}^\lambda\}$. In Eq. (64), the "basic matrix

element" $m_{0,0}$ corresponds to diagrams with only the detectable real photons and those virtual photons necessary for the process to occur. The terms m_{ij} for $i, j=0$ contain the effects of the noninfrared parts of the real and virtual soft photons to higher order in the coupling constant.

To lowest order in the noninfrared photons, the squared matrix element for a particular diagram $m_{0,0}$ is from Eq. (64):

$$|\tilde{M}|^2=\exp[2(\mathrm{Re}\alpha B+\alpha\tilde{B}_\Omega)]\times\exp[-\sum_{\lambda,a}|\epsilon_{fa}{}^\lambda-\epsilon_{ia}{}^\lambda|^2]|m_{0,0}|^2. \quad (87)$$

We can then sum over final states. The result (to lowest order) is independent of the initial state:

$$\sum_{\substack{\text{final}\\ \text{states}}}|\tilde{M}|^2=\exp[2(\mathrm{Re}\alpha B+\alpha\tilde{B}_\Omega)]|m_{0,0}|^2\times\lim_{n\to\infty}(\pi^{-1})^n\left[\int d^2\epsilon_f e^{-|\epsilon_f|^2}\right]^n$$
$$=\exp[2(\mathrm{Re}\alpha B+\alpha\tilde{B}_\Omega)]|m_{0,0}|^2. \quad (88)$$

The remaining exponential contains part of the effect of the choice of $\tilde{S}_i(k)$ and the region of resolution Ω, and we obtain a similar result to what Yennie *et al.* obtained (for a nonenergy-conserving potential). In fact, the "reason" why the results are the same is that, in the summation over all final states in the conventional treatment of the infrafed divergence, the main contributions came from states which were not in the usual Fock space, but were in a nonseparable space defined by Eq. (72) without any restrictions on the sequence $\{m\}$ of occupation numbers. In particular, the separable space of final states $\mathfrak{F}'$ is contained in this nonseparable space.

In the above computation, and in Sec. V, the resolution regions for the initial and final states were assumed to be the same. One can argue that the resolution region of the initial state can be made arbitrarily small, but finite, by waiting a sufficiently long time before the scattering experiment. The situation where the initial resolution region is smaller than the final state resolution region is discussed in the Appendix.

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APPENDIX A: GENERALIZATION OF THE CANCELLATION TO SEVERAL ELECTRON LINES

For simplicity, only the case of a single electron line interacting with a potential was treated in Sec. V. The generalization to several electron lines interacting with one another will be outlined here for completeness. It is possible to make an extension to positron scattering, pair production, and other processes, but we shall not do so here.

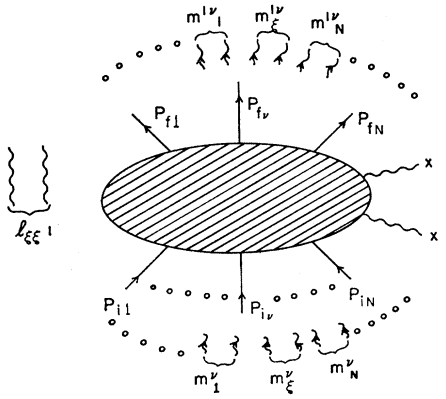

FIG. 4. Representation of l noninteracting real soft photons, $m_\xi{}^\nu$ real soft photons associated with the ξth electron and absorbed by the νth electron, and $m_\xi{}'^\nu$ real soft photons associated with the ξth electron and emitted by the νth electron.

Since the cancellation of the infrared divergences in this more complicated situation requires a proliferation of subscripts and superscripts, we drop all notation having to do with the polarization of the photons. Superscripts will now designate the electron line.

The initial state consists of N incoming electrons with momenta $p_{i1}, p_{i2}, \cdots, p_{i\nu}, \cdots, p_{iN}$, along with some photons. They scatter into a final state of N outgoing electrons with momenta $p_{f1}, \cdots, p_{f\nu}, \cdots, p_{fN}$, again with some photons. We have assumed that all resolution regions are identical.

