Determination of the Spins and Parities of Resonances*

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The properties of angular-momentum tensors described in a previous paper are used to develop tests for the spins and parities of resonances. Fermion resonances decaying into particles of spin zero and spin one-half or spin zero and spin three-halves and boson resonances decaying into particles of spin zero and spin one are considered in some detail. Attention is given to angular correlations between production and decay configurations, both generally and in special cases such as forward production, low-energy production, and peripheral collisions. A moment analysis of the decay distributions is developed.

I. INTRODUCTION

INVESTIGATION of the spins and parities of resonances has been simplified in the past by means of the principle "presumably, the spin is less than two." This principle no longer serves. In higher spin situations, any given experimental test is less likely to speak decisively, and the number of conceivable tests can be quite large. It becomes worthwhile to study the methodology of resonance analysis as a subject in itself. Among recent studies along these lines we may cite the formalisms of Byers and Fenster,[1] and Ademollo, Gatto, and Preparata.[2]

Our own approach has much in common with these, but relies on a tensor formulation of angular momentum (see preceding paper[3]) which, we feel, has special advantages of simplicity and versatility of application. This paper makes specific application, in Secs. III, IV, and V, to resonances with decay products of spin zero and spin one-half, zero and one, and zero and three-halves, respectively.

II. INGREDIENTS OF THE ANALYSIS

1. Types of Reactions to be Studied

The resonance X appears in a production process followed by a decay process:

$$A+B\rightarrow X+C+D+\cdots, \tag{2.1a}$$

$$X\rightarrow a+b+c+\cdots. \tag{2.1b}$$

The notation identifies *production particles* A, B, C, D, $\cdots$ and *decay particles* a, b, c, $\cdots$. The same symbols can be used for the four-momenta: $A\equiv(E_A,\mathbf{A})$, $A^2=E_A{}^2-\mathbf{A}^2=m_A{}^2$, etc.

In this paper, we study three-body productions

$$A+B\rightarrow X+C \tag{2.2a}$$

followed by two-body decays,

$$X\rightarrow a+b \tag{2.2b}$$

which may be followed by subsequent decays, e.g.,

$$b\rightarrow a'+b'. \tag{2.2c}$$

The physical data consist of the measured energies and momenta in the total process

$$A+B\rightarrow C+a+a'+b'. \tag{2.2d}$$

Amplitudes for different steps of the process will also depend on spins. Ultimately, for a comparison with experiment, the spin states of each particle are either averaged over, or identified in terms of a momentum in a subsequent reaction. If we make no use of dynamical principles, but rely only on rotation and reflection invariance, all useful data are expressible as angular correlations, or, at least, correlations in angle-dependent quantities.

2. Tensors

Systems of integral spin j and half-integral spin $j+\frac{1}{2}$ are described by tensors $T^j{}_m$, $T^{j+1/2}{}_{m,\alpha}$ as discussed in I. This notation manifests the rotation properties of the tensors. Manifest convariance under pure Lorentz transformations is not needed in our contemplated applications.

We use special notations for the spin wave functions of particles of low spin: (Pauli) spinors u, v for spin $\frac{1}{2}$, vectors $\mathbf{e}$, $\mathbf{f}$ for spin 1, and spinor-vectors $\mathbf{E}$, $\mathbf{F}$ for spin $\frac{3}{2}$. These symbols have components as follows:

$$u=(u_1,u_2)=u_\alpha,\quad \alpha=1,2, \tag{2.3}$$

$$\mathbf{e}=(e_1,e_2,e_3)=e_m,\quad m=1,2,3, \tag{2.4}$$

$$\mathbf{E}=E_{m\alpha},\quad \alpha=1,2;\quad m=1,2,3. \tag{2.5}$$

The spinor-vector obeys the constraint $\boldsymbol{\sigma}\cdot\mathbf{E}=0$, and hence also $\mathbf{E}^*\cdot\boldsymbol{\sigma}=0$. If we apply $\boldsymbol{\sigma}$ to these equations and use $\sigma_i\sigma_j=\delta_{ij}+i\sigma_k\epsilon^{ijk}$, we get

$$i\boldsymbol{\sigma}\times\mathbf{E}=\mathbf{E}, \tag{2.6a}$$

$$\mathbf{E}^*\times(i\boldsymbol{\sigma})=\mathbf{E}^*. \tag{2.6b}$$

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[1] N. Byers and S. Fenster, Phys. Rev. Letters **11**, 52 (1963). See also R. Gatto and H. Stapp, Phys. Rev. **121**, 1553 (1961).

[2] M. Ademollo, R. Gatto, and G. Preparata, Phys. Rev. Letters **12**, 462 (1964); M. Ademollo and R. Gatto, Phys. Rev. **133**, B531 (1964).

[3] C. Zemach, preceding paper, Phys. Rev. **139**, B97 (1965); hereafter called I.

3. Amplitudes

In order to make manifest the rotational invariance of a reaction amplitude M, one can write it as a sum over rotationally invariant terms built out of the three-momenta and spin wave functions of the particles, the different terms being multiplied by coupling constants or energy-dependent form factors. As emphasized in I, there is no need to refer all the variables to a common frame of reference. We prefer, following Stapp, to express M in terms of proper variables, by which is meant the following:

(a) Each spin wave function is referred to the particle's rest frame.

(b) Each three-momentum is referred to the center-of-mass frame of the reaction in which it occurs. Thus, **A**, **B**, **C** in (2.2) are referred to the $A+B$ center of mass; the spin wave function of b and the momentum of a' are referred to the b rest frame (called bRF for short).

(c) The relation between the center-of-mass frame (CMF) of a reaction and the rest frame (RF) of one of of the reacting particles whose spin state is being described must be that of a pure velocity transformation defined by the velocity of the particle in the CMF.

The motivation is this: Firstly, the sum or average over spin states is accomplished more easily; there are no relativistic projection operators to worry about. Secondly, the phase-space factor for the momentum distribution of a particle has the nonrelativistic form (see below).

The reason for (c) is inherent in the procedure by which an amplitude is expressed in terms of variables referred to different frames.

Now let $E^j{}_m$ denote a spin wave function for X. The production amplitude for (2.2a) will be of the form

$$E^{j*}:M_P \tag{2.7}$$

and the decay amplitude for (2.2b) or (2.2b) plus (2.2c) will be

$$M_D:E^j. \tag{2.8}$$

Then M_P and M_D will be tensor functions of the data. The combined amplitude for production and decay, summed over the spin states of X is proportional to the tensor product:

$$M_{DP}=M_D:M_P. \tag{2.9}$$

It is often convenient to call M_D, M_P themselves the decay and production amplitudes and omit mention of $E^j{}_m$.

Table I contains general forms for the more common amplitudes M_P, M_D for a resonance. The two-body processes are written as decays and the three-body processes as productions because our application is of this type. The notation already defined has been adhered to.

In the three-body productions in Table I, **P** represents any vector in the production plane, i.e., any linear combination of **A**, **B**, **C**. **Q** is the production normal; $\mathbf{Q}=\mathbf{A}\times\mathbf{C}$. The P's used in the *same* expression refer, in general, to *different* linear combinations. In this way, the complexity of the general form can be comprehended in compact expressions from which theorems on angular correlations (see below) can be derived.

The number of terms in an amplitude is fixed by simple angular momentum counting. Thus the $0^-+\frac{1}{2}^+\to\frac{1}{2}^++X$ entries have one nonspin-flip term with orbital angular momentum $l=j$ and three spin-flip tensors, with $l=j+1$, $l=j$, $l=j-1$. Combinations not appearing in the given enumeration can always be reduced to a sum of terms of the types considered. For example, tensors with two or more occurrences of **Q** can be reduced to tensors with one occurrence of **Q** or none.

4. Counting Rates and Phase Space

Let a, b be two particles among the products of a reaction. The phase space will include the factor

$$(d\mathbf{a}/E_a)(d\mathbf{b}/E_b)\delta^{(4)}(a+b-R), \tag{2.10}$$

where R includes all the other four-momenta. Let **a** be the three-momentum of a in the CMF of $a+b$ and $(p_{ab})_\mu=(E_{ab},\mathbf{p}_{ab})\equiv(a+b)_\mu$ be the momentum of the diparticle with mass m_{ab}; $m_{ab}{}^2=p_{ab}{}^2=(a+b)^2$. To re-express (2.10) in terms of $d\mathbf{a}d\mathbf{p}_{ab}$, first set

$$\begin{aligned}\delta^4(a+b-R)&=\int dm_{ab}{}^2\delta(m_{ab}{}^2-p_{ab}{}^2)d^4p_{ab}\\&\quad\times\delta^4(a+b-p_{ab})\delta^4(p_{ab}-R)\\&=\int(2m_{ab}dm_{ab})(d\mathbf{p}_{ab}/2E_{ab})\\&\quad\times\delta^4(a+b-p_{ab})\delta^4(p_{ab}-R).\end{aligned} \tag{2.11}$$

Putting this in (2.10) and integrating over **b** and the magnitude of **a**, we get the usual two-body phase-space formula:

$$\frac{d\mathbf{a}d\mathbf{b}}{E_aE_b}\delta^4(a+b-p_{ab})\to\frac{ad\Omega_a}{m_{ab}}, \tag{2.12}$$

where a and $d\Omega_a$ have a restricted meaning; they refer to the $(a+b)$ CMF. The general rule for "diparticle" phase space is

$$\frac{d\mathbf{a}d\mathbf{b}}{E_aE_b}\delta^{(4)}(a+b-R)=a\,d\Omega_a\,dm_{ab}\frac{d\mathbf{p}_{ab}}{E_{ab}}\delta^{(4)}(p_{ab}-R) \tag{2.13}$$

which can be used recursively for a many-body final state. Consider the whole process (2.2d). The resonances X,b, give factors like

$$[X^2-(m_X-\tfrac{1}{2}i\Gamma_X)^2]^{-1} \tag{2.14}$$

TABLE I. General forms for the amplitudes of two-particle decays and three-particle productions of resonances in the tensor formulation. For notation, see Sec. II.

Process	$j^P=0^-,1^+,2^-,\cdots$ $(j+\frac{1}{2})^P=\frac{1}{2}^-,\ \frac{3}{2}^+,\ \frac{5}{2}^-,\ \cdots$	$j^P=0^+,1^-,2^+,\cdots$ $(j+\frac{1}{2})^P=\frac{1}{2}^+,\ \frac{3}{2}^-,\ \frac{5}{2}^+,\ \cdots$
(1) $X\rightarrow 0^-+0^-$	none	$T^j(\mathbf{a})$
(2) $X\rightarrow 0^-+1^-$	$yT^j(\mathbf{a}\cdots\mathbf{a}\mathbf{e}^*)+z(\mathbf{a}\cdot\mathbf{e}^*)T^j(\mathbf{a})$ ($y=0$ for $j=0$)	$T^j(\mathbf{a}\cdots\mathbf{a}\ \mathbf{a}\times\mathbf{e}^*)$ (0^+ case absent)
(3) $X\rightarrow 0^-+\frac{1}{2}^+$	$u^*T^j(\mathbf{a})\mathcal{P}_\sigma$	$u^*(\boldsymbol{\sigma}\cdot\mathbf{a})T^j(\mathbf{a})\mathcal{P}_\sigma$ (or $u^*T^{j+1}(\mathbf{a})\cdot\boldsymbol{\sigma}$)
(4) $X\rightarrow 0^-+\frac{3}{2}^+$	$[yT^j(\mathbf{a}\cdots\mathbf{a}\ \mathbf{a}\times\mathbf{E}^*)$ $+z(\mathbf{E}^*\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})T^j(\mathbf{a})]\mathcal{P}_\sigma$ ($y=0$ for $j=0$)	$[yT^j(\mathbf{a}\cdots\mathbf{a}\mathbf{E}^*)$ $+z(\mathbf{E}^*\cdot\mathbf{a})T^j(\mathbf{a})]\mathcal{P}_\sigma$ ($y=0$ for $j=0$)
(5) $0^-+\frac{1}{2}^+\rightarrow 0^-+X$	$\mathcal{P}_\sigma[T^j(\mathbf{P}\cdots\mathbf{P}\mathbf{Q})+T^j(\mathbf{P})(\boldsymbol{\sigma}\cdot\mathbf{P})]u$	$\mathcal{P}_\sigma[T^j(\mathbf{P})+T^j(\mathbf{P})\boldsymbol{\sigma}\cdot\mathbf{Q}]u$
(6) $0^-+\frac{1}{2}^+\rightarrow\frac{1}{2}^++X$	$T^j(\mathbf{P})+(\boldsymbol{\sigma}\cdot\mathbf{Q})T^j(\mathbf{P})$ $+T^j(\mathbf{P}\cdots\mathbf{P}\boldsymbol{\sigma}\times\mathbf{P})$ $+T^j(\mathbf{P}\cdots\mathbf{P}\mathbf{Q}\boldsymbol{\sigma})$	$T^j(\mathbf{P}\cdots\mathbf{P}\mathbf{Q})+(\boldsymbol{\sigma}\cdot\mathbf{P})T^j(\mathbf{P})$ $+T^j(\mathbf{P}\cdots\mathbf{P}\boldsymbol{\sigma}\times\mathbf{Q})$ $+T^j(\mathbf{P}\cdots\mathbf{P}\boldsymbol{\sigma})$

in the amplitude for (2.2d). These produce δ functions like

$$\delta(X^2-m_X^2)$$

in the absolute square of the amplitude if the widths are small. In this case the counting rate is proportioned to

$$|M_{DP}|^2\delta(X^2-m_X^2)\delta(b^2-m_b^2)\frac{d\mathbf{C}}{E_C}\frac{d\mathbf{a}}{E_a}\frac{d\mathbf{a}'}{E_{a'}}\frac{d\mathbf{b}'}{E_{b'}}\times\delta^{(4)}\text{ (energy–momentum)}. \quad (2.15)$$

Applying (2.13) three times and dropping over-all constants and momentum factors fixed by kinematics, we obtain the physical counting rate:

$$d\sigma\sim|M_{DP}|^2 d\cos\theta_C\, d\Omega_a d\Omega_{a'}. \quad (2.16)$$

Some version of (2.16) is always valid in a cascade which proceeds via two-particle final states. In the case of a three-particle decay, the above approach leads to a "triparticle" phase-space formula

$$\frac{d\mathbf{a}d\mathbf{b}d\mathbf{c}}{E_aE_bE_c}\delta^{(4)}(a+b+c-R)=\frac{dm_{ab}^2dm_{ac}^2}{4m_{abc}^2}d\varphi\, d\cos\theta \times d\varphi' dm_{abc}\frac{d\mathbf{p}_{abc}}{E_{abc}}\delta^{(4)}(p_{abc}-R), \quad (2.17)$$

where φ, θ, φ' are Euler angles for the configuration $(\mathbf{a},\mathbf{b},\mathbf{c})$, viewed in the triparticle rest frame, and $dm_{ab}^2dm_{ac}^2$ is the phase space of the Dalitz plot.

