Field Theory of the Two-Pion System (S and D Waves)

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This is a sequel to a previous paper, designated as (I), which dealt primarily with the two-pion P wave in the $\lambda\phi^4$ model. Here, as in (I), we make use of a recently developed series expansion, applying the procedure to S and D waves. With an input of (π mass) $=1$, (ρ mass) $=5.53$, our results for the isospin-T, spin-J scattering lengths $a_J{}^T$ and the Chew-Mandelstam coupling constant λ are: $a_0{}^0=-0.72$, $a_0{}^2=-0.40$, $a_1{}^1=+0.032$, $a_2{}^0=+0.0032$, $a_2{}^2=0.0026$, and $\lambda=+0.22$. The results for $a_0{}^{0,2}$ and λ are improved here [as compared with those in (I)] by a higher order correction, but with little change; $a_1{}^1$ is as quoted in (*I*); the $a_2{}^{0,2}$ are new results. We also calculate the S-wave phase shifts $\delta_0{}^T$ and find that $\delta_0{}^0$ and $\delta_0{}^2$ go through $-\frac{1}{2}\pi$ on the way down at energies of 760 and 630 MeV, respectively. We finally calculate the D-wave phase shifts: The f_0 resonance is found at 1950 MeV, and a corresponding $T=2$, D-wave resonance is found at 1520 MeV. The substantial discrepancy with experiment for the f_0 is attributed to the need for considering more terms in our expansion as energy increases, as well as to a diminished reliability of the $\lambda\phi^4$ model at high energy.

1. INTRODUCTION

IN a previous paper,[1] to be designated as (I), we applied a recently developed series expansion[2] to the calculation of the P-wave phase shift for the two-pion system in the $\lambda\phi^4$ model. In particular, we obtained the ρ-meson width as a function of its position. Recent experimental data[3] have brought the measured and computed widths into good agreement. While a coincidence is not to be ruled out, we consider this agreement sufficiently encouraging to help justify a further exploration of the two-pion system by the same method.

An additional motivation for the present study is the availability of results concerning the third order Feynman diagrams for pion-pion scattering. These are computed in (I), but no use is made there of the S- and D-wave information they contain. This is done here.

Finally, one should mention the interest of the present calculations as a testing ground for the new series expansion being used. As explained later, much more insight into the general properties and limitations of the approximation scheme can be gained through these results than could be gained through (I). Several of these properties undoubtedly carry over to models other than $\lambda\phi^4$.

Our notation follows (I); equation or figure numbers labeled (I) refer to that article; those labeled (A) to (D) refer to the appropriate Appendix in the present paper. The reader is referred to (I) for most bibliographic references.

The rather detailed mathematical Appendices are partly meant as reference material for any subsequent work on the $\lambda\phi^4$ theory. Being the results of a straightforward perturbation expansion, they should be of use in most field-theoretic investigations, no matter what "improvement scheme" is being contemplated.

2. THE THREE-TERM APPROXIMATION

A prescription for optimizing the information contained in the first few terms of any Born series was given in Ref. 2. In (I), this prescription was applied to the first two Born terms only. In the present article it is applied to the case where three Born terms are used (S waves), as well as to the two-term case (D waves).

Let $G(g,x)$ be an unknown function of a coupling constant g and a dynamical variable x. If we assume the formal series expansion

$$G=gG^{(1)}+g^2G^{(2)}+g^3G^{(3)}+\cdots, \qquad (2.1)$$

then the optimized three-term approximation for G is obtained by solving the nonlinear ordinary differential equation

$$(G/G^{(1)})'=A(G/G^{(1)})^2+B(G/G^{(1)})^3, \qquad (2.2)$$

where A and B are the known functions

$$A=(G^{(2)}/G^{(1)})', \qquad (2.3)$$

$$B=[G^{(3)}/G^{(1)}-(G^{(2)}/G^{(1)})^2]', \qquad (2.4)$$

and where the prime denotes differentiation with respect to x. The following remarks are in order concerning (2.2):

(a) It is invariant under a change of variable of the type $x\rightarrow X(x)$.

