## IV. CONCLUSION

We conclude that there is no evidence for nonexponential behavior in the decay and that the mean life of the $K^+$ meson is $12.443 \pm 0.038$ nsec.

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# Normalization of Bethe-Salpeter Wave Functions

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In this paper is presented a brief and transparent derivation, which does not depend on the existence of a conserved quantity, of the normalization condition for Bethe-Salpeter bound-state wave functions. A comment on the structure of the condition is made. Application of the condition to the Bethe-Salpeter wave-function description of physical meson states in ordinary pseudoscalar-meson theory is also described in detail.

## 1. INTRODUCTION

Ever since the introduction into quantum-field theory of the Bethe-Salpeter equation,[1] treatment of the normalization of the bound state or Bethe-Salpeter wave function has presented considerable difficulty. Early derivations[2] required the existence of a conserved quantity such as baryon number or electric charge and are therefore inapplicable to neutral meson bound states, for example. Later authors, notably Allcock[3] and Cutkosky and Leon,[4] obtained normalization conditions without assuming the existence of a conserved quantity and showed[4] that their results were in agreement with the previous results. In the present paper, we give (a) a new method of derivation of the normalization of Bethe-Salpeter wave functions, which does not depend on the existence of a conserved quantity and which appears to be much more direct and transparent than those of Refs. 3 and 4; (b) a demonstration of the fact that, although the normalization condition obtained seems of somewhat odd appearance, its structure is very similar to that of the normalization conditions commonly used for one-particle wave functions in quantum field theory; (c) an application of the normalization condition obtained to the Bethe-Salpeter wave function description of the physical meson states in ordinary pseudoscalar meson theory. The discussion here is similar in spirit to but of more general nature than that given earlier by Okubo and Feldman,[5] and quite closely related to some recent work of Rowe.[6]

The material of the paper has been organized as follows. In Sec. 2, we present the work associated with (a) and (b) above, while treatment of (c) is to be found in Sec. 3.

## 2. NORMALIZATION OF BETHE-SALPETER WAVE FUNCTIONS

We illustrate our procedure by consideration of a convenient example, that of two fermion fields, $\psi_A$ and $\psi_B$, which describe distinguishable particles of the same mass, interacting with a neutral scalar meson field. This allows easy comparison with many important papers[1–3] on the Bethe-Salpeter formalism, as well as with the introduction to the subject given by Schweber.[7] Similar treatment of other interesting cases follows readily.

We begin with a brief review of those portions of

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[1] H. A. Bethe and E. E. Salpeter, Phys. Rev. **82**, 309 (1951); M. Gell-Mann and F. E. Low, *ibid.* **82**, 350 (1951).

[2] K. Nishijima, Prog. Theoret. Phys. (Kyoto) **10**, 549 (1953); **12**, 279 (1954); and **13**, 305 (1955); S. Mandelstam, Proc. Roy. Soc. (London) **A233**, 248 (1955); A. Klein and C. Zemach, Phys. Rev. **108**, 126 (1957).

[3] G. R. Allcock, Phys. Rev. **108**, 126 (1957); G. R. Allcock and D. J. Hooten, Nuovo Cimento **8**, 590 (1958).

[4] R. E. Cutkosky and M. Leon, Phys. Rev. **135**, B1445 (1964). See also I. Sato, J. Math. Phys. **4**, 24 (1963).

[5] S. Okubo and D. Feldman, Phys. Rev. **117**, 279 (1960).

[6] E. G. P. Rowe, Nuovo Cimento **32**, 1422 (1964), Sec. 3.

[7] S. S. Schweber, *Introduction to Relativistic Quantum-Field Theory* (Row, Peterson and Company, Evanston, Illinois, 1961), Sec. 17f, p. 705.

Bethe-Salpeter formalism which are required for the development of our work. The Heisenberg picture is employed throughout.

A central role in the theory is played by the two-fermion Green's function

$$K(x_1x_2x_3x_4)=-\langle 0|T\psi_A(x_1)\psi_B(x_2)\bar{\psi}_A(x_3)\bar{\psi}_B(x_4)|0\rangle,$$

which satisfies the inhomogeneous Bethe-Salpeter equation[1]

$$K(x_1x_2x_3x_4)=S'_{FA}(x_1x_3)S'_{FB}(x_2x_4)-\int d^4x_5d^4x_6d^4x_7d^4x_8S'_{FA}(x_1x_5)S'_{FB}(x_2x_6)\times G(x_5x_6x_7x_8)K(x_7x_8x_3x_4),\quad (2.1a)$$

or equivalently,

$$K(x_1x_2x_3x_4)=S'_{FA}(x_1x_3)S'_{FB}(x_2x_4)-\int d^4x_5d^4x_6d^4x_7d^4x_8K(x_1x_2x_5x_6)G(x_5x_6x_7x_8)\times S'_{FA}(x_7x_3)S'_{FB}(x_8x_4).\quad (2.1b)$$

In Eq. (2.1), $S'_F$ is given by

$$S'_F(x_ix_j)=\langle 0|T\psi(x_i)\bar{\psi}(x_j)|0\rangle,$$

and $G$ is the interaction function, corresponding to the sum of all Bethe-Salpeter irreducible graphs.

From invariance under space time translations, it follows that $K(x_1x_2x_3x_4)$ and $G(x_1x_2x_3x_4)$ are functions of only $x_1-x_2$, $x_3-x_4$ and $\frac{1}{2}(x_1+x_2)-\frac{1}{2}(x_3+x_4)$. Introducing Fourier transforms $K(pqP)$, $G(pqP)$ and $S'_F(p)$ by means of

$$K(x_1x_2x_3x_4)=(2\pi)^{-8}\int d^4pd^4qd^4Pe^{\frac{1}{2}iP(x_1+x_2-x_3-x_4)}\times e^{ip(x_1-x_2)}e^{-iq(x_3-x_4)}K(pqP),\quad (2.2)$$

an analogous equation for $G$, and

$$S'_F(x)=(2\pi)^{-4}\int d^4pe^{ipx}S'_F(p),\quad (2.3)$$

respectively, we may convert Eqs. (2.1a) and (2.1b) into the forms

$$\int d^4p'[I(pp'P)+G(pp'P)]K(p'qP)=\delta(p-q),\quad (2.4a)$$

$$\int d^4q'K(pq'P)[I(q'qP)+G(q'qP)]=\delta(p-q),\quad (2.4b)$$

where the quantity $I$ is given by

$$I(pqP)=\delta(p-q)[S'_{FA}(\tfrac{1}{2}P+p)]^{-1}\times[S'_{FB}(\tfrac{1}{2}P-p)]^{-1}.\quad (2.5)$$