5. Euler Angles and Internal Variables

Let $(\mathbf{N}^1,\mathbf{N}^2,\mathbf{N}^3)$ and $(\mathbf{n}^1,\mathbf{n}^2,\mathbf{n}^3)$ be orthogonal bases of unit vectors for the production particles and decay particles, respectively, of (2.2). For the moment analysis (see next subsection) we define $\mathbf{N}^1$ as the beam direction and $\mathbf{N}^3$ as the normal to the production plane, also $\mathbf{n}^3$ is the direction of $\mathbf{a}$ and $\mathbf{n}^1$ is in the plane of $\mathbf{a}$ and $\mathbf{a}'$. The Euler angles α,β,γ which relate $(\mathbf{n}^1,\mathbf{n}^2,\mathbf{n}^3)$ to $(\mathbf{N}^1,\mathbf{N}^2,\mathbf{N}^3)$ are defined in terms of the scalar products $\mathbf{n}^m\cdot\mathbf{N}^{m'}$ as in I, Sec. VI.

Let $\cos\theta=\hat{a}\cdot\hat{a}'$. The angular phase space of the previous section becomes

$$d\Omega_a d\Omega_{a'}=d\alpha\, d\cos\beta\, d\gamma\, d\cos\theta, \quad (2.18)$$

and counting rate for $A+B\rightarrow C+a+a'+b'$ is

$$d\sigma=\sum|M_D:M_P|^2 d\cos\theta_C\, d\alpha\, d\cos\beta\, d\gamma\, d\cos\theta, \quad (2.19)$$

where the summation is over spins, if any. If the decay of b is not observed, then we have

$$d\sigma=\sum|M_D:M_P|^2 d\cos\theta_C\, d\alpha\, d\cos\beta \quad (2.20)$$

as the rate for $A+B\rightarrow C+a+b$.

We emphasize again that θ_C is measured in the $A+B$ CMF; α,β,γ connect a coordinate system in the $A+B$ CMF and a coordinate system defined by $\mathbf{a}$ as seen in the XRF, and a' as seen in the bRF. Also, $\cos\theta=\hat{a}\cdot\hat{a}'$ relates a direction in the X RF to a direction in the bRF.

The internal variables of the production are θ_C and the incident center-of-mass energy E_0; the decay of (2.2) has only one internal variable θ.

6. Density Matrices and Tensor Moments

Let X have spin J (integral of half-integral) and let $\mathbf{S}$ be the angular-momentum operator on the spin space of X. We use the tensor operators $T^k(\mathbf{S})$, $0\leqq k\leqq 2J$, normalized to

$$\mathrm{Tr}\{T^k{}_m(\mathbf{S})T^{k'}{}_{m'}(\mathbf{S})\}=\delta_{kk'}\mathcal{P}^k{}_{mm'}d_{Jk} \quad (2.21)$$

as discussed in I.

Let $\lambda_{mm'}=(M_P)_m(M^*{}_P)_{m'}$ be called the production density matrix. The tensor moments $\Lambda^k{}_m$ of λ appear as coefficients in the expansion of λ in tensor operators:

$$\lambda = M_P M^*_P = \sum_{k=0}^{2J} d_{Jk}^{-1} T^k(\mathbf{S}) : \Lambda^k \qquad (2.22)$$

whence

$$\Lambda^k = M^*_P : T^k(\mathbf{S}) M_P . \qquad (2.23)$$

Similarly, there is a decay density matrix φ for X:

$$\varphi = M^*_D M_D = \sum_{k=0}^{2J} d_{Jk}^{-1} T^k(\mathbf{S}) : \Phi^k \qquad (2.24)$$

with

$$\Phi^k = M_D : T^k(\mathbf{S}) M^*_D . \qquad (2.25)$$

The full reaction amplitude squared is

$$|M_{DP}|^2 = \mathrm{Tr}(\lambda\varphi) = \sum d_{Jk}^{-1} \Lambda^k : \Phi^k . \qquad (2.26)$$

At some point, the spins of the production and decay particles must be summed over or represented by further momenta. When this is done Λ^k and Φ^k are tensor functions of momenta only. If the production is carried out on a polarized target, the Λ^k will also depend on the target polarization vector.

Λ^k and Φ^k can be further expanded in terms of the basic vectors $\mathbf{N}^1$, $\mathbf{N}^2$, $\mathbf{N}^3$ and $\mathbf{n}^1$, $\mathbf{n}^2$, $\mathbf{n}^3$ of Sec. II.5 above. The desired expansions define functions of the production variables E_0, θ_C and of the decay variable θ:

$$\Lambda^k = \sum_{M'=-k}^{+k} c_{kM'}^{-1/2} T^{j,M'}(\mathbf{N}) \lambda_{kM'}(E_0,\theta_C) , \qquad (2.27\text{a})$$

$$\Phi^k = \sum_{M=-k}^{+k} c_{kM}^{-1/2} T^{j,M}(\mathbf{n})^* \varphi_{kM}(\theta) . \qquad (2.27\text{b})$$

Section VI of I discusses the relation between rectangular bases $(\mathbf{N}^1,\mathbf{N}^2,\mathbf{N}^3)$ and spherical bases $(\mathbf{N}^+,\mathbf{N}^0,\mathbf{N}^-)$ and connects up the tensors $T^{j,M}$ with the familiar spherical harmonies and rotation functions $D^j_{MM'}(\alpha,\beta,\gamma)$.

Now substitute (2.27) into (2.26) and then into (2.16). With the aid of (6.17) and (6.42) of I, we get

$$d\sigma = \sum_{k,M,M'} d_{Jk}^{-1} \lambda_{kM'}(E_0,\theta_C) \varphi_{kM}(\theta) D^k_{MM'}(\alpha,\beta,\gamma) \times d\cos\theta_C \, d\alpha \, d\cos\beta d\gamma \, d\cos\theta . \qquad (2.28)$$

In this equation, the dependences of $d\sigma$ on the three different classes of variables are explicitly separated. These dependences were, of course, well defined before the introduction of the density matrix formalism. The purpose of the formalism is to express $d\sigma$ in terms of the $D^k_{MM'}$ in such a way that the coefficients can be calculated straightforwardly from M_P,M_D. The calculations are based on (2.23), (2.25), (2.27) and Table III of I.

As a step in analyzing (2.28), we defined projected cross sections $d\sigma_{kMM'}$:

$$d\sigma_{kMM'} = \int_{\alpha,\beta,\gamma,\theta_C} D^k_{MM'}{}^* d\sigma . \qquad (2.29)$$

Then

$$d\sigma_{kMM'} = R_{kM'} \varphi_{kM}(\theta) d\cos\theta ; \qquad (2.30)$$

$$R_{kM'} = 8\pi^2 d_{Jk}^{-1} \int \lambda_{kM'}(E_0,\theta_C) d\cos\theta_C . \qquad (2.31)$$

The $\varphi_{kM}(\theta)$ depend on the assumed spin J of X, on the coupling constants for the decays, and are otherwise completely defined. These $d\sigma_{kMM'}$ contain all and only that information about the reaction which depends on the decay mechanism but not on the production mechanism. The $k=0$ term is merely another expression for the decay distribution itself:

$$d\sigma_{000} \sim |M_D|^2 d\cos\theta . \qquad (2.32)$$

We recall the procedure for obtaining a quantity like $d\sigma_{kMM'}$ from a set of data: Each event i defines a set of values $(\theta_C)_i$, θ_i, α_i, β_i, etc. Separate the data into bins, each bin defined by values of θ in specified intervals. Then for each bin,

$$(d\sigma_{kMM'}/d\cos\theta)_{\text{expt}} = \sum_i D^k_{MM'}{}^*(\alpha_i,\beta_i,\gamma_i) , \qquad (2.33)$$

the sum being over all events in the given bin. The $D^k_{MM'}$'s are complex, as they have factors $e^{iM\alpha}$,$e^{iM'\gamma}$. As a practical matter one will want to work with linear combinations of them proportional to sines and cosines of α,γ, and with the corresponding combinations of the $\sigma_{kMM'}$'s which are real.

The projected cross sections with large k will have the least statistical significance for a given amount of data because of the oscillating character of the D's. In a clean experiment without background, the cases $k=0$, $k=1$, plus some other one-angle correlations as discussed below may provide fully adequate information and corraboration for the spin, parity, decay parameters, etc., of X. But these tests may fail if the data is contaminated with background. Now, suppose that much of the background consists of configurations of lower spin than the spin J of X. Then contamination will be absent, or greatly reduced, in the partial cross sections with the largest k values, $k=2J$, $k=2J-1$. This is a way of defeating the background problem, but at the cost of requiring more data for statistical significance.

III. FERMION RESONANCES WITH DECAY PRODUCTS OF SPIN 0 AND SPIN $\frac{1}{2}$

1. Preliminaries

We study angular correlations in the processes (2.2), especially their dependence on the spin $(j+\frac{1}{2})$ and the parity of the fermion resonance X. Suppose that A, a, and a' are 0^- particles (pions or kaons) and that B, b, and b' are $\frac{1}{2}^+$ particles (hyperons). Processes (2.2a), (2.2b) will be considered parity conserving and (2.2c) parity violating [as in $K+p\rightarrow\pi+Y^*$ (1385), $Y^*\rightarrow\pi+\Lambda$, $\Lambda\rightarrow p+\pi$]. The derived correlations which do not mention $\mathbf{a}'$ or $\mathbf{b}'$ are, of course, independent of any assumptions about b-decay.

A number of the topics considered in this section have a wider applicability and the results can be taken over in Secs. IV and V.

From now on, $\mathbf{A}$, $\mathbf{a}$, $\mathbf{a}'$, etc. denote unit vectors. The unit normal to the production plane is $\mathbf{Q}$. The amplitude for $b \to b'+a'$ is taken as

$$M_b = \langle u^*_{b'} | y + z\boldsymbol{\sigma}\cdot\mathbf{a}' | u_b \rangle \tag{3.1}$$

whence

$$\sum_{b'\,\text{spin}} |M_b|^2 = \langle u^*_b | 1+\rho\boldsymbol{\sigma}\cdot\mathbf{a}' | u_b\rangle (|y|^2+|z|^2) \tag{3.2}$$

with $\rho = 2\,\mathrm{Re}(y^*z)/(|y|^2+|z|^2)$. Hereafter, we shall usually omit explicit mention of the spinors for B, b, b'.

The form of the $X\to a+b$ amplitude depends (see Table I) on which parity sequence is being considered. We write this amplitude as

$$T^j(\mathbf{a})\Sigma_a\mathcal{P}_\sigma \tag{3.3}$$

for both cases, with the understanding that

$$\Sigma_a = 1 \quad \text{for} \quad \tfrac{1}{2}^+, \tfrac{3}{2}^-, \tfrac{5}{2}^+, \cdots, \tag{3.4a}$$

$$\Sigma_a = (\boldsymbol{\sigma}\cdot\mathbf{a}) \quad \text{for} \quad \tfrac{1}{2}^-, \tfrac{3}{2}^+, \tfrac{5}{2}^-, \cdots. \tag{3.4b}$$

In both cases,

$$\Sigma_a^2 = 1. \tag{3.5}$$

We also define a new unit vector $\boldsymbol{\alpha}$ related to $\mathbf{a}$ and $\mathbf{a}'$ by

$$\Sigma_a(1+\rho\boldsymbol{\sigma}\cdot\mathbf{a}')\Sigma_a = 1+\rho\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}. \tag{3.6}$$

Then

$$\boldsymbol{\alpha} = \mathbf{a}' \quad \text{if} \quad \Sigma_a = 1 \tag{3.7a}$$

$$\boldsymbol{\alpha} = -\mathbf{a}' + 2(\mathbf{a}'\cdot\mathbf{a})\mathbf{a} \quad \text{if} \quad \Sigma_a = \boldsymbol{\sigma}\cdot\mathbf{a}. \tag{3.7b}$$

Equation (3.7b) defines the familiar "magic direction" $\mathbf{a}_m$ obtained by rotating $\mathbf{a}'$ by 180° around an axis along $\mathbf{a}$.

In discussing counting rates, we shall not watch the over-all multiplicative constants too carefully. In view of the foregoing, the counting rate for $A+B\to C+a+a'+b'$, summed over the spins of B and b' is

$$d\sigma = \sum |M_{DP}|^2 d\cos\theta_C \, d\Omega_a \, d\Omega_{a'}, \tag{3.8}$$

where

$$\sum |M_{DP}|^2 = \mathrm{Tr}\{M_P^* : T^j(\mathbf{a})(1+\rho\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})T^j(\mathbf{a}) : M_P\}\,. \tag{3.9}$$

Integration of (3.9) over $d\Omega_{a'}$ removes the $\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}$ term and gives the counting rate for $A+B\to C+a+b$, summed over B and b spins. Equation (3.9) is our starting point for the detailed discussion of angular correlations.

2. Maximum Complexity Theorems

Before entering upon the details, we comment on some general features of correlations in a reaction like (2.1) with any number of particles and arbitrary spins. Let $\mathbf{V}$, $\mathbf{v}$ be directions defined in the production and decay configurations, respectively. Let coordinate frames be set up in these two configurations such that $\mathbf{V}$ and $\mathbf{v}$ are the polar axes and introduce Euler angles as in Sec. VI of I. The polar Euler angle will be defined by $\cos\beta = \mathbf{V}\cdot\mathbf{v}$. If the counting rate is integrated over α,γ and all other variables except β and summed over spins, one obtains the counting rate $Z(\beta)d\cos\beta$ as a function of the correlation between $\mathbf{V}$, $\mathbf{v}$ alone.[4]

M_P will be made up of the spin wave functions of $A, B, \cdots$, and of polar vectors $\mathbf{P}$ (like $\mathbf{A},\mathbf{B},\mathbf{C}$) and pseudovectors $\mathbf{Q}$ (like $\mathbf{A}\times\mathbf{C}$). In the same way M_D contains polar vectors $\mathbf{p}$ and pseudovectors $\mathbf{q}$. Thus, with regard to parity properties, there are four types of correlations and the corresponding types of angles can be labeled β_{Pp}, β_{Qp}, β_{Pq}, β_{Qq}.

The correlations have the following properties (the first two properties are unrelated to parity conservation):

(1) *Each $Z(\beta)$ is a polynomial in $\cos\beta$.* For we see by (6.15) of I that each factor of $\sin\beta$ is accompanied by one factor of cosine (or sine) of α or γ, while each $\cos\beta$ has either zero or two factors. Then the integral over $d\alpha d\gamma$ leaves only even powers of $\sin\beta$ which can be converted into powers of $\cos\beta$.

(2) *The highest power of $\cos\beta$ (the maximum complexity) in $Z(\beta)$ is $2j$ if X has spin j (and $2j+1$ if X has spin $j+\frac{1}{2}$).* For the density matrix $M_PM_P^*$ when summed over the spins of the production particles cannot have more than $2j$ (or $2j+1$) factors of momenta in it.