Thus the soft-photon initial state can, for example, be conveniently written as

$$\prod_a \frac{\exp\left[\sum_\nu(\beta_{ia}{}^\nu+\epsilon_{ia}{}^\nu)\int_\Omega d^3k\, f_a(k)a^\dagger(k)\right]}{\exp[\frac{1}{2}|\sum_\nu \beta_{ia}{}^\nu+\epsilon_{ia}{}^\nu|^2]}|0\rangle, \quad (A1)$$

where

$$\beta_{ia}{}^\nu\equiv(f_a{}^*,S_i(p_{i\nu}))_\Omega. \quad (A2)$$

The final-state soft photons are described by a similar expression.

The interaction is illustrated in Fig. 4. In this diagram, our attention is focused on the νth electron line. The integer $m_\xi{}^\nu$ denotes the number of photons in the initial state which "belongs" to the ξth electron and interact with the νth electron line. Similarly, $m_\xi{}'^\nu$ is the number of photons in the final state belonging to the ξth electron which interact with the νth electron line. The number of noninteracting photons coming from the ξth incoming electron, and, becoming part of the ξ'th outgoing electron, is given by the integer $l_{\xi\xi'}$.

We arrive at an equation analogous to Eq. (64) through the following considerations.

(a) The overlap of the $l=\sum_{\xi\xi'} l_{\xi\xi'}$ initial-state noninteracting photons with the l final-state noninteracting photon contributes a factor

$$\prod_{\xi\xi'}[l_{\xi\xi'}!(\sum \alpha_a{}^{\xi}\gamma_a{}^{*\xi'})^{l_{\xi\xi'}}],$$

where

$$\alpha_a{}^{\xi}\equiv\beta_{ia}{}^{\xi}+\epsilon_{ia}{}^{\xi}, \quad \gamma_a{}^{\xi}\equiv\beta_{fa}{}^{\xi}+\epsilon_{fa}{}^{\xi}.$$

Note again that the superscript ξ refers to the electron line, and not to the polarization.

(b) A contribution from the interaction of $m=\sum_{\nu,\xi}{}^{N} m_{\xi}{}^{\nu}$ initial-state photons, and the $m'=\sum_{\nu,\xi}{}^{N} m_{\xi}{}'^{\nu}$ final-state photons with the N electron lines, gives

$$\tilde{\rho}(\{m_{\xi}{}^{\nu}\}\{m_{\xi}{}'^{\nu}\})=\prod_{\nu}\left[\sum_{t=0}^{m^{\nu}+m'^{\nu}}\prod_{i=1}^{t}(-1)^{m_{\nu}}\tilde{S}(k_{\nu,i})\tilde{\xi}_{m^{\nu}+m'^{\nu}-t}(k_{\nu,t+1},\cdots,k_{\nu,m^{\nu}+m'^{\nu}})\frac{1}{t!(m^{\nu}+m'^{\nu}-t)!}\right],$$

where $m^{\nu}=\sum_{\xi} m_{\xi}{}^{\nu}$, $m'^{\nu}=\sum_{\xi} m_{\xi}{}'^{\nu}$.

(c) Contribution (b) must be integrated over the momentum distribution that is obtained from the formal expansion of the initial and final states,

$$\left\{\prod_{\xi}\left[\left(\prod_{r=1}^{m_{\xi}}\sum_{a}\alpha_a{}^{\xi}\int d^3k_{\xi,r}\, f_a(k_{\xi,r})\right)\left(\prod_{r'=m_{\xi}+1}^{m_{\xi}+m_{\xi}'}\sum_{c}\gamma_c{}^{*\xi}\int d^3k_{\xi,r'}\, f_a{}^{*}(k_{\xi,r'})\right)\right]\right\}\{\tilde{\rho}(\{m_{\xi}{}^{\nu}\},\{m_{\xi}{}'^{\nu}\};k_1,\cdots,k_m,\cdots,k_{m+m'})\},$$

where $m_{\xi}=\sum_{\nu} m_{\xi}{}^{\nu}$, $m_{\xi}{}'=\sum_{\nu} m_{\xi}{}'^{\nu}$.