An obvious operator notation for Eqs. (2.4a) and (2.4b) is

$$(I+G)K=1,\quad (2.6a)$$

$$K(I+G)=1.\quad (2.6b)$$

Now, if there exists a two-fermion bound state, possibly of $n$-fold degeneracy, of mass $M$, it can only give a contribution to $K(x_1x_2x_3x_4)$ for $t_1$, $t_2>t_3$, $t_4$, because of conservation of baryon number. For such time orderings, the contribution to

$$K(x_1x_2x_3x_4)=-\langle 0|T(\psi_A(x_1)\psi_B(x_2))T(\bar{\psi}_A(x_3)\bar{\psi}_B(x_4))|0\rangle$$

is given by[8]

$$-\sum_{r=1}^{n}\int d^4P\chi_{Pr}(x_1x_2)\bar{\chi}_{Pr}(x_3x_4)\theta(P_0)\delta(P^2+M^2),\quad (2.7)$$

where $\chi$ and $\bar{\chi}$ are Bethe-Salpeter amplitudes defined by[9]

$$\chi_{Pr}(x_1x_2)=\langle 0|T\psi_A(x_1)\psi_B(x_2)|Pr\rangle,\quad (2.8a)$$

$$\bar{\chi}_{Pr}(x_3x_4)=\langle Pr|T\bar{\psi}_A(x_3)\bar{\psi}_B(x_4)|0\rangle.\quad (2.8b)$$

Here $P$ is the energy momentum of the bound state and $r$ denotes the remaining quantum numbers necessary to describe the $n$-fold degeneracy (cf. Schweber,[7] p. 715). On the mass shell, $P_0=(\mathbf{P}^2+M^2)^{1/2}$, $\chi$ and $\bar{\chi}$ satisfy homogeneous Bethe-Salpeter equations

$$\chi_{Pr}(x_1x_2)=-\int d^4x_5d^4x_6d^4x_7d^4x_8S'_{FA}(x_1x_5)S'_{FB}(x_2x_6)\times G(x_5x_6x_7x_8)\chi_{Pr}(x_7x_8),\quad (2.9a)$$

$$\bar{\chi}_{Pr}(x_3x_4)=-\int d^4x_5d^4x_6d^4x_7d^4x_8\bar{\chi}_{Pr}(x_5x_6)\times G(x_5x_6x_7x_8)S'_{FA}(x_7x_3)S'_{FB}(x_8x_4).\quad (2.9b)$$

Using invariance under space-time translations, we may explicitly factor out of $\chi$ and $\bar{\chi}$ their center-of-mass coordinate dependence according to

$$\chi_{Pr}(x_1x_2)=(2\pi)^{-3/2}e^{iPX}\chi_{Pr}(x),\quad (2.10a)$$

$$\bar{\chi}_{Pr}(x_3x_4)=(2\pi)^{-3/2}e^{-iPX'}\bar{\chi}_{Pr}(x'),\quad (2.10b)$$

where

$$X=\tfrac{1}{2}(x_1+x_2),\quad x=x_1-x_2,$$

$$X'=\tfrac{1}{2}(x_3+x_4),\quad x'=x_3-x_4.$$

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[8] Cf. S. Mandelstam (Ref. 2). A correct derivation of this result has never been given. Mandelstam in his paper states that Eq. (2.7) gives the contribution of the bound state for $X_0>X_0'$ which is incorrect. A derivation of Eq. (2.14) which takes into account the fact that the contribution of the bound states vanishes for all time orderings other than $t_1$, $t_2>t_3$, $t_4$ is given in an appendix to this paper.

[9] Note that an alternative expression for $\bar{\chi}$ is

$$\bar{\chi}_{Pr}(x_3,x_4)=\langle 0|\bar{T}\psi_B(x_4)\psi_A(x_3)|Pr\rangle^*\gamma_4\gamma_4,$$

where $\bar{T}$ orders operators antichronologically, i.e., operators with later times stand to the right of those with earlier times.

Then, introducing Fourier transforms by means of

$$\chi_{Pr}(x)=(2\pi)^{-4}\int d^4p e^{ipx}\chi_{Pr}(p)\,, \qquad (2.11a)$$

$$\bar{\chi}_{Pr}(x')=(2\pi)^{-4}\int d^4q e^{-iqx'}\bar{\chi}_{Pr}(q)\,, \qquad (2.11b)$$

we may convert Eqs. (2.9) $a$ and $b$ into the forms

$$\int d^4p'[I(pp'P)+G(pp'P)]\chi_{Pr}(p')=0\,, \qquad (2.12a)$$

$$\int d^4q'\bar{\chi}_{Pr}(q')[I(q'qP)+G(q'qP)]=0\,, \qquad (2.12b)$$

for $P_0=(\mathbf{P}^2+M^2)^{1/2}$ where $I$ is given as before by Eq. (2.5). Corresponding operator forms are

$$(I+G)\chi=0\,, \qquad (2.13a)$$

$$\bar{\chi}(I+G)=0\,. \qquad (2.13b)$$

Introducing the notation of Eq. (2.10), we may express the bound-state contribution (2.7) to the two-fermion Green's function in the form

$$-(2\pi)^{-3}\sum_r \int d^4P\,\chi_{Pr}(x)\bar{\chi}_{Pr}(x') \times e^{iP(X-X')}\delta[P_0-(\mathbf{P}^2+M^2)^{1/2}]/(2P_0)\,,$$

and readily use Eqs. (2.2) and (2.11) to derive the result[8]

$$K(pqP)=-\frac{i}{(2\pi)^4 2P_0}\frac{1}{P_0-(\mathbf{P}^2+M^2)^{1/2}+i\epsilon} \times\sum_{r=1}^{n}\chi_{Pr}(p)\bar{\chi}_{Pr}(q) +\text{terms regular at } P_0=(\mathbf{P}^2+M^2)^{1/2}. \qquad (2.14)$$

This last result clearly exhibits the pole structure characteristic of a bound state. One can in similar fashion exhibit that $K(pqP)$ has a pole at $P_0=-(\mathbf{P}^2+M^2)^{1/2}$ corresponding to the expected bound state of two antifermions. Also, we display for later reference, the immediate consequence

$$[P_0-(\mathbf{P}^2+M^2)^{1/2}]K(pqP)|_{P_0=(\mathbf{P}^2+M^2)^{1/2}} =-i[(2\pi)^4 2P_0]^{-1}\sum_r \chi_{Pr}(p)\bar{\chi}_{Pr}(q)\,, \qquad (2.15)$$

of Eq. (2.14).