(3) *If parity is conserved in production, then $Z(\beta_P)$ and $Z(\beta_{Pq})$ are even functions of $\cos\beta_{Pp}$, $\cos\beta_{Pq}$.* For under all $\mathbf{P}\to-\mathbf{P}$, $\cos\beta\to-\cos\beta$ for these cases and $M_P\to+M_P$ or $-M_P$. Thus $Z(\beta)$, obtained from $|M_P{:}M_D|^2$, is unchanged when $\cos\beta\to-\cos\beta$ for these cases.

(4) *If parity is conserved in decay, $Z(\beta_{Pp})$ and $Z(\beta_{Qp})$ are even functions of $\cos\beta_{Pp}$, $\cos\beta_{Qp}$.* Same reasoning as in (3).

These theorems are not quite the same as the Johnson-Teller type of maximum complexity theorem derived for nuclear physics.[5] The latter states that the complexity of a distribution does not exceed $2L$, where L is the maximum orbital angular momentum of the incident state. This principle does not require a special proof in our tensor formalism as it is already imbedded in the notation.

3. Forward and Backward Production (Adair Analysis)

We return to (3.8) and (3.9). Suppose that only events with $|\cos\theta_C|\approx 1$ are counted, that is, events in

[4] Examples of this procedure are given in C. Zemach, Phys. Rev. **133**, B1201 (1964). There are minor differences between the definitions of α, γ in this reference and in the present paper.

[5] See, for example, E. Eisner and R. G. Sachs, Phys. Rev. **72**, 680 (1947) and C. N. Yang, Phys. Rev. **74**, 764 (1948).

which **A**, **B**, and **C** are aligned. We refer to this as forward and backward production or, simply, "forward" production even though events in which **C** goes directly backward ($\cos\theta_C \approx -1$) are also included.

Then M_P is characterized by a single vector **A**:

$$M_P = \mathcal{P}_\sigma T^j(\mathbf{A})\Sigma_A \qquad (3.10)$$

with Σ_A defined like Σ_a in (3.4), and obeying $\Sigma_A{}^2=1$. Recalling from I that

$$T^j(\mathbf{a}) : P_\sigma T^j(\mathbf{A}) = (2j+1)^{-1} \times[(j+1)P_j(x) - i\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{A}P_j'(x)], \qquad (3.11)$$

[where $x=\mathbf{a}\cdot\mathbf{A}$, $P_j(x)$ is the Legendre polynomial], we have, by (3.9)

$$\sum|M_{DP}|^2 = (2j+1)^{-2}\{(j+1)^2P_j(x)^2 + (1-x^2)[P_j'(x)]^2\}. \qquad (3.12)$$

The result is independent of $\boldsymbol{\alpha}$ and of the parity of X. The Legendre formulas

$$xP_j' = P_{j-1}' + jP_j = P_{j+1}' - (j+1)P_j, \qquad (3.13a)$$

$$xP_{j+1}' = P_j' + (j+1)P_{j+1}, \quad xP_{j-1}' = P_j' - jP_{j-1}, \qquad (3.13b)$$

$$(1-x^2)P_j' = (j+1)(xP_j - P_{j+1}) \qquad (3.13c)$$

often help to simplify expressions like (3.12). We have

$$(1-x^2)P_j'P_j' = (j+1)xP_jP_j' - (j+1)P_{j+1}P_j' = (j+1)P_j[P_{j+1}' - (j+1)P_j] - (j+1)P_{j+1}P_j' \qquad (3.14)$$

whence, dropping over-all factors,

$$\Sigma|M_{DP}|^2 = P_jP_{j+1}' - P_{j+1}P_j'. \qquad (3.15)$$

Equation (3.15) is the Adair distribution,[6] proportional to 1, $1+3x^2$, $1-2x^2+5x^4$, for spin $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc., but expressed in compact form.

Now consider the more general amplitude

$$M_{DP} = (y+z\boldsymbol{\sigma}\cdot\mathbf{a}')\Sigma_a\{\mathcal{A}T^j(\mathbf{a}):\mathcal{P}_\sigma T^j(\mathbf{A}) + \mathcal{B}T^k(\mathbf{a}):\mathcal{P}_\sigma T^k(\mathbf{A}) + \mathcal{C}(\boldsymbol{\sigma}\cdot\mathbf{a})T^l(\mathbf{a}):\mathcal{P}_\sigma T^l(\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{A})\}\Sigma_A. \qquad (3.16)$$

We write (3.16) in order to consider the effect of interference between a dominant resonant amplitude (the j term) and other, possibly weaker, nonresonant amplitudes. The k term is in the same parity sequence as the j term, while the l term is in the opposite parity sequence. As is known, useful information can be obtained by looking at interference contributions as a function of resonant-invariant mass, taking account of the special behavior expected of the coefficient functions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$.

The calculation of $|M_{DP}|^2$ proceeds as before. In calculating the $j-l$ interference, one encounters

$$(\boldsymbol{\sigma}\cdot\mathbf{a})[(j+1)P_j - i\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{A}P_j'] = (\boldsymbol{\sigma}\cdot\mathbf{a})[(j+1)P_j + xP_j'] - (\boldsymbol{\sigma}\cdot\mathbf{A})P_j' = (\boldsymbol{\sigma}\cdot\mathbf{a})P_{j+1}' - (\boldsymbol{\sigma}\cdot\mathbf{A})P_j' \qquad (3.17)$$

and

$$(\boldsymbol{\sigma}\cdot\mathbf{A})[(l+1)P_l + i\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{A}P_l] = (\boldsymbol{\sigma}\cdot\mathbf{A})P_{l+1}' - (\boldsymbol{\sigma}\cdot\mathbf{a})P_l' \qquad (3.18)$$

and finally the trace

$$\tfrac{1}{2}\mathrm{Tr}[(\boldsymbol{\sigma}\cdot\mathbf{A})P_{l+1}' - (\boldsymbol{\sigma}\cdot\mathbf{a})P_l'][(\boldsymbol{\sigma}\cdot\mathbf{a})P_{j+1}' - (\boldsymbol{\sigma}\cdot\mathbf{A})P_j'] = P_{l+1}'(xP_{j+1}' - P_j') - P_l'(P_{j+1}' - xP_j') = (j+1)[P_{l+1}'P_{j+1} - P_l'P_j]. \qquad (3.19)$$

The result following from (3.16), including the j term and interferences with it is

$$\sum|M_{DP}|^2 = (j+1)[|\mathcal{A}|^2(P_jP_{j+1}' - P_{j+1}P_j')/(2j+1) + 2\,\mathrm{Re}(\mathcal{A}\mathcal{B}^*)(P_jP_{k+1}' - P_{j+1}P_k')/(2k+1) + 2\,\mathrm{Re}(\mathcal{A}\mathcal{C}^*)(P_{j+1}P_{l+1}' - P_jP_l')/(2l+1)] + 2\rho(\boldsymbol{\alpha}\cdot\mathbf{a}\times\mathbf{A})\{\mathrm{Im}(\mathcal{A}\mathcal{B}^*)[(j+1)P_jP_k' + (k+1)P_kP_j']/(2k+1) + \mathrm{Im}(\mathcal{A}\mathcal{C}^*) \times(P_j'P_l' - P_{l+1}'P_{j+1}')/(2l+1)\}. \qquad (3.20)$$

4. Peripheral Production

Figures 1(a) and 1(b) illustrate peripheral mechanisms for producing X. Suppose that G has spin zero in 1(a) and spin $\frac{1}{2}$ in 1(b) and that one of these processes dominates the reaction.

In the first case, one may pretend that X is produced in the simplified reaction

$$B+G \rightarrow X \rightarrow a+b. \qquad (3.21)$$

The production is characterized by a single vector **B** in the XRF. Hence formulas (3.15) and (3.20) above are applicable to this case, except that the definition of x is $x=\mathbf{B}\cdot\mathbf{a}$ with **B** viewed in the XRF.

In the second case, one may use the model

$$A+G \rightarrow X \rightarrow a+b, \qquad (3.22)$$

assume that the G beam is unpolarized and again arrive at the Adair analysis formulas.[7]

To verify this, let w_α, $\alpha=1,2,3,4$ and u_α, $\alpha=1,2$ be the Dirac spinor and rest-frame spinor, respectively,

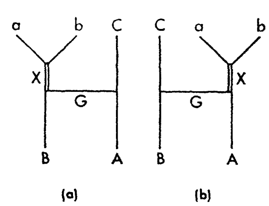

FIG. 1. Peripheral mechanisms for the production of a resonance X.

[6] R. K. Adair, Phys. Rev. **100**, 1540 (1955).

[7] This result has already been obtained by P. Schlein (private communication).

for B. Then

$$M_P=\mathcal{P}_\sigma T^j(\mathbf{A})F_\alpha\,,\ \alpha=1,2,3,4\,, \tag{3.23}$$

where $\mathbf{A}$ is in the XRF and F_α, expressed covariantly, is (we write only the spin-dependent factors)

$$F_\alpha\sim[(\gamma G+m_G)\gamma_5]_{\alpha\beta}w_\beta \quad\text{or}\quad F_\alpha\sim[\gamma G+m_G]_{\alpha\beta}w_\beta\,, \tag{3.24}$$

depending on the parity of G. Since M_P multiplies M_D which, in the XRF, has only upper spinor components, we need only the upper components of F_α. One finds,

$$F_\alpha=[f_0+f_1i\boldsymbol{\sigma}\cdot\mathbf{B}\times\mathbf{G}]_{\alpha\beta}u_\beta\,;\quad \alpha,\beta=1,2\,, \tag{3.25}$$

where f_0, f_1 are real functions of the energies. Hence the density matrix of G, given by

$$\sum_{B\text{ spin}} F_\alpha F_{\alpha'}{}^*=[f_0{}^2+f_1{}^2(\mathbf{B}\times\mathbf{G})^2]\delta_{\alpha\alpha'}\,, \tag{3.26}$$

is isotropic, justifying the model of G as unpolarized.

When G has spin greater than $\frac{1}{2}$, the model is still valid provided the polarization of G, determined by the (GAC) or (GBC) interaction and not necessarily isotropic is taken into account. The role of peripheral collisions in angular correlations is a subject in itself, which has received attention from a number of authors, and we shall not pursue it further here.

5. Correlations at Low Production Energies (Table II)

When X has spin $j+\frac{1}{2}$ with $j\geqq 1$, the production amplitude has two terms, corresponding to orbital angular momentum j and $j+1$. Suppose the incident energy so low that the De Broglie wavelength of the relative momentum (in the CMF) is large compared to the range of the interaction producing the X. Then the centrifugal barrier is important and the lower orbital term may dominate, simplifying the prediction of angular correlations.

To make this quantitative, we examine correlations between the production normal $\mathbf{Q}$, and the directions $\mathbf{a}$, $\mathbf{a}'$ of the decay system. We set up orthonormal vectors $\mathbf{N}^1$, $\mathbf{N}^2$, $\mathbf{N}^3$ with $\mathbf{Q}=\mathbf{N}^3$.

Any $\mathbf{P}$ in the plane of production can be written $\mathbf{P}=c_1\mathbf{N}^1+c_2\mathbf{N}^2$. Orthonormal vectors for the decay are defined by

$$\mathbf{a}=\mathbf{n}^3\,, \tag{3.27a}$$

$$\mathbf{a}'=\mathbf{n}^1\sin\theta+\mathbf{n}^3\cos\theta\,, \tag{3.27b}$$

$$\boldsymbol{\alpha}=\pm\mathbf{n}^1\sin\theta+\mathbf{n}^3\cos\theta\,, \tag{3.27c}$$

$$\mathbf{a}\times\boldsymbol{\alpha}=\pm\mathbf{a}\times\mathbf{a}'=\pm\mathbf{n}^2\sin\theta\,. \tag{3.27d}$$

The procedure is to calculate the reaction rate as a function of the Euler angles and average over γ. This leaves a function of angles β,α,θ, related to the physical directions by

$$\mathbf{Q}\cdot\mathbf{a}=\cos\beta\,, \tag{3.28a}$$

$$\mathbf{Q}\cdot\mathbf{a}\times\mathbf{a}'=\sin\beta\sin\alpha\sin\theta\,, \tag{3.28b}$$

$$\mathbf{Q}\cdot\mathbf{a}\times(\mathbf{a}\times\mathbf{a}')=-\sin\beta\cos\alpha\sin\theta\,. \tag{3.28c}$$

Let $Z(\beta,\mathbf{a}')d\cos\beta\, d\alpha\, d\cos\theta$ denote the counting rate as a function of the indicated variables. The distribution in $\cos\beta$ is $Z(\beta)$;

$$Z(\beta)=\int Z(\beta,\mathbf{a}')d\alpha\, d\cos\theta\,. \tag{3.29}$$

Both $Z(\beta,\mathbf{a}')$ and $Z(\beta)$ depend on θ_C also. We can now do two things: Firstly, we can define correlations Z_s by taking only sideward production, that is, only data with θ_C not too far from $\frac{1}{2}\pi$. Secondly, we can define correlations Z_f by taking only values of θ_C not too far from 0 or π, as has already been done above. From Table I, line 5, we see that for the sequence $\frac{3}{2}^-$, $\frac{5}{2}^+$, $\cdots$, both the forward-dominant and the low-energy-dominant term are the same, namely

$$M_P=\mathcal{P}_\sigma T^j(\mathbf{P})\,. \tag{3.30}$$

Thus, for low-energy production, $Z_f(\beta)$ and $Z_s(\beta)$ may be very similar (or identical). But for $\frac{3}{2}^+$, $\frac{5}{2}^-$, $\cdots$ the forward-dominant term is like (3.30), while the low-energy-dominant term is

$$M_P=\mathcal{P}_\sigma T^j(\mathbf{P}\cdots\mathbf{PQ})\,. \tag{3.31}$$

Thus Z_s and Z_f come from different terms and may be quite different. The situation provides a parity test as we shall show.

To obtain the forward correlations in $\cos\beta$, we substitute $x=-\cos\gamma\sin\beta$ in the Adair distributions 1, $1+3x^2$, $1-2x^2+5x^4$, etc., average over γ to get

$$Z_f(\beta)=1,\ 5-3\cos^2\beta,\ 1-22\cos^2\beta/15+\cos^4\beta,\cdots \tag{3.32}$$

for spins $\frac{1}{2}^\pm$, $\frac{3}{2}^\pm$, $\frac{5}{2}^\pm$, etc. The second two functions of (3.32) are "hill" distributions rather than "valleys," i.e., larger for $\cos\beta=0$ than for $\cos\beta=\pm 1$.

We shall not worry about contamination by nonresonant processes in this section. Then the forward correlations are independent of $\mathbf{a}'$ as we have seen.