(b) It is independent of g. This is undoubtedly connected with the fact that coupling renormalization is taken care of automatically. Anticipating slightly, we may state that all coefficients A and B in the two-pion

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† Research supported by the National Science Foundation.
[1] M. Alexanian and M. Wellner, Phys. Rev. **137**, B 155 (1965). This paper contains further references to the literature.
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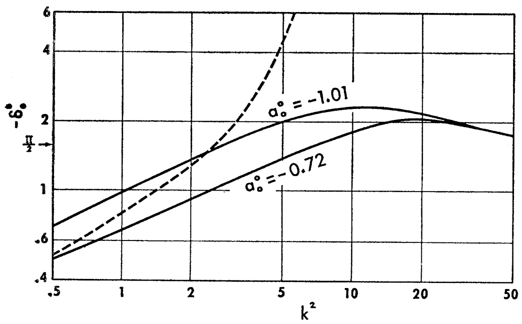

FIG. 1. The magnitude of the S-wave, $T=0$ phase shift δ_0^0, plotted against k^2. The horizontal and vertical scales are logarithmic. The solid curves are the result of integrating the three-term equation for scattering lengths a_0^0 chosen as -0.72 and -1.01, the first value corresponding to a present-day ρ-meson fit. The dashed curve shows the two-term result for $a_0^0=-0.72$. The small-k behavior is $\delta_0^0\approx a_0^0k$ for each curve. Both solid curves tend to $55\pi/148$ as $k^2\to\infty$. The dashed curve has a vertical asymptote at $k^2=7.2$.

case turn out to be cutoff-independent.[4] The role of a physical coupling constant is assumed by the (adjustable) constant of integration.

(c) One category of cases where (2.2) is likely to provide a good approximation is whenever x is not too far from a region where G is real and analytic. In detailed studies, reasonable estimates can usually be given for the meaning of "not too far."

(d) Equation (2.2) cannot in general be solved exactly. In the cases we are considering we must resort to a numerical method of solution.

3. CHEW-MANDELSTAM COUPLING CONSTANT; S-WAVE SCATTERING LENGTHS

Using the mass of the ρ meson as an input, the parameters λ and a_0^T were computed in (I) with the third-order Born contributions to the S waves being ignored. In this section we present the result of including these terms.[5]

In order to determine λ, we can use either one of the (exact) crossing relations (I.5.16), (I.5.17). Which one we chose made no difference in (I), and it will be seen that this feature persists here. For the sake of illustration, let us use (I.5.16):

$$\partial_I\gamma_1=\partial_{II}\gamma_0. \qquad (3.1)$$

We still obtain $\partial_I\gamma_1$ from the first relation (I.5.18). However, $\partial_{II}\gamma_0$ must now be related to the definition (I.5.23) via a third-degree differential equation of the type (2.2). Details of this equation are given by (B1) to (B7).

If expressed in terms of the symmetry-point quantities $a(\frac{4}{3})$, $a'(\frac{4}{3})$, $b'(\frac{4}{3})$, the equations for $T=0$ and $T=2$ yield upon comparison

$$\partial_{II}\gamma_0=-2\partial_{II}\gamma_2, \qquad (3.2)$$

which bears out the fact that either crossing relation may be used. For $T=0$ one obtains

$$-(\tfrac{1}{10})\partial_{II}\gamma_0=64\times0.115\lambda^2+256\times0.180\lambda^3/\pi. \qquad (3.3)$$

In solving for λ, we keep only the case which reduces to the perturbation limit as $\lambda\to0$. Inserting the first formula (I.5.18) for the left side of (3.3), we obtain

$$(2\lambda/\pi)[4+12.5(2\lambda/\pi)]^{1/2}=(\kappa+0.575)^{-1} \qquad (3.4)$$

[cf. the less accurate Eq. (I.5.24)], where κ is a parameter related to the ρ-meson mass (see Fig. I.4). Numerically, the result is

$$\lambda=0.22, \qquad (3.5)$$

to be compared with the value 0.24 found in (I). This change is almost entirely due to the change in formula rather than to the updating of the experimental ρ mass.