We are now in a position to deduce our normalization condition for the Bethe-Salpeter wave function $\chi_{Pr}(p)$ directly from Eqs. (2.6b), (2.13a), and (2.15). It is convenient to introduce an auxiliary quantity

$$Q(pqP)=\int d^4q'[P_0-(\mathbf{P}^2+M^2)^{1/2}]K(pq'P) \times\frac{\partial}{\partial P_0}[I(q'qP)+G(q'qP)]\,, \qquad (2.16)$$

defined off as well as on the mass shell. Using the more compact operator notation, the definition of $Q$ appears as

$$Q=[P_0-(\mathbf{P}^2+M^2)^{1/2}]K\frac{\partial}{\partial P_0}(I+G)\,. \qquad (2.17)$$

An alternative expression, which can be obtained with the aid of Eq. (2.6b), is

$$Q=1-\left[\frac{\partial}{\partial P_0}\{[P_0-(\mathbf{P}^2+M^2)^{1/2}]K\}\right](I+G)\,. \qquad (2.18)$$

If we operate on $\chi_{Pr}$ with $Q$ in the form (2.18), we obtain, using Eq. (2.13a), which is valid for $P_0=(\mathbf{P}^2+M^2)^{1/2}$, the simple equation

$$Q\chi_{Pr}=\chi_{Pr}\,, \qquad (2.19)$$

for $P_0=(\mathbf{P}^2+M^2)^{1/2}$. On the other hand, if we operate using $Q$ in the form (2.17), we obtain, using Eq. (2.15),

$$Q\chi_{Pr}=-i[(2\pi)^4 2P_0]^{-1}\sum_s \chi_{Ps}\bar{\chi}_{Ps}\frac{\partial}{\partial P_0}(I+G)\chi_{Pr}\,, \qquad (2.20)$$

for

$$(P_0=(\mathbf{P}^2+M^2)^{1/2})\,.$$

In the absence of degeneracy, comparison of Eqs. (2.19) and Eq. (2.20) would lead directly to the normalization condition

$$-(2\pi)^{-4}i\bar{\chi}_P\frac{\partial}{\partial P_0}(I+G)\chi_P=2P_0\,, \quad (P^2=-M^2)\,, \qquad (2.21)$$

which has previously been obtained by Allcock[3] and by Cutkosky and Leon[4] on the basis of much less straightforward considerations. When written in full, Eq. (2.20) takes on the appearance

$$-(2\pi)^{-4}i\int d^4q d^4q'\bar{\chi}_P(q')\frac{\partial}{\partial P_0}[I(q'qP)+G(q'qP)]\chi_P(q) =2P_0\,, \quad (P^2=-M^2)\,. \qquad (2.22)$$

In the case of $n$-fold degeneracy, we obtain in place of Eq. (2.21)

$$\sum_{s=1}^{n}\chi_{Ps}\left\{-i(2\pi)^{-4}(2P_0)^{-1}\bar{\chi}_{Ps}\frac{\partial}{\partial P_0}(I+G)\chi_{Pr}-\delta_{sr}\right\} =0\,, \quad (P^2=-M^2)\,,$$

so that, if we assume the linear independence of the

$\chi_{Ps}$,[10] we obtain the orthonormality relations

$$-(2\pi)^{-4} i\bar{\chi}_{Ps}\frac{\partial}{\partial P_0}(I+G)\chi_{Pr}=2P_0\delta_{sr}\,,\qquad (P^2=-M^2)\,. \tag{2.23}$$

It seems worthwhile to emphasize that the above normalization condition, although apparently of unfamiliar appearance, is, in fact, a close analog of the usual normalization conditions for free elementary-particle wave functions. To underline the formal correspondence, we may reverse the conventional procedure of quantum-field theory and derive the normalization condition for single-particle wave functions in a manner parallel at each step to the one used above in the derivation of Eq. (2.23). We illustrate for spin-$\frac{1}{2}$ particles, but it is evident that the procedure is quite general.

We consider the fermion propagator

$$S_F(x_1-x_2)=\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle\,. \tag{2.24}$$

It follows from the free-field equations and equal-time commutation rules that $S_F(x_1-x_2)$ satisfies

$$(\gamma\cdot\partial+m)S_F(x_1-x_2)=-i\delta(x_1-x_2)\,,$$

and hence its Fourier transform $S_F(p)$ satisfies

$$(\gamma\cdot p-im)S_F(p)=S_F(p)(\gamma\cdot p-im)=-1\,. \tag{2.25}$$

We now consider $t_1>t_2$, and obtain

$$S_F(x_1-x_2)=\sum_{r=1}^{2}\int d^4p\langle 0|\psi(x_1)|pr\rangle\langle pr|\bar{\psi}(x_2)|0\rangle \times\theta(p_0)\delta(p^2+m^2)\,, \tag{2.26}$$

since only one-fermion states contribute because of baryon number conservation. Using invariance under space-time translations, we write

$$\langle 0|\psi(x_1)|pr\rangle=(2m)^{1/2}(2\pi)^{-3/2}e^{ipx_1}u_{pr}\,, \tag{2.27a}$$

$$\langle pr|\bar{\psi}(x_2)|0\rangle=(2m)^{1/2}(2\pi)^{-3/2}e^{-ipx_2}\bar{u}_{pr}\,, \tag{2.27b}$$

thereby defining spinors $u_{pr}$ and $\bar{u}_{pr}$ which satisfy

$$(\gamma\cdot p-im)u_{pr}=0\,, \tag{2.28a}$$

$$\bar{u}_{pr}(\gamma\cdot p-im)=0\,, \tag{2.28b}$$

as a consequence of the free-field equations. From Eqs. (2.26), (2.27a) and (2.27b) and the fact that one-fermion states do not contribute for $t_2>t_1$, we deduce as above the result

$$S_F(p)=\frac{i}{p_0-(\mathbf{p}^2+m^2)^{1/2}+i\epsilon}\frac{m}{p_0}\sum_{r=1}^{2}u_{pr}\bar{u}_{pr} +\text{terms regular at } p_0=(\mathbf{p}^2+m^2)^{1/2}\,. \tag{2.29}$$

We now proceed exactly as above—introducing an auxiliary off-the-mass-shell quantity

$$[p_0-(\mathbf{p}^2+m^2)^{1/2}]S_F(p)\frac{\partial}{\partial p_0}(\gamma\cdot p-im)\,, \tag{2.30}$$

and using Eqs. (2.25), (2.28a), and (2.29), to deduce the orthonormality relations[11]

$$i\bar{u}_{pr}\frac{\partial}{\partial p_0}(\gamma\cdot p-im)u_{ps}=-\left(\frac{p_0}{m}\right)\delta_{rs}\,, \tag{2.31}$$

or simply

$$u^{\dagger}_{pr}u_{ps}=(p_0/m)\delta_{rs}\,, \tag{2.32}$$

in the familiar form.