The averaging over γ involves, in general, calculations of the following type:

$$\begin{aligned}\langle(\mathbf{a}\cdot\mathbf{P})(\mathbf{a}\cdot\mathbf{P}^*)\rangle_\gamma &=\langle|c_1|^2(\mathbf{a}\cdot\mathbf{N}^1)^2+2\,\mathrm{Re}c_1c_2{}^*\\ &\quad\times(\mathbf{a}\cdot\mathbf{N}^1)(\mathbf{a}\cdot\mathbf{N}^2)+|c_2|^2(\mathbf{a}\cdot\mathbf{N}^2)^2\rangle_\gamma\\ &=\langle|c_1|^2\cos^2\gamma\sin^2\beta-2\,\mathrm{Re}c_1c_2{}^*\\ &\quad\times\cos\gamma\sin\gamma\sin^2\beta+|c_2|^2\sin^2\gamma\sin\beta\rangle_\gamma\\ &=(|c_1|^2+|c_2|^2)\tfrac{1}{2}\sin^2\beta\,,\end{aligned} \tag{3.33a}$$

$$\begin{aligned}\langle(\mathbf{a}\times\mathbf{P})\cdot(\mathbf{a}\times\mathbf{P}^*)\rangle_\gamma &=(|c_1|^2+|c_2|^2)\tfrac{1}{2}(1+\cos^2\beta)\end{aligned} \tag{3.33b}$$

$$\langle(\mathbf{a}\cdot\mathbf{P})(\mathbf{a}\times\mathbf{a}'\cdot\mathbf{P}^*)\rangle_\gamma$$
$$=-\tfrac{1}{2}(|c_1|^2+|c_2|^2)\cos\beta\sin\beta\sin\alpha\sin\theta$$
$$+i\,\mathrm{Im}c_2c_1^*\sin\beta\cos\alpha\sin\theta$$
$$=-\tfrac{1}{2}(|c_1|^2+|c_2|^2)[\cos\beta(\mathbf{Q}\cdot\mathbf{a}\times\mathbf{a}')$$
$$+i\bar{c}\mathbf{Q}\cdot\mathbf{a}\times(\mathbf{a}\times\mathbf{a}')], \quad (3.33c)$$

where

$$\bar{c}=\frac{2\,\mathrm{Im}c_2c_1^*}{|c_1|^2+|c_2|^2};\quad |\bar{c}|\leqq 1. \quad (3.34)$$

We are now ready to consider the correlations $Z(\beta,\mathbf{a}')$ for the lower spin cases. Write M_D' for the amplitude of $X\to a+b$ so that $M_D=(y+z\boldsymbol{\sigma}\cdot\mathbf{a}')M_D'$. Over-all factors like $(|c_1|^2+|c_2|^2)$ will often be dropped without special comment.

Case $\frac{1}{2}^+$: We have $M_P=1$, $M_D'=\boldsymbol{\sigma}\cdot\mathbf{a}$, and $Z(\beta,\mathbf{a}')$ is isotropic. If we do not restrict the production to S wave we would have, more generally, $M_P=c_1+c_2\boldsymbol{\sigma}\cdot\mathbf{Q}$, leading to

$$Z(\beta,\mathbf{a}')=1+\rho\bar{c}\mathbf{Q}\cdot\boldsymbol{\alpha}=1+\rho\bar{c}[-\mathbf{Q}\cdot\mathbf{a}'+(\mathbf{a}\cdot\mathbf{a}')\mathbf{Q}\cdot\mathbf{a}]. \quad (3.35)$$

Case $\frac{1}{2}^-$: $M_P=\boldsymbol{\sigma}\cdot\mathbf{P}$,

$$\sum|M_{DP}|^2=\tfrac{1}{2}\,\mathrm{Tr}(\boldsymbol{\sigma}\cdot\mathbf{P}^*)(1+\rho\boldsymbol{\sigma}\cdot\mathbf{a}')(\boldsymbol{\sigma}\cdot\mathbf{P}).$$

Then

$$Z(\beta,\mathbf{a}')=1+\rho\bar{c}\mathbf{Q}\cdot\mathbf{a}'. \quad (3.36)$$

Again, $Z(\beta)$ is flat, but the $\mathbf{Q}\cdot\mathbf{a}'$ term distinguishes this case from spin $\frac{1}{2}^+$.

Case $\frac{3}{2}^-$: Take $M_P\equiv\mathbf{P}+\frac{1}{2}i\boldsymbol{\sigma}\times\mathbf{P}$ and $M_D'\equiv M_D'(\mathrm{raw})=(\boldsymbol{\sigma}\cdot\mathbf{a})\mathbf{a}$ so that

$$M_D':M_P=(\boldsymbol{\sigma}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{P}-\tfrac{1}{2}i\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{P}). \quad (3.37)$$

The rate depends on

$$\sum|M_{DP}|^2=\tfrac{1}{2}\,\mathrm{Tr}(\mathbf{a}\cdot\mathbf{P}^*+\tfrac{1}{2}i\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{P}^*)$$
$$\times(1+\rho\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})(\mathbf{a}\cdot\mathbf{P}-\tfrac{1}{2}i\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{P})$$
$$=[(\mathbf{a}\cdot\mathbf{P}^*)(\mathbf{a}\cdot\mathbf{P})+\tfrac{1}{4}(\mathbf{a}\times\mathbf{P})(\mathbf{a}\times\mathbf{P}^*)]$$
$$+\rho\,\mathrm{Im}[\tfrac{1}{2}(\mathbf{a}\cdot\mathbf{P}^*)(\mathbf{a}\times\mathbf{a}'\cdot\mathbf{P})$$
$$+\tfrac{1}{4}(\mathbf{a}\times\mathbf{P}^*)\times(\mathbf{a}\times\mathbf{P})\cdot\boldsymbol{\alpha}]. \quad (3.38)$$

Then, using (3.33), we have

$$Z(\beta,\mathbf{a}')=(5-3\cos^2\beta)+\rho\bar{c}[4\mathbf{Q}\cdot\mathbf{a}'-2\cos\beta\,\mathbf{a}\cdot\mathbf{a}']. \quad (3.39)$$

In this case and the two previous ones, there is no distinction between Z_s and Z_f.

Case $\frac{3}{2}^+$: To calculate the sideward correlation at low energy, we use $M_P=\mathbf{Q}+\frac{1}{2}i\boldsymbol{\sigma}\times\mathbf{Q}$ and proceed as before. No averaging over γ is needed and the result,

$$Z_s(\beta,\mathbf{a}')\equiv Z_s(\beta)=1+3\cos^2\beta, \quad (3.40)$$

is a valley with no dependence on $\mathbf{a}'$.

Case $\frac{5}{2}^-$: We can use the raw form of M_P, $M_P=\mathbf{PQ}$ if we also use the constrained form of M_D';

$$M_D'=T^2(\mathbf{a})\mathcal{P}_\sigma$$
$$=\mathbf{aa}-\tfrac{1}{3}\mathbf{I}-\tfrac{1}{3}i(\boldsymbol{\sigma}\times\mathbf{a})\mathbf{a}-\tfrac{1}{3}i\mathbf{a}(\boldsymbol{\sigma}\times\mathbf{a}). \quad (3.41)$$

The work is again straightforward with the result

$$Z_s(\beta,\mathbf{a}')=1+6\cos^2\beta-4\cos^4\beta+\rho\bar{c}[6\cos^2\beta(\mathbf{Q}\cdot\mathbf{a}')$$
$$-2(\cos^3\beta-\cos\beta)\mathbf{a}\cdot\mathbf{a}'], \quad (3.42)$$

again a valley for $Z_s(\beta)$.

Case $\frac{5}{2}^+$: The raw M_P is now $\mathbf{PP}'$ with $\mathbf{P}=c_1\mathbf{N}^1+c_2\mathbf{N}^2$ and $\mathbf{P}'=c_1'\mathbf{N}^1+c_2'\mathbf{N}^2$ and

$$M_D':M_P\equiv(\boldsymbol{\sigma}\cdot\mathbf{a})\{(\mathbf{a}\cdot\mathbf{P})(\mathbf{a}\cdot\mathbf{P}')$$
$$-\tfrac{1}{3}(\mathbf{P}\cdot\mathbf{P}')-\tfrac{1}{3}i[(\mathbf{a}\cdot\mathbf{P})(\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{P}')$$
$$+(\mathbf{a}\cdot\mathbf{P}')(\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{P})]\}. \quad (3.43)$$

The $\mathbf{a}'$ term of $Z_s(\beta,\mathbf{a}')$ is not uniquely determined, but is found to be a mixture of $\sin^2\beta\mathbf{Q}\cdot\mathbf{a}'$ and $\sin^2\beta\cos\beta\mathbf{a}\cdot\mathbf{a}'$. To obtain the character of $Z(\beta)$ we can write (3.43) as

$$M_D':M_P=c_1c_1'[(\mathbf{a}\cdot\mathbf{N}^1)^2-\tfrac{1}{3}]+c_2c_2'[(\mathbf{a}\cdot\mathbf{N}^2)^2-\tfrac{1}{3}]$$
$$+(c_1c_2'+c_2c_1')(\mathbf{a}\cdot\mathbf{N}^1)(\mathbf{a}\cdot\mathbf{N}^2)$$
$$+\boldsymbol{\sigma}\cdot(\text{a vector}). \quad (3.44)$$

When computing $Z_s(\beta)$, one finds that the three lines of (3.44) do not interfere and the square of each averages to a hill distribution, hence $Z_s(\beta)$ is a hill.

The conclusions on shape are listed in Table II. As a practical application consider the reaction

$$K^-+p\to Y^*(1385)+\pi;\quad Y^*\to\Lambda+\pi \quad (3.45)$$

as described by Ely *et al.*[8] and Shafer *et al.*[9] The simplest mechanisms for the Y^* production are peripheral interactions with K^* or Λ exchange. Thus the range of the interaction is probably not greater than a K^* Compton wavelength. A production with momentum of the order of a K^* mass corresponds to a center-of-mass energy of about 2.3 BeV and an incident lab momentum of about 2.1 BeV/c. Thus the Ely and Shafer reactions, at 1.15 and 1.22 BeV/c are low-energy productions in our sense. In both cases, the $Z_s(\beta)$ distribution, with β the angle between the production normal and the Λ direction, is a valley. This is consistent with the $\frac{3}{2}^+$ assignment, inferred by Shafer by other means, but inconsistent with a $\frac{3}{2}^-$ assignment, among others. The valley "washes out" if data with

TABLE II. Shapes of correlations in $\cos\beta=\mathbf{Q}\cdot\mathbf{a}$ in low-energy production of fermion resonances with decay products of spin 0^- and $\frac{1}{2}^+$. β is the angle between the production normal and the decay direction in the rest frame of the resonance. See Sec. III.5.

Spin and parity	Sideward $Z_s(\beta)$	Forward and back $Z_f(\beta)$
$\frac{1}{2}^\pm$	flat	flat
$\frac{3}{2}^+$, $\frac{5}{2}^-$	valley	hill
$\frac{3}{2}^-$, $\frac{5}{2}^+$	hill	hill

[8] R. P. Ely, S. Y. Fung, G. Gidal, Y. L. Pan, W. M. Powell, and H. S. White, Phys. Rev. Letters **7**, 461 (1961).
[9] J. B. Shafer, J. J. Murray, D. O. Huwe, Phys. Rev. Letters **10**, 179 (1963).

forward angles θ_C are mixed in[10] as it should, since the forward distribution is a hill. Ely's valley is deeper than Schafer's, which is reasonable since his production, at the lower energy, should have a purer sample of the lower orbital angular momentum. Moreover, the completely pure distribution (3.40) is a still deeper valley.

6. Moment Analysis

The analysis of the tensor moments of the decay distribution given here is similar in spirit, though not in detail, to the work of Refs. 1 and 2.

We first give a simple proof of the maximum complexity theorem for the polarization of the decay fermion, which was used in Ref. 9. The polarization in the direction $\mathbf{Q}$ is given by

$$P_Q=\int (\mathbf{a}'\cdot\mathbf{Q})\sum |M_{DP}|^2 d\Omega_{a'}. \qquad (3.46)$$

We ask how P_Q depends on $\cos\beta=\mathbf{a}\cdot\mathbf{Q}$. If we refer to (3.9) and also (3.7), (3.4), we see that P_Q is a polynomial in $\cos^2\beta$ with maximum term $(\cos\beta)^{2j}$ for X in the $\frac{1}{2}^+$ sequence. But the maximum term is $(\cos\beta)^{2j+2}$ for the $\frac{1}{2}^-$ sequence as $\mathbf{a}_m$ has two factors of $\mathbf{a}$ which can combine with $\mathbf{Q}$'s in M_P, M_P^*. On the other hand, if we look at P_{Q_m} with $\mathbf{Q}_m=-\mathbf{Q}+2(\mathbf{Q}\cdot\mathbf{a})\mathbf{a}$, this counteracts the effect. P_{Q_m} has maximum complexity $(\cos\beta)^{2j+2}$ for the $\frac{1}{2}^+$ sequence and $(\cos\beta)^{2j}$ for the $\frac{1}{2}^-$ sequence. As may be noted from the results of the previous section, the complexity may not reach its maximum value if only the lower orbital term of the production amplitude is observable.

We now calculate the projected cross sections $d\sigma_{kMM'}$ using (2.25), (2.27b), (2.30). We want Φ^k for the whole decay $X\to a+a'+b'$, summed over b' spin:

$$\Phi^k=\tfrac{1}{2}\,\mathrm{Tr}[(1+\rho\boldsymbol{\sigma}\cdot\boldsymbol{\alpha})T^{j+1/2}(\mathbf{a})^*:T^k(\mathbf{S})T^{j+1/2}(\mathbf{a})]. \qquad (3.47)$$

We refer to lines 7,8, Table III of I. This gives, dropping normalizations,

$$\Phi^k=T^k(\mathbf{a}), \quad (k \text{ even}) \qquad (3.48)$$

$$\Phi^k=(j+1)T^k(\mathbf{a}\cdots\mathbf{a}\boldsymbol{\alpha})+\tfrac{1}{2}(k-2j-1)\mathbf{a}\cdot\boldsymbol{\alpha}T^k(\mathbf{a}), \quad (k \text{ odd}). \qquad (3.49)$$

These are to be converted to the form (2.27b) using (3.27) and $\mathbf{n}^1=(\mathbf{n}^- - \mathbf{n}^+)/\sqrt{2}$. Then

$$d\sigma_{k0M'}=R_{kM'}\,d\cos\theta, \quad (k \text{ even}) \qquad (3.50)$$

$$d\sigma_{k0M'}=R_{kM'}\cos\theta\,d\cos\theta, \quad (k \text{ odd}) \qquad (3.51)$$

$$d\sigma_{k-1M'}=-d\sigma_{k1M'}=\pm R_{kM'}[k(k+1)]^{-1/2}(j+1)\sin\theta\,d\cos\theta, \quad (k \text{ odd}) \qquad (3.52)$$

$$\text{other } d\sigma_{kMM'}=0. \qquad (3.53)$$

[10] J. Shafer (private communication).

The upper sign in (3.52) is for the $\frac{1}{2}^-$ sequence, and the lower sign for the $\frac{1}{2}^+$ sequence. The spin and parity can be determined by seeing which $d\sigma_{kMM'}$ are nonvanishing and by considering the ratio of (3.51) to (3.52).

The situation may be summarized by saying that we have a series of test functions characterized by three quantum numbers and one continuous variable. The $d\sigma_{kMM'}$ are nonzero for k equal to or less than twice the spin of the resonance and for $M=0$, k even, and $M=0$, ±1, k odd. They also vanish for odd M', by parity considerations, if the production normal is taken as the polar axis of the production configuration. The experimental evaluation of those $d\sigma_{kMM'}$ predicted to vanish is useful as a check on the background.