Little need be said about the new S-wave scattering lengths. The relevant differential equations are discussed in Appendix B, Sec. (i). Numerically, the cubic term is so small that its inclusion modifies the result by something like 1%. Therefore, almost the whole change in the a_0^T is due to the change in λ, and we find

$$a_0^0=-0.72,\quad a_0^2=-0.40. \qquad (3.6)$$

For any interaction strengths in this neighborhood, the relation between the a_0^T and λ may be taken over unmodified from (I).

4. S-WAVE PHASE SHIFTS

In (I), Sec. 6, it was argued that the two-term formula was insufficient for a reliable calculation of the S-wave phase shifts above the immediate neighborhood of the elastic threshold. Here the three-term formula will be used to this end. The calculation consists of integrating Eq. (2.2) [or (B1)], specialized according to the pre-

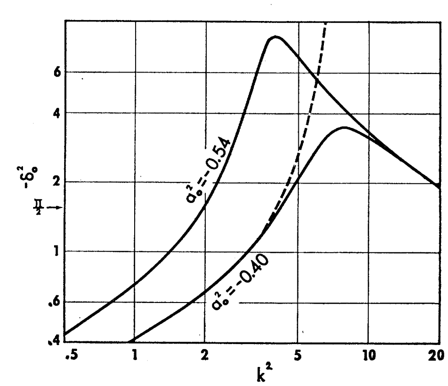

FIG. 2. Log-log plot of the S-wave, $T=2$ phase shift δ_0^2 against k^2. The meaning of the curves is as in Fig. 1. The scattering lengths a_0^2 are -0.40 and -0.54, the first being a ρ-meson fit. Both solid curves approach $11\pi/74$ for $k^2\to\infty$; the vertical asymptote for the dashed curve is at $k^2=7.0$.

[4] For a general discussion, see M. Alexanian, Lawrence Radiation Laboratory Progress Report, May 1965 (unpublished), available as UCRL-14360.

[5] See also M. Alexanian, doctoral dissertation, Indiana University, 1964 (unpublished).

scriptions of Appendix B, Sec. (iv). The constant of integration is determined by the "initial condition" at $k^2=0$, namely the scattering length a_0^T. The results of the integration for two different scattering lengths and for $T=0$, 2, are shown in Figs. 1 and 2. A result of the two-term approximation is also shown for comparison.

Within the context of the three-term formula, some exact statements can be made concerning the high-k behavior of the phase shifts. An examination of (B1) shows that, if the function $y=(2\pi)^{-2}\delta_0/\delta_0^{(1)}$ [see (B14)] monotonically approaches a finite nonzero value for $k^2\to\infty$, as the numerical evaluation suggests, then, asymptotically,

$$y\approx -A/B\ (k^2\to\infty). \tag{4.1}$$

In particular, using the explicit formulas (B18)–(B21), we obtain in this way for δ_0^T

$$\delta_0^0\to -55\pi/148,\quad \delta_0^2\to -11\pi/74 \tag{4.2}$$

as $k^2\to\infty$. These values do not depend on the strength of the interaction. The ratio y itself becomes curiously independent of isospin.

5. D-WAVE PHASE SHIFTS FOR GENERAL SCATTERING LENGTHS

The first-order Feynman diagram for π-π scattering contributes S waves only. Therefore, if our information consists of the diagrams illustrated in Fig. (I.1), then the formula for the D-wave phase shift is obtained via the two-term approximation in a fashion entirely similar to that for the P wave. The reader is referred to (I), Secs. 4 and 5, for a discussion of the method, the basic formula being (I.4.4). In the present case, we use the following perturbative results for the D-wave phase shifts δ_2.