## 3. BETHE-SALPETER WAVE-FUNCTION DESCRIPTION OF PHYSICAL MESON IN PSEUDOSCALAR MESON THEORY

In order to give an illustration of the use of the normalization condition for Bethe-Salpeter amplitudes derived in Sec. 2 and a demonstration of the consistency of the formalism developed there, we consider in this section a bound-state problem similar to that studied by Okubo and Feldman[5] and by Rowe.[6] In this problem we study the Bethe-Salpeter amplitudes

$$\chi_P(x_1x_2)_{\alpha\beta}=\langle 0|T\psi_\alpha(x_1)\bar{\psi}_\beta(x_2)|P\rangle\,, \tag{3.1a}$$

$$\bar{\chi}_P(x_2x_1)_{\beta\alpha}=\langle P|T\bar{\psi}_\alpha(x_1)\psi_\beta(x_2)|0\rangle\,, \tag{3.1b}$$

corresponding to a meson bound state $|P\rangle$ having the same spin and parity quantum numbers as the meson field $\phi$ in a Yukawa theory with interaction Lagrangian

$$\mathcal{L}=iG_0\bar{\psi}\gamma_5\psi\phi\,. \tag{3.2}$$

The amplitudes $\chi$ and $\bar{\chi}$ are shown to be determined (to within a factor since $\chi$ and $\bar{\chi}$ satisfy homogeneous equations) in terms of the basic Green's functions of the theory. We then determine the proportionality factors by means of the normalization condition for $\chi$ and $\bar{\chi}$, derived by the methods of Sec. 2. The possibility of obtaining expressions for $\chi$ and $\bar{\chi}$ in closed form in terms of the basic Green's functions stems from the fact that the bound state $|P\rangle$ can be shown in this case to be simply a dressed quantum of the field $\phi$. This feature also allows a direct evaluation of $\chi$ and $\bar{\chi}$ by means of the reduction formulas of Lehmann, Symanzik, and Zimmermann (LSZ).[12] Comparison of the results of the two approaches affords a demonstration of the consistency of the Bethe-Salpeter formalism developed here.

[10] This is not an additional assumption. It can be seen from reference to Gell-Mann and Low's derivation (Ref. 1) of the Bethe-Salpeter equation, that the linear independence is already required if one is to obtain Eqs. (2.9a) and (2.9b).

[11] The form Eq. (2.31) of the normalization condition for one-particle wave functions arises naturally in the formulation of quantum field theory developed by Y. Takahashi and H. Umezawa, Nucl. Phys. **51**, 193 (1964).

[12] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1955).

We begin by quoting the basic equations for $\chi$ and $\bar{\chi}$ in the present theory. Defining the Fourier transforms $\chi_{Pr}(p)$, $\bar{\chi}_{Pr}(q)$ of the Bethe-Salpeter amplitudes (3.1a) and (3.1b) by means of Eqs. (2.10) and (2.11), we may write the Bethe-Salpeter equations in the forms

$$\chi_P(p)_{\alpha\beta}=-S'_F(p+\tfrac{1}{2}P)_{\alpha\alpha'}\int d^4q'G(pq'P)_{\alpha'\beta'\gamma'\delta'} \times\chi_P(q')_{\gamma'\delta'}S'_F(p-\tfrac{1}{2}P)_{\beta'\beta}, \quad (3.3a)$$

$${}_P(q)_{\delta\gamma}=-S'_F(q-\tfrac{1}{2}P)_{\delta\delta'}\int d^4p'\bar{\chi}_P(p')_{\beta'\alpha'} \times G(p'qP)_{\alpha'\beta'\gamma'\delta'}S'_F(q+\tfrac{1}{2}P)_{\gamma'\gamma}. \quad (3.3b)$$

It is convenient[5,6] to write the interaction function $G(pqP)$ as the sum of two terms

$$G(pqP)=\bar{G}(pqP)+A(pqP), \quad (3.4)$$

where $A$ describes the annihilation graph of Fig. 1 and $\bar{G}$ contains the effect of all other Bethe-Salpeter irreducible graphs. An explicit form for $A$ is

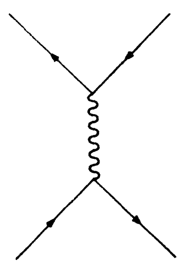

FIG. 1. The annihilation graph.

$$A(pqP)_{\alpha\beta\gamma\delta}=-(2\pi)^{-4}iG_0^2\Delta_F(P)\gamma_{5\alpha\beta}\gamma_{5\delta\gamma}, \quad (3.5)$$

where $\Delta_F(P)=(P^2+\mu_0^2-i\epsilon)^{-1}$ is the bare meson propagator with $\mu_0$ the bare meson mass. To order $G_0^2$, only the graph displayed in Fig. 2 contributes to $\bar{G}$, and the expression for it is

$$\bar{G}^{(2)}(pqP)_{\alpha\beta\gamma\delta}=iG_0^2\gamma_{5\alpha\gamma}\gamma_{5\delta\beta}(2\pi)^{-4}\Delta_F(p-q). \quad (3.6)$$

Hence, to order $G_0^2$, Eq. (3.3) appears as[13]

$$\chi_P(p)=-iS'_F(p+\tfrac{1}{2}P)\gamma_5 \times\int d^4q'\chi_P(q')\gamma_5S'_F(p-\tfrac{1}{2}P)(2\pi)^{-4}G_0^2\Delta_F(p-q') +iS'_F(p+\tfrac{1}{2}P)\gamma_5S'_F(p-\tfrac{1}{2}P) \times(2\pi)^{-4}G_0^2\Delta_F(P)\int d^4q'\,\mathrm{Tr}[\gamma_5\chi_P(q')]. \quad (3.7)$$

[13] We omit spinor indices wherever this can be done without loss of clarity.

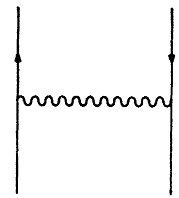

FIG. 2. The one-meson-exchange graph.

The alternative forms of Eqs. (3.3a) and (3.3b),

$$\int d^4q[I(pqP)+G(pqP)]_{\alpha\beta\gamma\delta}\chi_P(q)_{\gamma\delta}=0, \quad (3.8a)$$

$$\int d^4p\bar{\chi}_P(p)_{\beta\alpha}[I(pqP)+G(pqP)]_{\alpha\beta\gamma\delta}=0, \quad (3.8b)$$

where

$$I(pqP)_{\alpha\beta\gamma\delta}=[S'_F(p+\tfrac{1}{2}P)]^{-1}{}_{\alpha\gamma} \times[S'_F(p-\tfrac{1}{2}P)]^{-1}{}_{\delta\beta}\delta(p-q), \quad (3.9)$$

will also be useful later. Finally, we note that the relation

$$\int dq_0\bar{\chi}_P(q)=-\int dq_0\gamma_4\chi_P(q)^\dagger\gamma_4, \quad (3.10)$$

where $q=(q_0,\mathbf{q})$, can be deduced directly from the definitions, Eq. (3.1), provided that one recalls that the Heisenberg fields $\psi(x_1)$ and $\bar{\psi}(x_2)$ anticommute whenever $x_1-x_2$ is space-like.