In the Byers-Fenster approach, the total counting rate $d\sigma=\sum_k d_{Jk}^{-1}\Lambda^k:\Phi^k d\cos\theta_C d\Omega_a d\Omega_{a'}$ is multiplied by $Y_{k,M}(\Omega_a)$ and by $(\mathbf{a}'\cdot\mathbf{n})Y_{k,M}(\Omega_a)$ and integrated over $d\Omega_a d\Omega_{a'}$ to obtain the moment $\langle Y_{k,M}\rangle$ of the intensity and the moments $\langle \mathbf{P}\cdot\mathbf{n}Y_{k,M}\rangle$ of the components of the polarization $\mathbf{P}$ of the b particle. Here, Ω_a is referred to axes defined by the production process. With Φ^k given by (3.48), (3.49), the integration over $(\mathbf{a}'\cdot\mathbf{n})d\Omega_{a'}$ reduces Φ^k to zero for k even, while for k odd, reduces Φ^k to $\frac{1}{2}(k+1)T^k(\mathbf{a})$ if $\mathbf{n}=\mathbf{a}$ (longitudinal polarization) and to $\pm(j+1)T^k(\mathbf{a}\cdots\mathbf{a}\mathbf{n})$ if $\mathbf{n}\cdot\mathbf{a}=0$ (transverse polarization), with the sign depending on the parity of the resonance. It is seen that the Byers-Fenster intensity moment corresponds to our projected cross section for the $M=0$, k even case, their moment of longitudinal polarization to the $M=0$, k odd case, and their transverse polarization moments to our $M=\pm1$, k odd case.

IV. BOSON RESONANCES WITH DECAY PRODUCTS OF SPIN 0 AND SPIN 1

1. Preliminaries

We now discuss

$$A+B\to X+C, \quad X\to a+b, \qquad (4.1)$$

where A and a are 0^-, B and C are $\frac{1}{2}^+$, and b is 1^-, e.g., a ρ or ω meson. Let $\mathbf{e}$ be the spin wave function of b. The decay amplitude can be written

$$M_D=T^j(\mathbf{a}\cdots\mathbf{a}\mathbf{f}), \qquad (4.2)$$

where

$$\mathbf{f}=\mathbf{a}\times\mathbf{e} \quad \text{for} \quad 1^-, 2^+, 3^-, \cdots, \qquad (4.3a)$$

$$\mathbf{f}=y\mathbf{e}+z\mathbf{a}(\mathbf{a}\cdot\mathbf{e}) \quad \text{for} \quad 0^-, 1^+, 2^+, \cdots, \qquad (4.3b)$$

and y, z are possibly complex coupling constants; $y=0$ for 0^-. If the X decay is a low-energy decay, i.e., $|\mathbf{a}|\times$range of decay interaction <1, then the y term, carrying orbital angular momentum $l=j-1$ may dominate over the z term which includes $l=j+1$.

We consider two possible decay modes for b:

$$b\to a'+b' \quad \text{or} \quad b\to a'+b'+c', \qquad (4.4)$$

where all primed particles are 0^-. In the first case b is, or is like, the ρ meson and has a decay amplitude

$$M_b = \mathbf{a}'\cdot\mathbf{e}. \qquad (4.5a)$$

The second case corresponds to the ω meson with decay amplitude

$$M_b = \mathbf{q}\cdot\mathbf{e}, \qquad (4.5b)$$

where $\mathbf{q}$ is the normal to the decay plane; $\mathbf{q}=\mathbf{a}'\times\mathbf{b}'$.

The amplitude for the over-all decay is then proportional to

$$\sum_{b \text{ spin states}} M_b M_D. \qquad (4.6)$$

Since

$$\sum_{b \text{ spin states}} e_i e_j = \delta_{ij}, \qquad (4.7)$$

Eq. (4.6) is just the instruction to substitute $\mathbf{a}'$ or $\mathbf{q}$ for $\mathbf{e}$ in (4.3) to get the amplitude expressed in terms of observable vectors. We shall continue to use the symbole $\mathbf{e}$, and write $\cos\theta=\mathbf{a}\cdot\mathbf{e}$. Experimentally, this θ is the angle between $\mathbf{a}$, in the XRF and $\mathbf{a}'$, or $\mathbf{q}$ in the bRF.[11] The Euler angles β, α will relate $\mathbf{e}$ to the production configuration.

In these examples of b decay, $\mathbf{e}$ is effectively a real vector. If, on the other hand, b were, say, a 1^+ particle decaying into three 0^- particles, the decay amplitude might be

$$M_b = (x_1\mathbf{a}'+x_2\mathbf{b}')\cdot\mathbf{e} \qquad (4.8)$$

with the ratio x_1/x_2 complex. Then additional variables are needed to describe the b decay. We comment briefly on this more general situation at the end of the section.

2. Correlations in Special Circumstances

Complexity of correlations. The maximum complexity theorems derived in Sec. III are applicable here.

Forward and backward production. The Adair-type distributions are less interesting here than in Sec. III because even if **A**, **B**, and **C** are aligned, there are still two independent production amplitudes, namely

$$x_1T^j(\mathbf{A})+x_2T^j(\mathbf{A}\cdots\mathbf{A}\boldsymbol{\sigma}\times\mathbf{A}) \quad \text{for } 0^-, 1^+, 2^- \cdots \quad (x_2=0 \text{ for } 0^-) \qquad (4.9a)$$

and

$$x_1T^j(\mathbf{A}\cdots\mathbf{A}\boldsymbol{\sigma})+x_2(\boldsymbol{\sigma}\cdot\mathbf{A})T^j(\mathbf{A}) \quad \text{for } 1^-, 2^+, 3^- \cdots. \qquad (4.9b)$$

These are not likely to be helpful in determining the spin and parity of X unless something is known about the fermion polarizations. But if the X quantum numbers are known, fitting the data to these amplitudes may clarify the production mechanisms.

[11] C. Zemach, Nuovo Cimento **32**, 1605 (1964). In this reference, **e** is taken in the XRF. This has the advantage that the correlation angle is given as the difference between two directions in the same frame. It has the disadvantage that the density-of-states factor is not simple $d\cos\theta$, but rather $[1-a^2\cos^2\theta/E_b^2]^{-5/2}d\cos\theta$. This reference also gives correlations appropriate to coherent production as may occur in a heavy-liquid bubble chamber.

If a "low-energy production" approximation is applicable, and X is in the 1^- sequence, then (see line 4 of Table I of I)

$$\begin{aligned} M_{DP} &= T^j(\mathbf{a}\cdots\mathbf{a}\;\mathbf{a}\times\mathbf{e}):T^j(\mathbf{A}\cdots\mathbf{A}\;\boldsymbol{\sigma}) \\ &= (\mathbf{A}\cdot\mathbf{a}\times\mathbf{e})[(\boldsymbol{\sigma}\cdot\mathbf{a})P_j''(x)-(\boldsymbol{\sigma}\cdot\mathbf{A})P_{j-1}''(x)] \\ &\quad +\boldsymbol{\sigma}\cdot\mathbf{a}\times\mathbf{e}P_j'(x), \quad x=\mathbf{a}\cdot\mathbf{A}. \end{aligned} \qquad (4.10)$$

Summing over fermion spins,

$$\begin{aligned} \sum|M_{DP}|^2 &= (\mathbf{A}\cdot\mathbf{a}\times\mathbf{e})^2 \\ &\quad \times\{(P_j'')^2+(P_{j-1}'')^2 -2xP_j''P_{j-1}''-2P_{j-1}''P_j'\} \\ &\quad +(\mathbf{a}\times\mathbf{e})^2(P_j')^2. \end{aligned} \qquad (4.11)$$

For the lower spin values, we then have

$$d\sigma/d\Omega_a d\Omega_e = (\mathbf{a}\times\mathbf{e})^2, \quad (1^-), \qquad (4.12a)$$

$$= x^2(\mathbf{a}\times\mathbf{e})^2+(\mathbf{A}\cdot\mathbf{a}\times\mathbf{e})^2, \quad (2^+) \qquad (4.12b)$$

$$= (5x^2-1)^2(\mathbf{a}\times\mathbf{e})^2+8(5x^2-1)\times(\mathbf{A}\cdot\mathbf{a}\times\mathbf{e})^2, \quad (3^-) \qquad (4.12c)$$

and so on. Integrating over $d\Omega_e$—this is equivalent to applying (4.7)—gives the rate as a function of one variable:

$$d\sigma/dx = 1,1,9+22x^2-15x^4 \quad \text{for} \quad 1^-,2^+,3^-, \qquad (4.13)$$

respectively.

Peripheral production. Referring to Fig. 1(b), if G is 0^- then x must be in the 1^- sequence, and the amplitude for (3.22) is

$$M_{DP} = T^j(\mathbf{a}\cdots\mathbf{a}\;\mathbf{a}\times\mathbf{e}):T^j(\mathbf{A}) = \mathbf{A}\cdot\mathbf{a}\times\mathbf{e}P_j'(x) \qquad (4.14)$$

so that

$$d\sigma/d\Omega_a d\Omega_e = (\mathbf{A}\cdot\mathbf{a}\times\mathbf{e})^2|P_j'(x)|^2; \qquad (4.15)$$

$$d\sigma/dx = (1-x^2)(P_j'(x))^2. \qquad (4.16)$$

Notice that (4.15) results from the second, rather than the first term of (4.9b). This serves as a warning that when the reaction goes by a specific mechanism, preference may be given to the higher rather than the lower orbital term, even in a low-energy production.

If the exchanged particle has spin $\frac{1}{2}$, the possible amplitudes for the simplified process has the same generality as (4.9). But there is a special case which may be interesting. Suppose B and G in (3.21) are nucleon and antinucleon. For a nucleon-antinucleon system,

$$G \text{ parity (triplet spin state)} = (-1)^{l+I}, \qquad (4.17a)$$

$$G \text{ parity (singlet spin state)} = (-1)^{l+I+1}. \qquad (4.17b)$$

Thus if X has definite I and G and the production conserves these quantum numbers, only one term in (4.9a) is present. The amplitude for the 0^-, 1^+, 2^-, $\cdots$ cases is then either

$$\begin{aligned} M_{DP} &= T^j(\mathbf{a}\cdots\mathbf{a}\mathbf{f}):T^j(\mathbf{A}\cdots\mathbf{A}\boldsymbol{\sigma}\times\mathbf{A}) \\ &= (\boldsymbol{\sigma}\cdot\mathbf{A}\times\mathbf{a})[(\mathbf{A}\cdot\mathbf{f})P_j''(x)-(\mathbf{a}\cdot\mathbf{f})P_{j-1}''(x)] \\ &\quad +(\boldsymbol{\sigma}\cdot\mathbf{A}\times\mathbf{f})P_j'(x) \end{aligned} \qquad (4.18a)$$

or

$$M_{DP}=T^j(\mathbf{a}\cdots\mathbf{af}):T^j(\mathbf{A})$$
$$=\mathbf{f}\cdot\mathbf{A}P_j'(x)-(\mathbf{f}\cdot\mathbf{a})P_{j-1}'(x) \quad (4.18b)$$

depending on the G and I of X. These expressions are easily converted into cross sections and summed over b spin if that is desired. They serve as further examples of how the tensor notation, with a few key formulas, allows us to go from a well-defined spin situation to a calculated angular correlation with a minimum of difficulty.

3. Moment Analysis

In order to obtain projected cross sections $d\sigma_{kMM'}$, first define the $\mathbf{n}^1$, $\mathbf{n}^2$, $\mathbf{n}^3$ basis vectors for the decay configuration:

$$\mathbf{a}=\mathbf{n}^3 \quad (4.19a)$$

$$\mathbf{e}=\mathbf{n}^1\sin\theta+\mathbf{n}^3\cos\theta \quad (4.19b)$$

$$\mathbf{a}\times\mathbf{e}=\mathbf{n}^2\sin\theta. \quad (4.19c)$$

Second, write the decay amplitude in this basis:

$$M_D=(\sin\theta)T^j(\mathbf{n}^3\cdots\mathbf{n}^3\mathbf{n}^2)\,,\quad (1^-,2^+,3^-,\cdots) \quad (4.20a)$$

and

$$M_D=(y+z)\cos\theta T^j(\mathbf{n}^3)+y\sin\theta T^j(\mathbf{n}^3\cdots\mathbf{n}^3\mathbf{n}^1)\,.$$
$$(0^-,1^+,2^-,\cdots)\,. \quad (4.20b)$$

Third, specialize the relevant equations of Table III of I to the case where $\mathbf{p}\cdot\mathbf{q}=0$, and drop the over-all normalization factor a_{nj}:

$$T^j(\mathbf{p}):T^{2n}(\mathbf{S})T^j(\mathbf{p})=T^{2n}(\mathbf{p}) \quad (4.21a)$$

$$T^j(\mathbf{p}):T^{2n}(\mathbf{S})T^j(\mathbf{p}\cdots\mathbf{pq})=T^j(\mathbf{p}\cdots\mathbf{pq}):T^{2n}(\mathbf{S})T^j(\mathbf{p})$$
$$=(n/j)T^{2n}(\mathbf{p}\cdots\mathbf{pq}) \quad (4.21b)$$

$$T^j(\mathbf{p}\cdots\mathbf{pq}):T^{2n}(\mathbf{S})T^j(\mathbf{p}\cdots\mathbf{pq})$$
$$=(n(j+1)/j(n+1))T^{2n}(\mathbf{p}\cdots\mathbf{pqq})$$
$$+\frac{(j-n)(2n+1)(j+n+1)}{2j^2(n+1)}\mathbf{q}^2T^{2n}(\mathbf{p}) \quad (4.21c)$$

$$T^j(\mathbf{p})T^{2n-1}(\mathbf{S})T^j(\mathbf{p})$$
$$=T^j(\mathbf{p}\cdots\mathbf{pq})T^{2n-1}(\mathbf{S})T^j(\mathbf{p}\cdots\mathbf{pq})=0 \quad (4.21d)$$

$$T^j(\mathbf{p}):T^{2n-1}(\mathbf{S})T^j(\mathbf{p}\cdots\mathbf{pq})$$
$$=-T^j(\mathbf{p}\cdots\mathbf{pq}):T^{2n-1}(\mathbf{S})T^j(\mathbf{p})$$
$$=iT^{2n-1}(\mathbf{p}\cdots\mathbf{p}\;\mathbf{p}\times\mathbf{q})\,. \quad (4.21e)$$