Second order:

$$\delta_2^{(2)}=-\begin{pmatrix}15\\9\end{pmatrix}\frac{ka_2(k^2)}{16(2\pi)^3(k^2+1)^{1/2}},\quad T=\begin{pmatrix}0\\2\end{pmatrix}. \tag{5.1}$$

Third order:

$$\delta_2^{(3)}=\frac{-55}{16(2\pi)^5}\frac{k}{(k^2+1)^{1/2}}[b_2(k^2)+\tfrac{1}{2}(a^2)_2(k^2)]+\text{const}\times a_2(k^2),\quad (T=0),$$

$$\delta_2^{(3)}=\frac{-7}{4(2\pi)^5}\frac{k}{(k^2+1)^{1/2}}[b_2(k^2)+(43/56)(a^2)_2(k^2)]+\text{const}\times a_2(k^2).\quad (T=2). \tag{5.2}$$

The constants multiplying a_2 in the last two equations are cutoff-dependent but turn out to be irrelevant in our method. The functions a_2, $(a^2)_2$, and b_2 are defined by (A1) and are given explicitly by (A7). Rather than work with the functions in square brackets in (5.2), it is more convenient to define related functions with a higher order threshold behavior, i.e., k^6 instead of k^4. This is done by adding a suitable multiple of a_2. Expansions (C8) are used as a guide. We thus define

$$Y_0(k^2)=b_2+\tfrac{1}{2}(a^2)_2-(1/12)(4\pi^2+7)a_2,$$
$$Y_2(k^2)=b_2+(43/56)(a^2)_2-(1/48)(16\pi^2+43)a_2, \tag{5.3}$$

with threshold behavior

$$Y_0(k^2)\approx-(23-2\pi^2)k^6/1575,$$
$$Y_2(k^2)\approx-(1011-84\pi^2)k^6/66150. \tag{5.4}$$

The resulting phase shifts can now be written as

$$\delta_2^0=\frac{-135\pi}{242}\frac{k}{(k^2+1)^{1/2}}\frac{D}{(\zeta_0-Y_0/D)^2},$$
$$\delta_2^2=\frac{-729\pi}{1568}\frac{k}{(k^2+1)^{1/2}}\frac{D}{(\zeta_2-Y_2/D)^2}, \tag{5.5}$$

for $T=0$, 2, respectively. The notation

$$D(k^2)\equiv a_2(k^2) \tag{5.6}$$

is introduced so as to give only isospin indices in (5.5), and to bring out the analogy with the P-wave result (I.5.1). The real parameters ζ_0 and ζ_2 are independent of k and are still to be determined. If the two-term Born limit is to be continuously reachable from (5.5) by making $|\zeta|$ very large, then we must always take the same sign for ζ_0, ζ_2, and λ. Hence ζ_0 and ζ_2 are positive. The functions $Y_{0,2}/D$ are shown in Fig. 3.

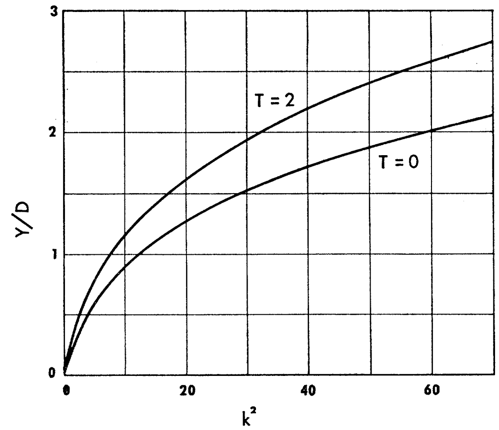

FIG. 3. The functions Y_0/D and Y_2/D, plotted against k^2. [See Eqs. (5.3)–(5.5).]

Even without the knowledge of $\zeta_{0,2}$, we can already conclude from their positive sign and from the qualitative behavior of Y/D that the phase shifts $\delta_2^{0,2}$ exhibit a monotonic increase with energy, reaching infinity at a finite value of k^2. For the interpretation of this behavior, we refer the reader to (I), Sec. 5b: we consider the passage through $\pi/2$ to indicate a true resonance, and discard all results above that energy as being outside the range of validity of our approximation.