We now turn to the relation of $\chi$ and $\bar{\chi}$ to the generalized vertex function $\Gamma_5$ of the Yukawa theory. The Fourier transform $\Gamma_5(p+\frac{1}{2}P,\ p-\frac{1}{2}P)$, corresponding to the diagram of Fig. 3, satisfies the inhomogeneous integral equation[13]

$$\Gamma_5(p+\tfrac{1}{2}P,\ p-\tfrac{1}{2}P)_{\alpha\beta} =\gamma_{5\alpha\beta}-\int d^4q'\bar{G}(pq'P)_{\alpha\beta\gamma\delta}[S'_F(q'+\tfrac{1}{2}P) \times\Gamma_5(q'+\tfrac{1}{2}P,\ q'-\tfrac{1}{2}P)S'_F(q'-\tfrac{1}{2}P)]_{\gamma\delta}. \quad (3.11a)$$

It is crucial for what follows that the $\bar{G}$ which appears here is exactly the same function as occurs in the Bethe-Salpeter equations, (3.3) with (3.4), given above. The truth of this statement can be checked to any desired order of $G_0^2$ by means of explicit developments of Eqs.

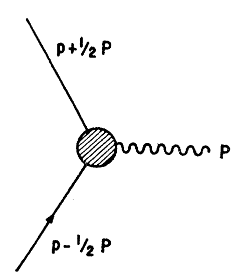


FIG. 3. Complete vertex in momentum space.

(3.3a) and (3.11a). When one recalls that the generalized vertex function $\Gamma_5$ is obtained from $\gamma_5$ by adding to it contributions from all proper vertex diagrams,[14] the reason for the occurrence of $\bar{G}$ rather than $G$ in Eq. (3.11a), or alternatively for the explicit separation of $A$ from $\bar{G}$ is understood: insertion of $A$ in place $\bar{G}$ in Eq. (3.11a) would give rise to improper vertex diagrams. In course of the above-mentioned explicit development of Eq. (3.3), one can also verify, to any desired order of $G_0^2$, the identity

$$\bar{G}(pqP)_{\alpha\beta\gamma\delta}=\bar{G}(qp-P)_{\delta\gamma\beta\alpha}, \qquad (3.12)$$

which allows one to transcribe Eq. (3.11a) into the form

$$\Gamma_5(q-\tfrac{1}{2}P,\, q+\tfrac{1}{2}P)_{\delta\gamma} =\gamma_{5\delta\gamma}-\int d^4p'\bar{G}(p'qP)_{\alpha\beta\gamma\delta}[S'_F(q'-\tfrac{1}{2}P) \times\Gamma_5(q'-\tfrac{1}{2}P,\, q'+\tfrac{1}{2}P)S'_F(q'+\tfrac{1}{2}P)]_{\beta\alpha}. \qquad (3.11b)$$

To proceed further toward the relation of $\chi$ and $\bar{\chi}$ to $\Gamma_5$, it is convenient to introduce an auxiliary function $\psi_P(p)$, proportional to $\chi_P(p)$ by means of

$$\chi_P(p)=\left\{-(2\pi)^{-4}iG_0^2\Delta_F(P) \times\int d^4p'\,\mathrm{Tr}[\gamma_5\chi_P(p')]\right\}\psi_P(p). \qquad (3.13a)$$

It is easily seen from Eq. (3.3), the $A$ contribution to this equation being the same as the $A$-contribution to Eq. (3.7), that $\psi_P(p)$ satisfies the inhomogeneous equation

$$\psi_P(p)_{\alpha\beta}=-S'_F(p+\tfrac{1}{2}P)_{\alpha\alpha'}\left\{\gamma_{5\alpha'\beta'}+\int d^4q'\bar{G}(pq'P)_{\alpha'\beta'\gamma'\delta'} \times\psi_P(q')_{\gamma'\delta'}\right\}S'_F(p-\tfrac{1}{2}P)_{\beta'\beta}, \qquad (3.14a)$$

and, comparison of Eqs. (3.14a) and (3.11a) leads to the immediate identification

$$\psi_P(p)=-S'_F(p+\tfrac{1}{2}P) \times\Gamma_5(p+\tfrac{1}{2}P,\, p-\tfrac{1}{2}P)S'_F(p-\tfrac{1}{2}P). \qquad (3.15a)$$

This important result will allow now the determination of the values of the bound-state mass $(-P^2)$ for which Eq. (3.3) has solutions. Indeed, if, for any solution $\chi_P(p)$, we multiply Eq. (3.13) by $\gamma_5$, take the trace, integrate over $p$, and divide by $\int d^4p\,\mathrm{Tr}\{\gamma_5\chi_P(p)\}$, we get

$$P^2+\mu_0^2=-(2\pi)^4iG_0^2\int d^4p\,\mathrm{Tr}\{\pi_5\psi_P(p)\}, \qquad (3.16)$$

and, hence, using (3.15a)

$$P^2+\mu_0^2=(2\pi)^4iG_0^2\int d^4p\,\mathrm{Tr}\{\gamma_5S'_F(p+\tfrac{1}{2}P) \times\Gamma_5(p+\tfrac{1}{2}P,\, p-\tfrac{1}{2}P)S'_F(p-\tfrac{1}{2}P)\} \qquad (3.17)$$

$$=G_0^2\Pi^*(P^2) \qquad (3.18)$$

where $\Pi^*(P^2)$ is the well-known expression[15] for the fermion loop shown in Fig. 4. Thus we see that the allowed values of $(-P^2)$ are just the solutions $\mu^2$ of the familiar meson mass-renormalization equation[15]

$$\mu^2-\mu_0^2=-G_0^2\Pi^*(-\mu^2)=\delta\mu^2(\mu^2). \qquad (3.19)$$

In other words, the bound states $|P\rangle$ of the problem are the physical (dressed) quanta of the field $\phi$ occurring in the Lagrangian. Result (3.15a) also allows the writing of the desired equation relating $\chi$ to $\Gamma_5$, namely

$$\chi_P(p)=cS'_F(p+\tfrac{1}{2}P)\Gamma_5(p+\tfrac{1}{2}P,\, p-\tfrac{1}{2}P)S'_F(p-\tfrac{1}{2}P), \quad (P^2=-\mu^2), \qquad (3.20a)$$

where the constant $c$ is given by

$$c=(2\pi)^{-4}iG_0^2\Delta_F(P)\int d^4p\,\mathrm{Tr}[\gamma_5\chi_P(p)] \quad (P^2=-\mu^2) \qquad (3.21a)$$

and is at present undetermined: This is because we have required only that $\chi_P$ be a solution of the homogeneous Bethe-Salpeter equation. Before proceeding to use the appropriate normalization condition for Bethe-Salpeter wave functions to determine the value of $c$, we note that $\bar{\psi}_P(p)$ defined by