(d) The formal expansion of Eq. (A1) also leads to the factors

$$\prod_{\xi}\left[\frac{1}{(m_{\xi}+l_{\xi})!}\frac{1}{(m_{\xi}'+l_{\xi}')!}\right]\exp[-\tfrac{1}{2}\sum_{a}|\sum_{\xi}\alpha_a{}^{\xi}|^2]\exp[-\tfrac{1}{2}\sum_{c}|\sum_{\xi}\gamma_c{}^{\xi}|^2],$$

where $l_{\xi}=\sum_{\xi'} l_{\xi\xi'}$, $l_{\xi}'=\sum_{\xi} l_{\xi\xi'}$.

(e) In addition to the above, there is a combinatorial factor which accounts for the number of ways that the $(m+l)$ initial-state photons, $(m'+l)$ final-state photons, and the l noninteracting photons, can be distributed among themselves:

$$\left[\prod_{\xi}\frac{(m_{\xi}+l_{\xi})!}{\prod_{\nu}(m_{\xi}{}^{\nu})!\prod_{\xi'}(l_{\xi\xi'})!}\right]\left[\prod_{\xi'}\frac{(m_{\xi}'+l_{\xi}')!}{\prod_{\nu}(m_{\xi}{}'^{\nu})!\prod_{\xi}(l_{\xi\xi'})!}\right].$$

(f) There is an infrared divergent factor due to virtual photon corrections (see Yennie *et al.*[4]):

$$F=\exp[\alpha B].$$

To get the matrix element $\tilde{M}$, the contributions are summed over all values of $\{m_{\xi}{}^{\nu}\}$, $\{m_{\xi}{}'^{\nu}\}$, $\{l_{\xi\xi'}\}$,

$$\tilde{M}=F\sum_{\{l_{\xi\xi'}\}=0}^{\infty}\sum_{\{m_{\xi}{}^{\nu}\}=0}^{\infty}\sum_{\{m_{\xi}{}'^{\nu}\}=0}^{\infty}\left[\prod_{\xi}\left(\frac{1}{(m_{\xi}+l_{\xi})!}\frac{1}{(m_{\xi}'+l_{\xi}')!}\right)\right]\left[\prod_{\xi}\frac{(m_{\xi}+l_{\xi})!}{\prod_{\nu}(m_{\xi}{}^{\nu})!\prod_{\xi'}(l_{\xi\xi'})!}\prod_{\xi'}\frac{(m_{\xi}'+l_{\xi}')!}{\prod_{\nu}(m_{\xi}{}'^{\nu})!\prod_{\xi}(l_{\xi\xi'})!}\right]$$

$$\times[\exp[-\tfrac{1}{2}\sum_{a}|\sum_{\xi}\alpha_a{}^{\xi}|^2]\exp[-\tfrac{1}{2}\sum_{a}|\sum_{\xi}\gamma_a{}^{\xi}|^2]][\prod_{\xi\xi'} l_{\xi\xi'}!(\sum_{a}\alpha_a{}^{\xi}\gamma_a{}^{*\xi'})^{l_{\xi\xi'}}]$$

$$\times\left\{\prod_{\xi}\left[\left(\prod_{r=1}^{m_{\xi}}(-1)^{m_{\xi}}\sum_{a}\alpha_a{}^{\xi}\int d^3k_{\xi,r}\, f_a(k_{\xi,r})\right)\left(\prod_{r'=m_{\xi}+1}^{m_{\xi}+m_{\xi}'}\sum_{c}\gamma_c{}^{*\xi}\int d^3k_{\xi,r'}\, f_c{}^{*}(k_{\xi,r'})\right)\right]\tilde{\rho}(k_1\cdots,k_{m+m'})\right\}.$$