Fourth, calculate the tensor moments of M_D. For the 1^- sequence, they are

$$M_D^*:T^k(\mathbf{S})M_D$$
$$=\{(j-\tfrac{1}{2}k)(j+1+\tfrac{1}{2}k)(k+1)T^k(\mathbf{n}^3)$$
$$+kj(j+1)T^k(\mathbf{n}^3\cdots\mathbf{n}^3\mathbf{n}^2\mathbf{n}^2)\}\sin^2\theta\,,$$
$$(k\text{ even}) \quad (4.22a)$$

and

$$M_D^*:T^k(\mathbf{S})M_D=0\,,\quad (k\text{ odd})\,. \quad (4.22b)$$

For the 0^- sequence, they are

$$M_D^*:T^k(\mathbf{S})M_D$$
$$=|y+z|^2\cos^2\theta T^k(\mathbf{n}^3)+\mathrm{Re}\,y(y+z)^*\cos\theta$$
$$\times\sin\theta(k/j)T^k(\mathbf{n}^3\cdots\mathbf{n}^3\mathbf{n}^1)+|y|^2\sin^2\theta$$
$$\times\left\{\frac{(j-\tfrac{1}{2}k)(j+1+\tfrac{1}{2}k)(k+1)}{j^2(k+2)}T^k(\mathbf{n}^3)\right.$$
$$\left.+\frac{k(j+1)}{j(k+2)}T^k(\mathbf{n}^3\cdots\mathbf{n}^3\mathbf{n}^1\mathbf{n}^1)\right\}\,,\quad (k\text{ even}) \quad (4.23a)$$

and

$$M_D^*:T^k(\mathbf{S})M_D=\mathrm{Im}(y^*z)\cos\theta\sin\theta T^k(\mathbf{n}^3\cdots\mathbf{n}^3\mathbf{n}^2)\,,$$
$$(k\text{ odd})\,. \quad (4.23b)$$

Fifth, convert the T^k to the spherical basis by putting $\mathbf{n}^1=(\mathbf{n}^- - \mathbf{n}^+)/\sqrt{2}$, $\mathbf{n}^2=i(\mathbf{n}^+ + \mathbf{n}^-)/\sqrt{2}$. Thus,

$$T^k(\mathbf{n}^3\cdots\mathbf{n}^3\mathbf{n}^1)$$
$$=2^{-1/2}[T^{k,-1}(\mathbf{n})-T^{k,+1}(\mathbf{n})] \quad (4.24a)$$

$$T^k(\mathbf{n}^3\cdots\mathbf{n}^3\mathbf{n}^1\mathbf{n}^1)$$
$$=\tfrac{1}{2}[T^{k,-2}(\mathbf{n})+T^{k,2}(\mathbf{n})-2T^k(\mathbf{n}^3\cdots\mathbf{n}^3\mathbf{n}^+\mathbf{n}^-)]$$
$$=\tfrac{1}{2}[T^{k,-2}(\mathbf{n})+T^{k,2}(\mathbf{n})-T^{k,0}(\mathbf{n})] \quad (4.24b)$$

$$T^k(\mathbf{n}^3\cdots\mathbf{n}^3\mathbf{n}^2\mathbf{n}^2)$$
$$=\tfrac{1}{2}[-T^{k,-2}(\mathbf{n})-T^{k,2}(\mathbf{n})-T^{k,0}(\mathbf{n})]\,. \quad (4.24c)$$

Sixth and last, use (2.25), (2.27b), (2.30) to obtain the projected cross sections. The results for $(1^-,2^+,3^-\cdots)$ are

for k even:

$$d\sigma_{k0M'}=R_{kM'}[j(j+1)(k+2)$$
$$+k(k+1)(j-k-1)]\sin^2\theta\,d\cos\theta\,, \quad (4.25a)$$

$$d\sigma_{k2M'}=d\sigma_{k-2M'}=-R_{kM'}\tfrac{1}{2}j(j+1)$$
$$\times[k(k+1)(k+2)(k-1)^{-1}]^{1/2}$$
$$\times\sin^2\theta\,d\cos\theta\,, \quad (4.25b)$$

$$\text{other}\quad d\sigma_{kMM'}=0\,, \quad (4.25c)$$

and for k odd:

$$d\sigma_{kMM'}=0. \quad (4.25d)$$

The results for $(0^-,1^+,2^-,\cdots)$ are

for k even:

$$d\sigma_{k0M'}=R_{kM'}\{|y+z|^2 2j^2(k+2)\cos^2\theta$$
$$+|y|^2[j(j+1)(k+2)+k(k+1)(j-k-1)]$$
$$\times\sin^2\theta\}d\cos\theta\,, \quad (4.26a)$$

$$d\sigma_{k-1M'}=-d\sigma_{k1M'}$$
$$=R_{kM'}\,\mathrm{Re}\,y(y+z)^*j(k+2)[k(k+1)]^{1/2}$$
$$\times\cos\theta\sin\theta\,d\cos\theta\,, \quad (4.26b)$$

$$d\sigma_{k-2M'} = d\sigma_{k2M'} = R_{kM'}|y|^2 \tfrac{1}{2} j(j+1) \times [k(k+1)(k+2)(k-1)^{-1}]^{1/2} \times \sin^2\theta \, d\cos\theta, \quad (4.26c)$$

and for k odd:

$$d\sigma_{k-1M'} = d\sigma_{k1M'} = R_{kM'} \,\mathrm{Im}(y^* z) \cos\theta \sin\theta \, d\cos\theta. \quad (4.26d)$$

All other $d\sigma_{kMM'} = 0$.

It is easy to see how the different equations of (4.26) can be compared and combined to yield, in a number of different ways, not only the spin of X but also the decay constants. But $\mathrm{Im}(y^*z)(|y|^2+|z|^2)^{-1}$ cannot be determined. The experimental calculation of those $d\sigma_{kMM'}$ predicted to be zero is also useful as it gives a check on the amount of background. The 1^- and 0^- sequences are distinguished, among other things, by the sign of $d\sigma_{k\pm 2M'}$.

The $k=0$ terms refer to the unprojected decay correlations:

$$d\sigma_{000} = M_D^* M_D d\cos\theta = \sin^2\theta \, d\cos\theta, \qquad (1^-, 2^+, \cdots), \quad (4.27a)$$

$$= \{|y+z|^2 2j \cos^2\theta + |y|^2 (j+1) \sin^2\theta\} d\cos\theta, \qquad (0^-, 1^+, \cdots) \quad (4.27b)$$

already given earlier.[10]

The tests of Ademollo, Gatto, and Preparata[2] carry the same information as (4.25), (4.26) but catalogued by a slightly different set of quantum numbers. Consider, for example, the two simplest cases:

$$(d\sigma_{000}/d\cos\theta)(1^-) = \sin^2\theta = \tfrac{2}{3}[1 - P_2(\cos\theta)], \quad (4.28a)$$

$$(d\sigma_{000}/d\cos\theta)(0^-) = \cos^2\theta = \tfrac{1}{3}[1 + 2P_2(\cos\theta)]. \quad (4.28b)$$

The corresponding information in Tables 1, 2 of Ademollo, Gatto, and Preparata is expressed this way:

$$A(20;00)/A(00;00) = -1/\sqrt{2} \quad (\text{for } 1^-), \quad (4.29a)$$

$$A(20;00)/A(00;00) = +2/\sqrt{2} \quad (\text{for } 0^-). \quad (4.29b)$$

We see that the coefficients of $P_2(\cos\theta)$ in (4.28) correspond, apart from a $\sqrt{2}$ normalization factor, with the ratios in (4.29).

Finally, we note the appropriate procedure if b follows a decay mode like (4.8). It is necessary to add to the list (4.19)

$$\mathbf{b}' = \sin\theta' \cos\varphi' \, \mathbf{n}^1 + \sin\theta' \sin\varphi' \, \mathbf{n}^2 + \cos\theta' \, \mathbf{n}^3, \quad (4.30)$$

where θ', φ' are the coordinates of $\mathbf{b}'$ in the coordinate frame defined by $\mathbf{a}, \mathbf{a}'$. One proceeds as before, obtaining cross sections $d\sigma_{kMM'}$ proportional to $d\cos\theta \, d\cos\theta' d\varphi'$. It is no longer true that $\mathbf{e}$ can be treated as real; this means there is a possibility of calculating $\mathrm{Im}(y^*z) \times (|y|^2+|z|^2)^{-1}$ which could not be done by means of (4.26).

V. FERMION RESONANCES WITH DECAY PRODUCTS OF SPIN 0 AND SPIN $\frac{3}{2}$

1. Preliminaries

Our model reaction will be

$$A+B \to C+X, \quad X \to a+b, \quad b \to a'+b', \quad b' \to a''+b'', \quad (5.1)$$

where A, a, a', a'', and C are 0^-, b is $\frac{3}{2}^+$ and B, b', b'' are $\frac{1}{2}^+$. The b' decay will be taken as parity nonconserving. We have in mind a sequence such as $K+p \to K + \Xi^*(1820)$, $\Xi^*(1820) \to \Xi^*(1530)+\pi$, $\Xi^*(1530) \to \Lambda+\pi$, $\Lambda \to N+\pi$, which has recently been observed.[12] As other examples, we mention reactions which involve $N^*(1520) \to N^*(1238)+\pi$ and the decay into $N^*(1238)+\pi$ of a possible $I=\frac{5}{2}$ resonance[13] at 1560 MeV. The b' decay is not relevant to the latter two possibilities.

The study of correlations for (5.1) is more complicated than in the previous cases because there are three decay directions, $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{a}''$ about which something can be said. Moreover, there are two orbital terms for either parity sequence for X and the spinor-vector $\mathbf{E}$ combines whatever complexity there is in spinors and vectors. The current experimental situation would seem to call for an investigation of correlations involving a spin-$\frac{3}{2}$ decay product, but not an exhaustive one.

We shall follow, in part, the scheme of the previous sections, looking at Adair distributions and forward versus sideward distributions, but assuming that only the lower orbital term in the X decay is retained. This may well be a good approximation for the $\Xi^*(1820)$ and $N^*(1520)$ decays, both of which have a Q value of about one pion mass. Then we shall do the moment analysis for the lowest order term—which gives the angular correlation of $\mathbf{a}$ and $\mathbf{a}'$—and the dipole term which also depends on $\mathbf{a}''$.

In the by now customary notation, the decay amplitudes for $b \to a'+b'$, $b' \to a''+b''$ take the forms

$$M_b = \langle u_{b'}^* | \mathbf{a}' \cdot \mathbf{E} \rangle, \quad (5.2)$$

$$M_{b'} = \langle u_{b''}^* | y' + z' \boldsymbol{\sigma} \cdot \mathbf{a}'' | u_{b'} \rangle. \quad (5.3)$$

The sum over b spin states [see (3.17) of I] is

$$\sum_{b \text{ spin}} E_{i\alpha} E^*_{j\beta} = (\mathcal{P}_\sigma)_{i\alpha;j\beta} \quad (5.4)$$

which is consistent with the constraints $\boldsymbol{\sigma} \cdot \mathbf{E} = 0$, $\mathbf{E}^* \cdot \boldsymbol{\sigma} = 0$. But it is more convenient to insert the projection operator directly into M_b (we again ignore a proportionality factor):

$$M_b = \langle u_{b'}^* | \mathbf{a}' \cdot \mathcal{P}_\sigma | \mathbf{E} \rangle \equiv \langle u_{b'}^* | (\mathbf{a}' + \tfrac{1}{2} i \mathbf{a}' \times \boldsymbol{\sigma}) \cdot | \mathbf{E} \rangle. \quad (5.5)$$

[12] G. A. Smith, J. S. Lindsay, J. B. Shafer, and J. J. Murray, Phys. Rev. Letters 14, 25 (1965).

[13] G. Goldhaber, invited address, American Physical Society, Washington, May, 1964; G. Goldhaber, S. Goldhaber, T. A. O'Halloran, B. C. Shen, Lawrence Radiation Laboratory Report No. 11445, 1964 (unpublished).

Then, we can use

$$\sum_{b\ \text{spin}} E_{i\alpha}E^*_{j\beta}=(1)_{i\alpha;j\beta}=\delta_{ij}\delta_{\alpha\beta}. \tag{5.6}$$

The amplitude $M_{bb'}$ for $b\to a'+a''+b''$ must now be computed, squared, and summed over b'' spin:

$$\sum|M_{bb'}|^2=\mathbf{E}^*\cdot(\mathbf{a}'+\tfrac{1}{2}i\boldsymbol{\sigma}\times\mathbf{a}')(1+\rho\boldsymbol{\sigma}\cdot\mathbf{a}'')\times(\mathbf{a}'+\tfrac{1}{2}i\mathbf{a}'\times\boldsymbol{\sigma})\cdot\mathbf{E}. \tag{5.7}$$

Equation (5.7) is somewhat similar to (3.9). If b were $\frac{3}{2}^-$ rather than $\frac{3}{2}^+$, $\mathbf{a}''$ in (5.7) would be replaced by the magic direction, $(\mathbf{a}'')_m=-2\mathbf{a}''+(\mathbf{a}'\cdot\mathbf{a}'')\mathbf{a}'$. Now the squared amplitude for $A+B\to C+a+b$, averaged over B spin will look like

$$\sum|M_{DP}|^2=\mathbf{E}^*\cdot\mathbf{M}\cdot\mathbf{E}, \tag{5.8}$$

where $\mathbf{M}$ is some function of $\mathbf{a}$, and the production angles. Correlations are computed by multiplying (5.7) by (5.8), using (5.6) and then integrating over appropriate variables.

The symbol $\mathbf{E}$ expresses the b spin as the sum of a spin-1 part and a spin-$\frac{1}{2}$ part. Experimentally, the momentum $\mathbf{a}'$ gives the direction of the spin-1 part and $\mathbf{a}''$ gives the direction of the spinor polarization associated with the spin-$\frac{1}{2}$ part.

We shall not attempt to find correlations in $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{a}''$ simultaneously. To obtain correlations of $\mathbf{a}'$, $\mathbf{a}$ and production vectors, one should integrate (5.7) over $\Omega_{a''}$, getting

$$\sum_{b''\ \text{spin}}|M_{bb'}|^2d\Omega_{a''}=\sum_{b'\ \text{spin}}|M_b|^2=\tfrac{3}{4}(\mathbf{E}^*\cdot\mathbf{a}')(\mathbf{a}'\cdot\mathbf{E})+\tfrac{1}{4}\mathbf{E}^*\cdot\mathbf{E}-\tfrac{1}{4}i\mathbf{E}^*\times\mathbf{a}'\cdot(\boldsymbol{\sigma}\cdot\mathbf{a}')\mathbf{E}+\tfrac{1}{2}i[(\mathbf{E}^*\cdot\boldsymbol{\sigma}\times\mathbf{a}')\mathbf{a}'\cdot\mathbf{E}-(\mathbf{E}^*\cdot\mathbf{a}')(\boldsymbol{\sigma}\times\mathbf{a}'\cdot\mathbf{E}). \tag{5.9}$$

To obtain correlations of $\mathbf{a}$ alone with production vectors, integrate (5.9) over $\Omega_{a'}$ to get

$$\int\sum_{b'\ \text{spin}}|M_b|^2d\Omega_{a'}=\mathbf{E}^*\cdot\mathcal{P}_\sigma\mathbf{E}\equiv\mathbf{E}^*\cdot\mathbf{E}+\tfrac{1}{2}i\mathbf{E}^*\cdot\boldsymbol{\sigma}\times\mathbf{E}, \tag{5.10}$$

which amounts to using (5.4) directly.