$$\bar{\chi}_P(p)=\left\{-(2\pi)^{-4}iG_0^2\Delta_F(P) \times\int d^4p'\,\mathrm{Tr}[\gamma_5\bar{\chi}_P(p')]\right\}\bar{\psi}_P(p) \qquad (3.13b)$$

obeys

$$\bar{\psi}_P(p)_{\delta\gamma}=-S'_F(p-\tfrac{1}{2}P)_{\delta\delta'}\left\{\gamma_{5\delta'\gamma'}+\int d^4q'\bar{G}(pq'P)_{\alpha'\beta'\gamma'\delta'} \times\bar{\psi}_P(q')_{\beta'\alpha'}\right\}S'_F(p+\tfrac{1}{2}P)_{\gamma'\gamma} \qquad (3.14b)$$

so that comparison with Eq. (3.11b) yields

$$\bar{\psi}_P(p)=-S'_F(p-\tfrac{1}{2}P) \times\Gamma_5(p-\tfrac{1}{2}P,\, p+\tfrac{1}{2}P)S'_F(p+\tfrac{1}{2}P), \qquad (3.15b)$$

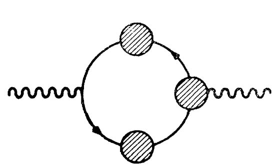

FIG. 4. Fermion loop.

[14] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[15] Our notation here coincides with that used in D. Lurié and A. J. Macfarlane, Phys. Rev. **136**, B816 (1964).

and, hence,

$$\bar{\chi}_P(p)=c'S'_F(p-\tfrac{1}{2}P)\Gamma_5(p-\tfrac{1}{2}P,\,p+\tfrac{1}{2}P)\times S'_F(p+\tfrac{1}{2}P),\quad (P^2=-\mu^2),\quad (3.20\text{b})$$

where the constant $c'$ is given by

$$c'=(2\pi)^{-4}iG_0^2\Delta_F(P)\int d^4p\,\mathrm{Tr}[\gamma_5\bar{\chi}_P(p)],\quad (P^2=-\mu^2).\quad (3.21\text{b})$$

We also need the following alternative form of Eq. (3.14):

$$\int d^4q[I(pqP)+\bar{G}(pqP)]_{\alpha\beta\gamma\delta}\psi_P(q)_{\gamma\delta}=-\gamma_{5\alpha\beta},\quad (3.22\text{a})$$

$$\int d^4p\bar{\psi}_P(p)_{\beta\alpha}[I(pqP)+\bar{G}(pqP)]_{\alpha\beta\gamma\delta}=-\gamma_{5\delta\gamma},\quad (3.22\text{b})$$

where $I$ is given by Eq. (3.9).

We now turn to the determination of $c$ and $c'$, which completes the specification of the relation of the Bethe-Salpeter wave functions $\chi$ and $\bar{\chi}$ to the vertex function $\Gamma_5$. We first note that although the previous discussion does not determine $c$ and $c'$, it does relate them. Insertion of Eq. (3.10) into Eq. (3.21b) and comparison with Eq. (3.21a) immediately provides the relation

$$c'=-c^*.\quad (3.23)$$

We now use a normalization condition for the Bethe-Salpeter wave functions $\chi$ and $\bar{\chi}$ to determine $c$ to within a phase. By means of a procedure similar to that followed in Sec. 2, we obtain such a condition in the form

$$(2\pi)^{-4}i\int d^4p\int d^4q\bar{\chi}_P(p)_{\beta\alpha}\frac{\partial}{\partial P_0}[I(pqP)+G(pqP)]_{\alpha\beta\gamma\delta}\times\chi_P(q)_{\gamma\delta}=2P_0,\quad (P^2=-\mu^2),\quad (3.24)$$

with the same $I$ and $G$ as in Eqs. (3.9) and (3.3). We consider the $A$ and $(I+\bar{G})$ contributions to the left side separately. In the former case, insertion of Eqs. (3.5), (3.20a) and (3.20b), and direct performance of the differentiation yields ($P^2=-\mu^2$)

$$(2\pi)^{-4}ic'\int d^4p\,\mathrm{Tr}[\gamma_5\bar{\psi}_P(p)](2P_0)(P^2+\mu_0^2)^{-2}\times(2\pi)^{-4}(-iG_0^2)c\int d^4q\,\mathrm{Tr}[\gamma_5\psi_P(q)].$$

Inserting Eqs. (3.15a) and (3.15b) for $\psi$ and $\bar{\psi}$, and using Eq. (3.17) twice immediately reduces this to

$$-2P_0cc'G_0^{-2},\quad (P^2=-\mu^2).\quad (3.25)$$

Turning next to the $I+\bar{G}$ contribution, we use Eqs. (3.20a) and (3.20b) and write it (still with $P^2=-\mu^2$) as

$$cc'(2\pi)^{-4}i\int d^4p\int d^4q\,\bar{\psi}_P(p)_{\beta\alpha}\times\frac{\partial}{\partial P_0}[I(pqP)+\bar{G}(pqP)]_{\alpha\beta\gamma\delta}\psi_P(q)_{\gamma\delta}$$
$$=cc'(2\pi)^{-4}i\int d^4p\int d^4q\,\bar{\psi}_P(p)_{\beta\alpha}[I(pqP)+\bar{G}(pqP)]_{\alpha\beta\gamma\delta}\times(-\partial\psi_P(q)_{\gamma\delta}/\partial P_0)$$
$$=cc'(2\pi)^{-4}i\int dq\frac{\partial}{\partial P_0}\{\mathrm{Tr}[\gamma_5\psi_P(q)]\}.$$

Herein, use of Eqs. (3.22a) and (3.22b) has been made in the second and third lines, respectively. Inserting the form (3.15a) for $\psi$ converts this contribution into

$$-cc'(2\pi)^{-4}i\int d^4q\frac{\partial}{\partial P_0}\{\mathrm{Tr}[\gamma_5S'_F(p+\tfrac{1}{2}P)\times\Gamma_5(p+\tfrac{1}{2}P,\,p-\tfrac{1}{2}P)S'_F(p-\tfrac{1}{2}P)]\},\quad (P^2=-\mu^2),$$
$$=-cc'\frac{\partial}{\partial P_0}\Pi^*(P^2)|_{P^2=-\mu^2}$$
$$=2P_0cc'\Pi^{*\prime}(-\mu^2),\quad (P^2=-\mu^2),\quad (3.26)$$

where $\Pi^*$ is defined by Eqs. (3.17) and (3.18) and $\Pi^{*\prime}$ is its derivative. Final evaluation of the contribution now follows use of the result[15]

$$Z_3^{-1}=1-G_0^2\Pi^{*\prime}(-\mu^2),\quad (3.27)$$

and, putting together of Eqs. (3.24) to (3.27) yields

$$cc'=-Z_3G_0^2.\quad (3.28)$$

Hence, if we demand that $c$ be real and positive,[16] Eqs. (3.23) and (3.28) give

$$c=Z_3^{1/2}G_0,\quad c'=-Z_3^{1/2}G_0,\quad (3.29)$$

and the relation of $\chi$ and $\bar{\chi}$ to $\Gamma_5$ is fully specified.