The last factor in the braces can be reduced by separating out the divergent terms:

$$\left\{\prod_{\nu\xi}\left[\sum_{j_{\xi}{}^{\nu}=0}^{m_{\xi}{}^{\nu}}\sum_{j_{\xi}{}'^{\nu}=0}^{m_{\xi}{}'^{\nu}}\frac{(m_{\xi}{}^{\nu})!}{(j_{\xi}{}^{\nu})!(m_{\xi}{}^{\nu}-j_{\xi}{}^{\nu})!}\frac{(m_{\xi}{}'^{\nu})!}{(j_{\xi}{}'^{\nu})!(m_{\xi}{}'^{\nu}-j_{\xi}{}'^{\nu})!}(-\sum_{a}\alpha_a{}^{\xi}(\tilde{S}_{\nu},f_a))^{j_{\xi}{}^{\nu}}(\sum_{a}\gamma_a{}^{*\xi}(f_a{}^{*},\tilde{S}_{\nu}))^{j_{\xi}{}'^{\nu}}P\right]\right\},$$

where

$$P(\{j_{\xi}{}^{\nu}\}\{j_{\xi}{}'^{\nu}\};\{p_{i\nu}\},\{p_{f\nu}\})=\left[\left(\prod_{r=1}^{j_{\xi}{}^{\nu}}\int d^3k_r\sum_{a}\alpha_a{}^{\xi}f_a(k)\right)\left(\prod_{r'=j_{\xi}{}^{\nu}+1}^{j_{\xi}{}^{\nu}+j_{\xi}{}'^{\nu}}\int d^3k_{r'}\sum_{c}\gamma_c{}^{*\xi}f_c{}^{*}(k)\right)\tilde{\xi}_{j_{\xi}{}^{\nu}+j_{\xi}{}'^{\nu}}(k_1,\cdots,k_{j_{\xi}{}^{\nu}+j_{\xi}{}'^{\nu}})\right].$$

Again, one defines a divergence-free "residual" by

$$m(j,j')=\frac{P}{j!j'!},$$

so that

$$\tilde{M}=F[\prod_{\xi\xi'}\exp[\sum_a \alpha_a{}^\xi\gamma_a{}^{*\xi'}]]\exp[-\sum_a|\sum_\xi \alpha_a{}^\xi|^2]\exp[-\sum_a|\sum_\xi \gamma_a{}^\xi|^2]$$
$$\times[\prod_{\nu\xi}\exp[-\sum_a \alpha_a{}^\xi\beta_a{}^{*\nu}]\exp[\sum_a \gamma_a{}^{*\xi}\beta_a{}^\nu]][\prod_{\nu\xi}\sum_{j_\xi{}^\nu=0}^{\infty}\sum_{j_\xi{}'^\nu=0}^{\infty} m(\{j_\xi{}^\nu\},\{j_\xi{}'^\nu\})].$$

Making the translation to a new coordinate system, we get

$$\alpha_a{}^\xi=\beta_{ia}{}^\xi+\epsilon_{ia}{}^\xi,\quad \gamma_a{}^\xi=\beta_{fa}{}^\xi+\epsilon_{fa}{}^\xi,\quad \beta_a{}^\nu=\beta_{fa}{}^\nu-\beta_{fa}{}^\nu,$$

and with a little additional algebra, the final result is exhibited:

$$\tilde{M}=F\exp[\tfrac{1}{2}\sum_a|\sum_\xi \beta_{fa}{}^\xi-\beta_{ia}{}^\xi|^2]\exp[-\tfrac{1}{2}\sum_a|\sum_\xi \epsilon_{fa}{}^\xi-\epsilon_{ia}{}^\xi|^2]e^{i\phi}[\prod_{\nu\xi}\sum_{j_\xi{}^\nu=0}^{\infty}\sum_{j_\xi{}'^\nu=0}^{\infty} m(\{j_\xi{}^\nu\},\{j_\xi{}'^\nu\})]. \tag{A3}$$