To obtain correlations of $\mathbf{a}$, $\mathbf{a}''$, but not $\mathbf{a}'$, one integrates (5.7) over $\mathbf{a}'$, with the result:

$$\sum|M_{bb'}|^2d\Omega_{a'}=(\tfrac{3}{2})(\mathbf{E}^*\cdot\mathbf{E}+\tfrac{1}{2}i\mathbf{E}^*\cdot\boldsymbol{\sigma}\times\mathbf{E})+\rho[(5i/4)(\mathbf{E}^*\times\mathbf{a}''\cdot\mathbf{E})+\mathbf{E}^*(\boldsymbol{\sigma}\cdot\mathbf{a}'')\mathbf{E}-\tfrac{1}{4}(\mathbf{E}^*\cdot\boldsymbol{\sigma})(\mathbf{a}''\cdot\mathbf{E})-\tfrac{1}{4}(\mathbf{E}^*\cdot\mathbf{a}'')(\boldsymbol{\sigma}\cdot\mathbf{E})]. \tag{5.11}$$

In these equations, one must not apply $\boldsymbol{\sigma}\cdot\mathbf{E}=0$ since the $\mathcal{P}_\sigma$ which enforces this constraint has been put explicitly into the matrix element. One can still apply $\boldsymbol{\sigma}\cdot\mathbf{E}=0$, and hence $i\boldsymbol{\sigma}\times\mathbf{E}=\mathbf{E}$ in M_D.

The maximum complexity theorems are as valid here as in Sec. III, but less interesting since, with higher particle spins, reactions can be dominated by lower orbital terms, giving complexities less than the theoretical maximum.

2. Forward and Backward Production (Table III)

We calculate the correlation in $x=\mathbf{A}\cdot\mathbf{a}$ for the case where the production vectors are aligned and the lower orbital term of the decay dominates. For spin $\frac{1}{2}$, $M_P=1$ or $\boldsymbol{\sigma}\cdot\mathbf{A}$ and the distribution in x is flat. For $(\frac{3}{2}^-, \frac{5}{2}^+, \cdots)$ the reaction amplitude is

$$\begin{aligned}M_{DP}&=T^j(\mathbf{a}\cdots\mathbf{a}\mathbf{E}^*):\mathcal{P}_\sigma T^j(\mathbf{A})\\&=(j+1)[(\mathbf{E}^*\cdot\mathbf{A})P_j'(x)-(\mathbf{E}^*\cdot\mathbf{a})P_{j-1}'(x)]\\&\quad+(\mathbf{E}^*\cdot\mathbf{A})(i\boldsymbol{\sigma}\cdot\mathbf{A}\times\mathbf{a})P_j''(x)\\&\quad-(\mathbf{E}^*\cdot\mathbf{a})(i\boldsymbol{\sigma}\cdot\mathbf{A}\times\mathbf{a})P_{j-1}''\\&\quad+i(\mathbf{E}^*\cdot\boldsymbol{\sigma}\times\mathbf{A})P_j'(x)\end{aligned} \tag{5.12}$$

with the help of Table 1 of Ref. 1. The last term reduces to $(\mathbf{E}^*\cdot\mathbf{A})P_j'(x)$ by (2.6b). This amplitude is of the general type

$$M_{DP}=\mathbf{E}^*\cdot(\mathfrak{A}+i\mathfrak{B}\boldsymbol{\sigma}\cdot\mathfrak{C}). \tag{5.13}$$

Then the counting rate, averaged over spins of B and b, is

$$\begin{aligned}\sum|M_{DP}|^2&=\tfrac{1}{2}\,\mathrm{Tr}(\mathfrak{A}-i\mathfrak{B}\boldsymbol{\sigma}\cdot\mathfrak{C})\cdot\mathcal{P}_\sigma(\mathfrak{A}+i\mathfrak{B}\boldsymbol{\sigma}\cdot\mathfrak{C})\\&=\mathfrak{A}\cdot\mathfrak{A}+\mathfrak{A}\cdot\mathfrak{B}\times\mathfrak{C}+\mathfrak{B}\cdot\mathfrak{B}\mathfrak{C}\cdot\mathfrak{C}.\end{aligned} \tag{5.14}$$

This gives the rate in terms of Legendre polynomials and their derivatives.

For the opposite parity sequence $\frac{3}{2}^+$, $\frac{5}{2}^-$, $\cdots$, one replaces $\mathbf{E}^*$ by $(\mathbf{a}\times\mathbf{E}^*)$ and $T^j(\mathbf{A})$ by $T^j(\mathbf{A})\boldsymbol{\sigma}\cdot\mathbf{A}$ in (5.12) and proceeds the same way. The extra $\boldsymbol{\sigma}\cdot\mathbf{A}$ factor does not affect the calculation, as already noted in Sec. III.

The calculated distributions for the lower spins are given in Table III. We mention again that these distributions are also applicable to peripheral productions if the exchanged particle has spin zero or one-half. They also apply to two particle production processes such as $\pi+N\to X\to\pi+N^*(1238)$.

TABLE III. Adair distributions for fermion resonances with decay products of spin 0^- and $\frac{3}{2}^+$. Only the lower orbital term of the decay is taken into account. See Sec. V.2.

Spin and parity	Distribution in $x=\mathbf{A}\cdot\mathbf{a}$
$\frac{1}{2}^\pm$, $\frac{3}{2}^-$	1
$\frac{3}{2}^+$	$1-6x^2/7$
$\frac{5}{2}^+$	$1+2x^2$
$\frac{5}{2}^-$	$1+10x^2-10x^4$
$\frac{7}{2}^+$	$1-40x^2/7+65x^4/3-50x^6/3$
$\frac{7}{2}^-$	$1+x^2/5+19x^4/10$

3. Correlations at Low-Production Energies (Table IV)

We shall calculate $Z(\beta)$, the angular correlation between the production normal $\mathbf{Q}$ and the X decay direction $\mathbf{a}$. Taking only the $l=j$ term of the decay amplitude, and following the general scheme of Sec. III.5, we expect again that the sideward rate $Z_s(\beta)$ and the forward-backward rate $Z_f(\beta)$ will be similar for $\frac{3}{2}^-$, $\frac{5}{2}^+$ and different for $\frac{3}{2}^+$, $\frac{5}{2}^-$ for the same reasons.

Setting $x=-\cos\gamma\sin\beta$ in Table III and averaging over γ, we get $Z_f(\beta)=1$ for $\frac{1}{2}^+$, $\frac{1}{2}^-$, $\frac{3}{2}^-$, and $Z_f(\beta)=4+3\cos^2\beta$, $9+10\cos^2\beta-15\cos^4\beta$, $2-\cos^2\beta$ for $\frac{3}{2}^+$, $\frac{5}{2}^-$, $\frac{5}{2}^+$. The last three are hill, valley, and hill distributions, respectively.

To obtain $Z_s(\beta)$, we use the M_P's of Sec. III.5, and the M_D's of Sec. V.2. The latter must be written out in the traceless, symmetric, transverse to $\boldsymbol{\sigma}$ form, e.g.,

$$\begin{aligned}M_D(\tfrac{5}{2}^+)\equiv&\,\mathbf{a}\mathbf{E}^*+\mathbf{E}^*\mathbf{a}-(\tfrac{2}{3})(\mathbf{E}^*\cdot\mathbf{a})\mathbf{I}\\&+(i/3)[\mathbf{a}(\mathbf{E}^*\times\boldsymbol{\sigma})+(\mathbf{E}^*\times\boldsymbol{\sigma})\mathbf{a}\\&+(\mathbf{a}\times\boldsymbol{\sigma})\mathbf{E}^*+\mathbf{E}^*(\mathbf{a}\times\boldsymbol{\sigma})]\\\equiv&\,4\mathbf{a}\mathbf{E}^*+4\mathbf{E}^*\mathbf{a}-2(\mathbf{E}^*\cdot\mathbf{a})\mathbf{I}\\&+i(\mathbf{a}\times\boldsymbol{\sigma})\mathbf{E}^*+i\mathbf{E}^*(\mathbf{a}\times\boldsymbol{\sigma})\,.\end{aligned}\quad(5.15)$$

The calculations follow the rules already developed. For $\frac{3}{2}^+$, $Z_s(\beta)$ is easily found to be $3-2\cos^2\beta$, a hill. The $\frac{5}{2}^-$ calculation of $Z_s(\beta)$ is somewhat longer, but has a definite answer, $19-16\cos^2\beta+27\cos^4\beta$, a valley. The $\frac{5}{2}^+$ distribution, as in III.5, is not completely determined. It is a hill in several special cases: $2-\cos^2\beta$ if $\mathbf{P}$ parallel to $\mathbf{P}'$, $21-18\cos^2\beta$ if $\mathbf{P}$, $\mathbf{P}'$ real and perpendicular. It is probably a hill in general, but we do not have a general proof. These shapes are summarized in Table IV.

4. Further Correlations with the Production Normal for $\frac{1}{2}^\pm$, $\frac{3}{2}^-$

When X has quantum numbers $\frac{1}{2}^+$, $\frac{1}{2}^-$, $\frac{3}{2}^-$, all the distributions considered above are flat. To distinguish these cases, we look more closely at correlations among $\mathbf{Q}$, $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{a}''$.

Case $\frac{1}{2}^+$. Here $M_P=1$, $M_D=\mathbf{E}^*\cdot\mathbf{a}$ so $|M_{DP}|^2=(\mathbf{E}^*\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{E})$. Invoking (5.7) and (5.6), the total counting rate is

$$d\sigma=(1+3\cos^2\theta)\,d\cos\theta;\quad\cos\theta=\mathbf{a}\cdot\mathbf{a}'.\quad(5.16)$$

There is no dependence on $\mathbf{Q}$ or $\mathbf{a}''$.

Case $\frac{1}{2}^-$. In this case $M_P=\boldsymbol{\sigma}\cdot\mathbf{P}$, $M_D=(\mathbf{E}^*\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})$. Then

$$\begin{aligned}\sum_{A\text{ spin}}M_{DP}M_{DP}^*&=(\mathbf{E}^*\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{P})(\boldsymbol{\sigma}\cdot\mathbf{P}^*)(\boldsymbol{\sigma}\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{E})\\&=(\mathbf{E}^*\cdot\mathbf{a})(1+\bar{c}\boldsymbol{\sigma}\cdot\mathbf{Q}_m)(\mathbf{a}\cdot\mathbf{E})\,,\end{aligned}\quad(5.17)$$

where we follow the notation of (3.34), (3.7b). Then, by (5.9), the part of the correlation independent of a'' is

$$d\sigma=(1+3\cos^2\theta)\,d\cos\theta\quad(5.18)$$

as in (5.16). But there are also correlations among $\mathbf{Q}$, $\mathbf{a}$, $\mathbf{a}''$ given by (5.11):

$$\begin{aligned}d\sigma&=\{1+\tfrac{2}{3}\bar{c}\rho[\mathbf{Q}_m\cdot\mathbf{a}''-\tfrac{1}{2}(\mathbf{a}\cdot\mathbf{Q}_m)(\mathbf{a}\cdot\mathbf{a}'')]\}d\Omega_a d\Omega_{a''}\\&=\{1+\bar{c}\rho[(\mathbf{a}\cdot\mathbf{Q})(\mathbf{a}\cdot\mathbf{a}'')-\tfrac{2}{3}(\mathbf{Q}\cdot\mathbf{a}'')]\}d\Omega_a d\Omega_{a''}\,.\end{aligned}\quad(5.19)$$

Case $\frac{3}{2}^-$. We restrict the discussion to the approximate production amplitudes used in the previous subsection, so that $M_{DP}=\mathbf{E}^*\cdot\mathbf{P}$. Then (5.9) gives a rate proportional to $3|\mathbf{a}'\cdot\mathbf{P}|^2+\mathbf{P}^*\cdot\mathbf{P}$. Averaging this over angles in the production plane as in (3.33), we obtain the correlation in $\mathbf{Q}$, $\mathbf{a}'$:

$$d\sigma=[1-\tfrac{3}{4}(\mathbf{Q}\cdot\mathbf{a}')^2]d\Omega_{a'}\,.\quad(5.20)$$

Using (5.11), we derive the a'-independent correlation

$$d\sigma=[(1+\tfrac{5}{6}\bar{c}\rho\mathbf{Q}\cdot\mathbf{a}'')]d\Omega_{a''}\,.\quad(5.21)$$

These correlations are, moreover, independent of $\mathbf{a}$, distinguishing them from (5.19).

5. Decay Correlations

Consider now correlations in the three decay momenta $\mathbf{a}$, $\mathbf{a}'$, $\mathbf{a}''$, summed over all production information. For parity reasons, these correlations, are, in fact, independent of $\mathbf{a}''$, leaving us with a counting rate $d\sigma(\theta)$ which is a function of a single variable $\cos\theta=\mathbf{a}\cdot\mathbf{a}'$ and the coupling constants y, z. This $d\sigma(\theta)$ is identical with the projected cross section for the zeroth-tensor moment, previously called $d\sigma_{000}$, and is given by

$$d\sigma(\theta)=M_D{:}M^*{}_D d\cos\theta\,.\quad(5.22)$$

For the $\frac{1}{2}^+$ sequence,

$$\begin{aligned}M_D{:}M^*{}_D=&[yT^j(\mathbf{a}\cdots\mathbf{a}\mathbf{E}^*)+z(\mathbf{E}^*\cdot\mathbf{a})T^j(\mathbf{a})]{:}\mathcal{P}_\sigma\\&\times[y^*T^j(\mathbf{a}\cdots\mathbf{a}\mathbf{E})+z^*T^j(\mathbf{a})(\mathbf{a}\cdot\mathbf{E})]\,.\end{aligned}\quad(5.23)$$

When the right side is multiplied out, there are eight separate matrix elements, if $\mathcal{P}_\sigma\sim(j+1)+\boldsymbol{\sigma}\cdot\mathbf{S}$ is counted as two terms, some of which are equal to one another. All of them are evaluated in Tables I and II of Paper I.

TABLE IV. Shapes of correlations in $\cos\beta=\mathbf{Q}\cdot\mathbf{a}$ in low-energy production of fermion resonances with decay products of spin 0^- and spin $\frac{1}{2}^+$. Only the lower orbital term in the decay is taken into account. See Sec. V.3.

Spin and parity	Sideward $Z_s(\beta)$	Forward and back $Z_f(\beta)$
$\frac{1}{2}^\pm$, $\frac{3}{2}^-$	flat	flat
$\frac{3}{2}^+$	valley	hill
$\frac{5}{2}^-$	hill	valley
$\frac{5}{2}^+$	hill	hill

They are easily gathered together to yield

$$M_D:M_D^* = (\mathbf{E}^*\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{E}) \times[(3j-3)|y|^2+3j(yz^*+zy^*)+2j|z|^2] + (\mathbf{E}^*\cdot\mathbf{E})(j+2)|y|^2. \quad (5.24)$$

To express the counting rate in terms of observable momenta, we multiply (5.23) by (5.7) and average over spins. One sees that because (5.23) is independent of $\boldsymbol{\sigma}$ and symmetric under $\mathbf{E}\leftrightarrow\mathbf{E}^*$ the result is independent of $\mathbf{a}''$.