Before proceeding to the use of the LSZ formalism to confirm the results just obtained, we may note the following argument which sheds considerable light upon them. In the theory defined by Eq. (3.2), the fermion-antifermion Green's function is known to have a pole at $P_0=(\mathbf{P}^2+\mu^2)^{1/2}$ arising from a term of the form

$$(2\pi)^{-4}iG_0^2\Delta'_F(P^2)\times[S'_F(p+\tfrac{1}{2}P)\Gamma_5(p+\tfrac{1}{2}P,\,p-\tfrac{1}{2}P)S'_F(p-\tfrac{1}{2}P)]\times[S'_F(q-\tfrac{1}{2}P)\Gamma_5(q-\tfrac{1}{2}P,\,q+\tfrac{1}{2}P)S'_F(q+\tfrac{1}{2}P)].$$

This term corresponds to the one-boson exchange dia-

[16] This choice of phase leads to agreement with the alternative method of obtaining $\chi$ and $\bar{\chi}$ in terms of $\Gamma_5$ employed below.

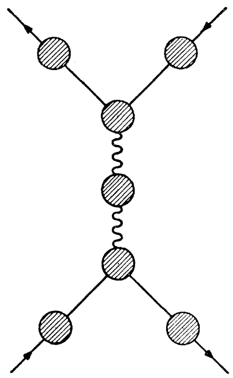

FIG. 5. Diagram giving pole in fermion-antifermion Green's function.

gram shown in Fig. 5. The residue at the pole $P_0 = (\mathbf{P}^2+\mu^2)^{1/2}$ is simply

$$(2\pi)^{-4}G_0^2 iZ_3(-2P_0)^{-1} \times[S'_F(p+\tfrac{1}{2}P)\Gamma_5(p+\tfrac{1}{2}P,\, p-\tfrac{1}{2}P)S'_F(p-\tfrac{1}{2}P)] \times[S'_F(q-\tfrac{1}{2}P)\Gamma_5(q-\tfrac{1}{2}P,\, q+\tfrac{1}{2}P)S'_F(q+\tfrac{1}{2}P)],$$

evaluated at $P_0=(\mathbf{P}^2+\mu^2)^{1/2}$. By virtue of Eqs. (3.20a), (3.20b), and Eq. (3.28), this equals the quantity $(2\pi)^{-4}i(2P_0)^{-1}\chi_P(p)\bar{\chi}_P(q)$, evaluated at $P_0=(\mathbf{P}^2+\mu^2)^{1/2}$, in agreement with the analog in the present theory of Eq. (2.14). We now conclude the section by employing the reduction formula technique of Lehmann, Symanzik, and Zimmermann[12] to evaluate the Bethe-Salpeter amplitudes (3.1a) and (3.1b). The availability of this approach stems from the fact that the boson bound state $|P\rangle$ is just the dressed quantum of the field $\phi$, appearing in the Lagrangian.

Direct application of the reduction formula techniques to $\chi$ and $\bar{\chi}$ as defined by Eqs. (3.1a) and (3.1b) yields the results[17]

$$\chi_P(x_1x_2)=-(2\pi)^{-3/2}iZ_3^{-1/2}\int d^4y e^{iPy}(\Box_y-\mu^2) \times\langle 0|T\psi(x_1)\bar{\psi}(x_2)\phi(y)|0\rangle, \quad (3.30a)$$

$$\bar{\chi}_P(x_2x_1)=-(2\pi)^{-3/2}iZ_3^{-1/2}\int d^4y e^{-iPy}(\Box_y-\mu^2) \times\langle 0|T\bar{\psi}(x_2)\psi(x_1)\phi(y)|0\rangle, \quad (3.30b)$$

with $P^2=-\mu^2$, $\mu$ being the physical meson mass. A comment on the occurrence of the factor $Z_3^{-1/2}$ in these equations seems necessary.[18] It stems from our use of the asymptotic condition

$$\lim_{t\to\mp\infty}\langle a|\phi(x)|b\rangle=Z_3^{1/2}\langle a|\phi_{\substack{\text{out}\\ \text{in}}}(x)|b\rangle, \quad (3.31)$$

$\phi(x)$, like $\psi(x)$ and $\bar{\psi}(x)$, being an unrenormalized Heisenberg field operator.

By standard procedures, we can evaluate the vacuum expectation value $\langle 0|T\psi(x_1)\bar{\psi}(x_2)\phi(y)|0\rangle$ obtaining

$$\langle 0|T\psi(x_1)\bar{\psi}(x_2)\phi(y)|0\rangle =G_0(2\pi)^{-8}i\int d^4k\int d^4k' e^{ikx_1}e^{-ik'x_2}e^{-i(k-k')y} \times S'_F(k)\Gamma_5(k,k')S'_F(k')\Delta'_F(k-k'), \quad (3.32)$$

where $\Delta'_F(P)$ is defined by

$$\langle 0|T\phi(x)\phi(y)|0\rangle=-i\Delta'_F(x-y) =-(2\pi)^{-4}i\int d^4p e^{ip(x-y)}\Delta'_F(p).$$

We may combine Eqs. (3.30a) and (3.32) and derive the result

$$\chi_P(x_1x_2)=(2\pi)^{-3/2}G_0Z_3^{-1/2}(2\pi)^{-4}\int d^4p e^{\frac{1}{2}iP(x_1+x_2)}e^{ip(x_1-x_2)} \times S'_F(p+\tfrac{1}{2}P)\Gamma_5(p+\tfrac{1}{2}P,\, p-\tfrac{1}{2}P) \times S'_F(p-\tfrac{1}{2}P)\Delta'_F(P)(P^2+\mu^2), \quad (3.33)$$

which is to be evaluated at $P^2=-\mu^2$, using the well-known result

$$\lim_{P^2\to-\mu^2}(P^2+\mu^2)\Delta'_F(P)=Z_3. \quad (3.34)$$

Introducing the Fourier transform of $\chi_P(x_1x_2)$ by means of Eqs. (2.10a) and (2.11a) now gives

$$\chi_P(p)=G_0Z_3^{1/2}S'_F(p+\tfrac{1}{2}P) \times\Gamma_5(p+\tfrac{1}{2}P,\, p-\tfrac{1}{2}P)S'_F(p-\tfrac{1}{2}P), \quad (3.35)$$

which agrees with the previous results, cf. Eqs. (3.20a) and (3.29).