The divergences in the factor F and the first exponential cancel (Yennie *et al.*[4]). Again the condition for finite matrix elements is

$$\textstyle\sum_a|\sum_\xi \epsilon_{fa}{}^\xi-\epsilon_{ia}{}^\xi|^2<\infty. \tag{A4}$$

APPENDIX B: THE PROJECTION OF ENERGY-MOMENTUM EIGENSTATES

For completeness, the procedure for projecting out energy-momentum eigenstates will be given here. Consider a typical soft photon state described by a principal vector $|\{\alpha_a{}^\xi\}\rangle$:

$$|\{\alpha_a{}^\lambda\}\rangle=\exp[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^\lambda|^2]\exp\Big[\sum_{\lambda,a}\alpha_a{}^\lambda\int_\Omega d^3k\, f_a(k)e^{(\lambda)}(k)a^{(\lambda)\dagger}(k)\Big]|0\rangle$$
$$=\exp[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^\lambda|^2]\sum_{n=0}^{\infty}\frac{1}{n!}\Big[\sum_{a,\lambda}\alpha_a{}^\lambda\int_\Omega d^3k_1\, f_a(k_1)e^{(\lambda)}a^{(\lambda)\dagger}(k)\Big]\cdots\Big[\sum_{a,\lambda}a_a{}^\lambda\int_\Omega d^3k_n\, f_a(k_n)e^{(\lambda)}a^{(\lambda)\dagger}(k_n)\Big]|0\rangle. \tag{B1}$$

Using the formula

$$\frac{1}{2\pi}\int e^{ixy}dy=\delta(x), \tag{B2}$$

it is then possible to project out from $|\{\alpha_a{}^\lambda\}\rangle$ the eigenstates of energy momentum:

$$P(E,\mathbf{K})|\{\alpha_a{}^\lambda\}\rangle=\exp[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^\lambda|^2]\sum_{n=0}^{\infty}\frac{1}{n!}$$
$$\times\Big\{\int d^3k_i\cdots d^3k_n\prod_{i=1}^{n}[\sum_{a,\lambda}\alpha_a{}^\lambda f_a(k_i)e^{(\lambda)}(k_i)a^{(\lambda)\dagger}(k_i)]\delta(E-\sum_i\omega_i)\delta^3(\mathbf{K}-\sum_i\mathbf{k}_i)\Big\}|0\rangle$$
$$=\exp[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^\lambda|^2]\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{(2\pi)^4}\int dy d^3\mathbf{x}\, e^{iEy}e^{i\mathbf{K}\cdot\mathbf{x}}$$
$$\times\Big\{\int d^3k_i\cdots d^3k_n\prod_i[\sum_{\lambda,a}\alpha_a{}^\lambda e^{-i\omega y}e^{-ik_i\cdot x}f_a(k_i)e^{(\lambda)}(k_i)a^{(\lambda)\dagger}(k_i)]\Big\}|0\rangle$$
$$=\frac{1}{(2\pi)^4}\int dy d^3\mathbf{x}\exp[-\tfrac{1}{2}\sum_{\lambda,a}|\alpha_a{}^\lambda|^2]e^{iEy+i\mathbf{K}\cdot\mathbf{x}}\exp\Big[\sum_{\lambda,a}\alpha_a{}^\lambda\int d^3k e^{-i\omega y}e^{-ik\cdot x}f_a(k)e^{(\lambda)}(k)a^{(\lambda)\dagger}(k)\Big]. \tag{B3}$$

APPENDIX C: INITIAL AND FINAL STATES WITH DIFFERENT RESOLUTION REGIONS

Suppose that we are dealing with the situation where the resolution region Ω for the initial state is smaller than the resolution region Ω' for the final state ($\Omega \subset \Omega'$). In other words, the threshold for detecting low-energy photons is lower before the scattering experiment than afterwards. Then an infrared divergenceless matrix element in a form analogous to Eq. (64) may be obtained with very little additional complication.