Hence, (5.9) can be used directly instead of (5.7). We see that

$$\sum_{b\text{ spin}} (\mathbf{a}\cdot\mathbf{E})[3(\mathbf{E}^*\cdot\mathbf{a}')(\mathbf{a}'\cdot\mathbf{E})+(\mathbf{E}^*\cdot\mathbf{E})](\mathbf{E}^*\cdot\mathbf{a}) = 1+3\cos^2\theta, \quad (5.25)$$

$$\sum_{b\text{ spin}} \mathbf{E}\cdot[3(\mathbf{E}^*\cdot\mathbf{a}')(\mathbf{a}'\cdot\mathbf{E})+(\mathbf{E}^*\cdot\mathbf{E})]\mathbf{E}^* = 6. \quad (5.26)$$

Hence

$$d\sigma(\theta) = \{(1+3\cos^2\theta)[(3j-3)|y|^2 +3j(yz^*+zy^*)+2j|z|^2] +6(j+2)|y|^2\}\, d\cos\theta. \quad (5.27)$$

To compute correlations for the $\frac{1}{2}^-$ sequence, it is convenient to use, *instead* of the form in Table I, the following:

$$M_D = \{yT^{j+1}(\mathbf{a}\cdots\mathbf{a}\mathbf{E}^*)+z(\mathbf{E}^*\cdot\mathbf{a})T^{j+1}(\mathbf{a})\}\mathcal{Q}_\sigma \quad (5.28)$$

where, acting upon tensors of rank $j+1$, $\mathcal{Q}_\sigma$ is the "lowering" operator

$$\mathcal{Q}_\sigma = 1-\mathcal{P}_\sigma = (j+1-\boldsymbol{\sigma}\cdot\mathbf{S})(2j+3)^{-1}. \quad (5.29)$$

The advantage of representing spin $j+\frac{1}{2}$ with tensors of rank $j+1$ is that the eight matrix elements of $M_D:M_D^*$ are the same as those for the previous parity case (with $j\to j+1$) and have already been evaluated. The distribution for the $\frac{1}{2}^-$ sequence, with y, z defined by (5.28), is

$$d\sigma(\theta) = \{(1+3\cos^2\theta)[-j(j+3)|y|^2 +j(j+1)(yz^*+y^*z)+2(j+1)^2|z|^2] +6j(j+2)|y|^2\}\, d\cos\theta. \quad (5.30)$$

For the cases $\frac{1}{2}^+$, $\frac{1}{2}^-$, $y=0$ and the correlation is $1+3\cos^2\theta$ in agreement with (5.16), (5.18).

If, for spins $\geqq\frac{3}{2}$, we drop the z terms, as has been done in the previous subsections, we get

$$d\sigma(\theta) = \left(1+\frac{j-1}{j+1}\cos^2\theta\right) d\cos\theta; \quad (\tfrac{3}{2}^-, \tfrac{5}{2}^+, \cdots), \quad (5.31)$$

and

$$d\sigma(\theta) = \left(1-\frac{3j+9}{5j+9}\cos^2\theta\right) d\cos\theta; \quad (\tfrac{3}{2}^+, \tfrac{5}{2}^-, \cdots). \quad (5.32)$$

6. The First Moment of the Decay Distribution

We shall not attempt a general moment analysis of the decay distribution as was done in Secs. III and IV. The zeroth-order moment was calculated in the previous subsection. We now calculate the first moment and the projected cross sections $\sigma_{1MM'}$. For parity reasons, these will be zero for $M'=\pm1$, and nonzero for $M'=0$, corresponding to a first moment of the production density matrix in the direction of the production normal.

We first separate $\mathbf{E}$ into two parts:

$$\mathbf{E} = \mathbf{a}(\mathbf{a}\cdot\mathbf{E})+\mathbf{F}; \quad \mathbf{F} = -\mathbf{a}(\mathbf{a}\cdot\mathbf{E})+\mathbf{E}, \quad (6.1)$$

so that $\mathbf{a}\cdot\mathbf{F}=0$, $\mathbf{a}\times\mathbf{F}=\mathbf{a}\times\mathbf{E}$. Then, the decay amplitudes are taken as

$$M_D = \mathfrak{M}^j\mathcal{P}_\sigma, \quad (\tfrac{1}{2}^+, \tfrac{3}{2}^-, \cdots) \quad (6.2a)$$

$$M_D = \mathfrak{M}^{j+1}\mathcal{Q}_\sigma, \quad (\tfrac{1}{2}^-, \tfrac{3}{2}^+, \cdots) \quad (6.2b)$$

with

$$\mathfrak{M}^j = (y+z)(\mathbf{E}^*\cdot\mathbf{a})T^j(\mathbf{a})+yT^j(\mathbf{a}\cdots\mathbf{a}\mathbf{F}^*). \quad (6.3)$$

Let $\mathbf{S}$ be the angular-momentum operator on tensors of rank j or $j+1$, depending on which parity sequence is being treated, so that the total angular-momentum operator is represented by $\mathbf{S}+\frac{1}{2}\boldsymbol{\sigma}$ for either sequence. The component of the first moment of the decay density matrix along $\mathbf{Q}$ is then

$$\mathbf{Q}\cdot\boldsymbol{\Phi} = \mathfrak{M}^j:\mathcal{P}_\sigma\mathbf{Q}\cdot(\mathbf{S}+\tfrac{1}{2}\boldsymbol{\sigma})\mathfrak{M}^{j*}, \quad (\tfrac{1}{2}^+, \tfrac{3}{2}^-, \cdots) \quad (6.4a)$$

or

$$\mathbf{Q}\cdot\boldsymbol{\Phi} = \mathfrak{M}^{j+1}:\mathcal{Q}_\sigma\mathbf{Q}\cdot(\mathbf{S}+\tfrac{1}{2}\boldsymbol{\sigma})\mathfrak{M}^{j+1}, \quad (\tfrac{1}{2}^-, \tfrac{3}{2}^+, \cdots). \quad (6.4b)$$

(Note that $\mathbf{S}+\frac{1}{2}\boldsymbol{\sigma}$ must commute with $\mathcal{P}_\sigma$ and $\mathcal{Q}_\sigma$.) Using the commutation properties of $\mathbf{S}$, we find that

$$(2j+1)\mathcal{P}_\sigma\mathbf{Q}\cdot(\mathbf{S}+\tfrac{1}{2}\boldsymbol{\sigma}) = \tfrac{1}{6}(2j+3)(j+1)\mathbf{Q}\cdot\boldsymbol{\sigma}+\tfrac{1}{2}(2j+3)(\mathbf{Q}\cdot\mathbf{S}) +T^2(\mathbf{Q}\boldsymbol{\sigma}):T^2(\mathbf{S}), \quad (6.5a)$$

$$(2j+3)\mathcal{Q}_\sigma\mathbf{Q}\cdot(\mathbf{S}+\tfrac{1}{2}\boldsymbol{\sigma}) = -\tfrac{1}{6}(2j+1)(j+1)\mathbf{Q}\cdot\boldsymbol{\sigma}+\tfrac{1}{2}(2j+1)\mathbf{Q}\cdot\mathbf{S} -T^2(\mathbf{Q}\boldsymbol{\sigma}):T^2(\mathbf{S}), \quad (6.5b)$$

where (6.5a) operates on the jth-rank tensors and (6.5b) operates on $(j+1)$th-rank tensors.

The evaluation of $\mathbf{Q}\cdot\boldsymbol{\Phi}$ is then reduced to the evaluation of matrix elements of tensors in $\mathbf{S}$. These are given in Table III of I, lines one through six. We find, apart from a common normalization,

$$\mathfrak{M}^j:\mathfrak{M}^{j*} = (\mathbf{E}^*\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{E})|y+z|^2 +|y|^2(j+1)(2j)^{-1}\mathbf{F}^*\cdot\mathbf{F}, \quad (6.6)$$

$$\mathfrak{M}^j:\mathbf{S}\mathfrak{M}^j = -\tfrac{1}{2}j(j+1) \times\{2\,\mathrm{Im}[y(y+z)^*(\mathbf{E}^*\times\mathbf{a})(\mathbf{a}\cdot\mathbf{E})] +ij^{-1}|y|^2\mathbf{F}^*\times\mathbf{F}\}, \quad (6.7)$$

$$\mathfrak{M}^j:T^2(\mathbf{S})\mathfrak{M}^j = -\tfrac{1}{2}j(j+1) \times\{|y+z|^2(\mathbf{E}^*\cdot\mathbf{a})(\mathbf{a}\cdot\mathbf{E})T^2(a) +2\,\mathrm{Re}[y^*(y+z)(\mathbf{E}^*\cdot\mathbf{a})j^{-1}T^2(\mathbf{a}\mathbf{F})] +|y|^2[(j+1)(2j)^{-1}T^2(\mathbf{F}^*\mathbf{F}) +3(j-1)(j+2)(4j^2)^{-1} \times(\mathbf{F}^*\cdot\mathbf{F})T^2(\mathbf{a})]\}. \quad (6.8)$$

Of course the $\boldsymbol{\sigma}$'s in (6.5) fit in between $\mathbf{E}^*$ and $\mathbf{E}$ in (6.6)–(6.8).

If these three equations are multiplied by (5.7) and averaged over b spin, it is found that only terms linear in $\mathbf{a}''$ survive. Hence, it is reasonable to look at the dependence of $\mathbf{Q}\cdot\boldsymbol{\Phi}$ and σ_{1M0} on $\mathbf{a}$, $\mathbf{a}''$, but not on $\mathbf{a}'$ by using (5.11) rather than (5.7). We obtain

$$\mathbf{Q}\cdot\boldsymbol{\Phi}=X^{(1)}|y+z|^2+X^{(2)}\,\mathrm{Re}y(y+z)^*+X^{(3)}|y|^2 \quad (6.9)$$

where, for the $\frac{1}{2}^+$ sequence,

$$X^{(1)}=8(j+1)(\mathbf{Q}\cdot\mathbf{a}'')-4(2j+1)(\mathbf{Q}\cdot\mathbf{a})\mathbf{a}''\cdot\mathbf{a}, \quad (6.10a)$$

$$X^{(2)}=(20j^2+30j+2)(\mathbf{Q}\cdot\mathbf{a}'')-(20j^2+30j-\tfrac{2}{3})(\mathbf{Q}\cdot\mathbf{a})(\mathbf{a}''\cdot\mathbf{a}), \quad (6.10b)$$

$$X^{(3)}=j^{-1}(-3j^2+22j+15)(\mathbf{Q}\cdot\mathbf{a})(\mathbf{a}''\cdot\mathbf{a})+j^{-1}(20j^2+16j-9)(\mathbf{Q}\cdot\mathbf{a}''), \quad (6.10c)$$

and, for the $\frac{1}{2}^-$ sequence,

$$X^{(1)}=-4(j+1)^3(\mathbf{Q}\cdot\mathbf{a}'')+2(j+1)^2(j+2)(\mathbf{Q}\cdot\mathbf{a})(\mathbf{a}''\cdot\mathbf{a}), \quad (6.11a)$$

$$X^{(2)}=(j+1)(j+2)[(10j^2+15j+6)(\mathbf{Q}\cdot\mathbf{a}'')-(10j^2+15j+14/3)(\mathbf{Q}\cdot\mathbf{a})(\mathbf{a}''\cdot\mathbf{a})], \quad (6.11b)$$

$$X^{(3)}=(j+2)[(2j^2+3j-\tfrac{1}{2})(\mathbf{Q}\cdot\mathbf{a}'')-(j^2+3j-\tfrac{7}{2})(\mathbf{Q}\cdot\mathbf{a})(\mathbf{a}''\cdot\mathbf{a})]. \quad (6.11c)$$

Let a basis be defined in the decay configuration by

$$\mathbf{a}=\mathbf{n}^3,$$
$$\mathbf{a}''=\mathbf{n}^1\sin\theta''+\mathbf{n}^3\cos\theta'', \quad \cos\theta''=\mathbf{a}''\cdot\mathbf{a}.$$

Then the projected cross sections for the $\frac{1}{2}^-$ sequence are

$$d\sigma_{100}=R\cos\theta''[4|y+z|^2+(8/3)\,\mathrm{Re}y(y+z)^*+j^{-1}(17j^2+38j+6)|y|^2]\,d\cos\theta'', \quad (6.12a)$$

$$d\sigma_{1-10}=-d\sigma_{110}=(R/\sqrt{2})\sin\theta''[8(j+1)|y+z|^2+(20j^2+30j+\tfrac{2}{3})\,\mathrm{Re}y(y+z)^*\times j^{-1}(20j^2+16j-9)|y|^2]\,d\cos\theta''. \quad (6.12b)$$

The formulas for the $\frac{1}{2}^-$ sequence are

$$d\sigma_{100}=R\cos\theta''[-2j(j+1)^2|y+z|^2+\tfrac{4}{3}(j+1)(j+2)\,\mathrm{Re}y(y+z)^*+(j+2)(j^2+3)|y|^2]\,d\cos\theta'', \quad (6.13a)$$

$$d\sigma_{1-10}=d\sigma_{110}=(R/\sqrt{2})\sin\theta''[-4(j+1)^3|y+z|^2+(j+1)(j+2)(10j^2+15j+9)\,\mathrm{Re}y(y+z)^*+\tfrac{1}{2}(j+2)(4j^2+6j-1)|y|^2]\,d\cos\theta''. \quad (6.13b)$$

The ratios of the $M=0$ to the $M=\pm1$ terms are independent of the production and give information on the decay constants. One may hope that the spin and parity determination need not rely on formulas of this complexity. Once the spin and parity are known, however, such formulas may be needed if one wishes to determine the decay constants. The reader will appreciate why we do not wish to present the complete moment analysis.

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CP Violation in Nonleptonic K^0 Decays*

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The nonleptonic K^0 decays are examined on the basis of the CPT theorem and unitary symmetry without the requirement of CP invariance. It is shown that the present model (based on the CPT theorem and unitary symmetry) is consistent with the various experimental branching ratios of $K\to2\pi$ modes, if CP invariance is almost maximally violated. Further, the decays $K_1^0\to3\pi^0$ and $K_2^0\to3\pi^0$ are forbidden by unitary symmetry in the framework of the boson pole model, even if CP invariance is violated.

Apparent violation of CP invariance which appears in the decay mode $K_2^0\to\pi^+\pi^-$ has been reported.[1] This led to a number of attempts to explain the experimental result without CP violation.[2] We examine the interrelation between CP invariance and unitary symmetry (SU_3 invariance) in which the nonleptonic Lagrangian behaves as a member of the **8** and **27** representations and the strong interactions are invariant under the transformations of the group SU_3. It is convenient to introduce spurions of $I=\frac{1}{2}$ and $I=\frac{3}{2}$ so as to express the K decay modes in terms of SU_3 channel

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