We may also handle Eq. (3.30b) in a like manner and thereby confirm the previous results, cf. (3.20b) and (3.29), for $\bar{\chi}_P(q)$.

## APPENDIX: DERIVATION OF EQ. (2.14)

The contribution (2.7) of the bound states to the two-fermion Green's function may be written as

$$-(2\pi)^{-3}\sum_{r=1}^{n}\int\frac{d^3P}{2\omega_P}\chi_{Pr}(x)\bar{\chi}_{Pr}(x')e^{i\mathbf{P}\cdot(\mathbf{X}-\mathbf{X}')}e^{-i\omega_P(X_0-X_0')} \times\theta(X-X'-\tfrac{1}{2}|x|-\tfrac{1}{2}|x'|), \quad (A1)$$

where $\omega_P=(\mathbf{P}^2+M^2)^{1/2}$. The above expression represents the contribution of the bound states for *any* time ordering, since, as is easily verified, the factor $\theta(X-X'-\frac{1}{2}|x|-\frac{1}{2}|x'|)$ equals 1 when $t_1$, $t_2>t_3$, $t_4$ and

[17] Our normalization of single-particle states is $\langle P'|P\rangle=2P_0\delta(\mathbf{P}-\mathbf{P}')$. This is consistent with the conventions followed in Sec. 2.

[18] Cf. Y. Takahashi and H. Umezawa, Ref. 11, and H. Ezawa, Ann. Phys. (N. Y.) 24, 46 (1963).

zero otherwise. We now insert the formula

$$\theta(y) = -(2\pi i)^{-1} \int dK_0 (K_0 + i\epsilon)^{-1} e^{-iK_0 y} \qquad \text{(A2)}$$

into (A1) and make the change of variabl es $K_0 \to P_0 - \omega_P$ to obtain

$$-(2\pi)^{-4} i \sum_{r=1}^{n} \int d^3P \frac{dP_0}{2\omega_P} \chi_{Pr}(x) \times \bar{\chi}_{Pr}(x') e^{i\mathbf{P}\cdot(\mathbf{X}-\mathbf{X}')} e^{-iP_0(X_0-X_0')} \times (P_0 - \omega_P + i\epsilon)^{-1} e^{\frac{1}{2}i(P_0-\omega_P)(|x_0|+|x_0'|)}. \qquad \text{(A3)}$$

If we now define new amplitudes $\chi'$ and $\bar{\chi}'$ by

$$\chi'_{Pr}(x) = e^{\frac{1}{2}i(P_0-\omega_P)|x_0|} \chi_{Pr}(x),$$
$$\bar{\chi}'_{Pr}(x') = e^{\frac{1}{2}i(P_0-\omega_P)|x_0'|} \bar{\chi}_{Pr}(x'), \qquad \text{(A4)}$$

and their Fourier transforms as in (2.11), we find, using (2.2), that the contribution of the bound state to $K(pqP)$ has the form

$$\frac{-i}{(2\pi)^4 2\omega_P} \frac{1}{P_0 - \omega_P + i\epsilon} \sum_{r=1}^{n} \chi'_{Pr}(p) \bar{\chi}'_{Pr}(q). \qquad \text{(A5)}$$

As is evident from (A4), the amplitudes $\chi'_{Pr}(x)$ and $\bar{\chi}'_{Pr}(x')$ go over into $\chi_{Pr}(x)$ and $\bar{\chi}_{Pr}(x')$ on the mass shell $P_0 = \omega_P$. The same is true of the Fourier transforms $\chi'_{Pr}(p)$ and $\bar{\chi}'_{Pr}(q)$ as can be verified through use of the formula

$$e^{iA|x_0|} = -(2\pi i)^{-1} \times \int_{-\infty}^{\infty} d\omega\, e^{-i\omega x_0} [(A+\omega+i\epsilon)^{-1} + (A-\omega+i\epsilon)^{-1}].$$

This completes the proof of Eq. (2.14).

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# Coupled-Equations Method for the Scattering of Identical Particles*

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The completely antisymmetric solution $\Psi^A$ to the problem of the scattering of a fermion by a finite system of identical fermions is studied by means of the expansion $\Psi^A = \sum \varphi_\alpha \psi_\alpha$, where $\{\varphi_\alpha\}$ is a complete set of antisymmetric states for the target and $\psi_\alpha$ are one-particle functions. Coupled equations for the $\psi_\alpha$ are found that obey the proper boundary conditions. This is done by means of the integral equation for $\Psi^A$ and the use of projection operators. The elastic-scattering (optical-model) wave function $\psi_0$ is shown to obey an inhomogeneous differential equation, rather than a Schrödinger equation. The homogeneous solution is identical to the elastic wave function obtained when the projectile is distinguishable, while the inhomogeneous solution is due entirely to exchange effects. The function $\psi_0$ is identical to the optical-model wave function found by Bell and Squires, who showed that $\psi_0$ obeys a Schrödinger equation with an optical potential containing direct and exchange contributions. It is shown that $\psi_0$ yields the exact elastic amplitude including direct and exchange contributions, and a phase-shift analysis of the exchange term is given. The more standard form of solution $\Psi^A = \sum \mathcal{A}\{\varphi_\alpha f_\alpha\}$, where $\mathcal{A}$ is an antisymmetrizer, is briefly discussed. The extension to the cases of inelastic scattering and deuteron elastic scattering is made.

## I. INTRODUCTION

MANY of the collision phenomena encountered in physics involve projectiles containing particles identical with those in the scattering system. In such cases, proper account must be taken of the relevant statistics. The purpose of this paper is to study the exact, symmetrized scattering wave function using the eigenstate-expansion method. This leads to a simple set of coupled-channel equations which take account of the relevant symmetry. Only the case of Fermi statistics is treated, although the extension to Bose particles is indicated.

The eigenfunction-expansion method is well known when the projectile is a single distinguishable particle.[1] If $\{\varphi_\alpha(\xi)\}$ is a complete set of states for the target (whose coordinates are $\{\xi\}$) and $\Psi$ is the exact scattering solution satisfying the Schrödinger equation $(E-H)\Psi = 0$, then the expansion $\Psi = \sum_\alpha \varphi_\alpha(\xi) u_\alpha(r)$ leads to a set of coupled equations for the $u_\alpha(r)$. Here, $r$ is the coordinate of the incident particle. The $u_\alpha$ obey

* Work performed under the auspices of the U. S. Atomic Energy Commission.
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[1] A comprehensive discussion is given by H. Feshbach, Ann. Phys. (N. Y.) 5, 357 (1958). An alternative approach is given by G. E. Brown, Rev. Mod. Phys. 31, 893 (1959). See also, P. G. Burke and K. Smith, Rev. Mod. Phys. 34, 458 (1962), and references cited therein, for a discussion of the electron-hydrogen scattering problem, including exchange effects.