Let us define a domain D of momentum space such that

$$\Omega' = \Omega \bigcup D,$$

where $\Omega \bigcup D = 0$. Then we suppose that there exists a complete set of orthonormal functions $\{g_j(k)\}$ defined on D. A typical final state is now given by

$$|f\rangle = |\{\gamma_i,\gamma_{j'}\}\rangle = \frac{\exp\left[\prod_i \gamma_i \int_\Omega d^3k\, f_i(k) a^\dagger(k)\right]}{\exp\left[\frac{1}{2}\sum_i |\gamma_i|^2\right]} \times \frac{\exp\left[\prod_j \gamma_j' \int_D d^3k\, g_j(k) a^\dagger(k)\right]}{\exp\left[\frac{1}{2}\sum_j |\gamma_j|^2\right]} |0\rangle. \quad \text{(C1)}$$

The indices having to do with polarization have been suppressed. The result of the modification is that in the derivation of Eq. (55) one must make the substitutions

$$\gamma_a^\lambda \int_\Omega f_a e^{(\lambda)} \rightarrow \gamma_a^\lambda \int_\Omega f_a e^{(\lambda)} + \gamma_a'^\lambda \int_D g_a e^{(\lambda)},$$

so that Eq. (55) is correct only if we have on the right-hand side the additional factor

$$\exp\left[-\tfrac{1}{2}\sum_{\lambda,c} |\gamma_c'^\lambda|^2\right] \exp\left[\sum_{\lambda,c} \gamma_c'^{*\lambda} (g_c^*, \tilde{S}^{(\lambda)})_D\right],$$

and the divergenceless sum

$$\sum_{m,m'=0}^{\infty} m_{m,m'}$$

contains integrals of the functions $\{g_c\}$ over the addition region D.

We now define new coefficients $\beta_a'^\lambda$ and variables $\epsilon_{fa}'^\lambda$ in a similar way to Eqs. (59) and (61):

$$\beta_a'^\lambda = (g_a^*, \tilde{S}^{(\lambda)})_D, \quad \gamma_a'^\lambda = \beta_a'^\lambda + \epsilon_{fa}'^\lambda. \quad \text{(C2)}$$

Then the additional (noninfrared-divergent) factor becomes

$$\exp\left[\sum_{\lambda,a} \left(\tfrac{1}{2}|\beta_a'^\lambda|^2 - |\epsilon_{fa}'^\lambda|^2 + i\,\mathrm{Im}\,\epsilon_{fa}^{*\prime\lambda}\beta_a'^\lambda\right)\right].$$

It is natural to define

$$\alpha\tilde{B}_D = \exp\left[\sum_{\lambda,a} \tfrac{1}{2}|\beta_a'^\lambda|^2\right], \quad \text{(C3)}$$

so that Eq. (64) becomes

$$\tilde{M} = \exp(\alpha B + \alpha\tilde{B}_\Omega + \alpha\tilde{B}_D) \exp\left[-\tfrac{1}{2}\sum_{\lambda,a} |\epsilon_{fa}^\lambda - \epsilon_{ia}^\lambda|^2\right] \times \exp\left[-\tfrac{1}{2}\sum_{\lambda,a} |\epsilon_{fa}'^\lambda|^2\right] e^{i\phi} \sum_{m,m'=0}^{\infty} m_{m,m'}. \quad \text{(C4)}$$

In this expression the infrared divergences cancel in the sum $\alpha B + \alpha B_\Omega$ of the argument of the first exponential. The term $\exp(\alpha\tilde{B}_D)$ accounts for the difference in resolution regions. The condition for finite matrix elements is now

$$\sum_{\lambda,a} |\epsilon_{fa}^\lambda - \epsilon_{ia}^\lambda|^2 < \infty, \quad \sum_{\lambda,a} |\epsilon_{fa}'^\lambda|^2 < \infty. \quad \text{(C5)}$$