6. *D*-WAVE SCATTERING LENGTHS AND RESONANCE PARAMETERS

We next turn to the problem of determining the parameters $\zeta_{0,2}$. This amounts to finding the D-wave scattering lengths

$$a_2^{0,2} \equiv \lim_{k\to 0} (\sin\delta_2^{0,2})/k^5. \tag{6.1}$$

The relation between $\zeta_{0,2}$ and $a_2^{0,2}$ is obtained from (5.5):

$$\zeta_0 = (1/11)(3\pi/5a_2^0)^{1/2}, \quad \zeta_2 = (9/140)(\pi/a_2^2)^{1/2}, \tag{6.2}$$

where the threshold behavior of $D(=a_2)$ is taken from (C8).

We start out by specializing the partial-wave expansion (I.2.1) to the single variable s (or k): setting $t=u$, i.e., $\theta=\pi/2$, we obtain

$$(k^2+1)^{1/2} f_k(\tfrac{1}{2}\pi) = (k^2+1)^{1/2} k^{-1} \times[e^{i\delta_0}\sin\delta_0 - (5/2)e^{i\delta_2}\sin\delta_2 + \cdots] \tag{6.3}$$

valid for isospins 0 and 2. [The over-all factor $(k^2+1)^{1/2}$ has been inserted for later convenience.] We then expand both sides of (6.3) in powers of k. To order k^4, only the $l=0$ and $l=2$ terms contribute to the right side. Setting

$$(k^2+1)^{1/2} f_k(\tfrac{1}{2}\pi) \equiv \psi(k), \quad (k^2+1)^{1/2} k^{-1} e^{i\delta_0}\sin\delta_0 \equiv \chi(k), \tag{6.4}$$

we obtain, for successive derivatives at $k=0$,

$$\psi = \chi (= a_0^T), \tag{6.5}$$

$$\psi' = \chi', \quad \psi'' = \chi'', \quad \psi''' = \chi''', \tag{6.6}$$

$$\psi^{\text{IV}} = \chi^{\text{IV}} - 60 a_2^T. \tag{6.7}$$

Equations (6.5)–(6.7) are still exact. Equation (6.7) shows how, if we possess trustworthy approximations for ψ and χ near threshold, the required a_2^T can be determined. We note that, according to Sec. 2, the threshold constitutes the approximate upper end of the favorable region in which to apply the improved-convergence scheme to ψ and χ. These have no adjustable parameters left if we use (6.5), and hence can be fully determined. Summarizing the procedure in one sentence, we say that a_2^T is determined in terms of a_0^T by matching the derivatives of full and partial amplitudes at threshold.

Three further remarks should be made at this point: (a) The specialization to $\theta=\frac{1}{2}\pi$ (rather than to some other angle) is not compulsory, but turns out to be calculationally most convenient in the present case. (b) Equations (6.6), if not satisfied automatically, cannot be enforced by any adjustment whatsoever, and thus constitute a powerful check on the consistency of our procedure. (c) All threshold results, including the solutions of the nonlinear differential equations, will turn out to have the usual analyticity properties, and therefore it does not matter whether we perform the matching just below or just above threshold. We shall choose the latter alternative for clarity: the presence of real and imaginary parts provides an easy classification of terms [see (C9) and (C10)].

We now proceed to a calculation of the quantities occurring in (6.5)–(6.7). We note that ψ may be written

$$\psi = -(5/16\pi)(\gamma_0/\gamma_0^{(1)}), \quad (T=0) \qquad \psi = -(1/8\pi)(\gamma_2/\gamma_2^{(1)}), \quad (T=2) \tag{6.8}$$

[see (I.3.13)], and that, similarly,

$$\chi = -\begin{pmatrix} 5/16\pi \\ 1/8\pi \end{pmatrix} \frac{e^{i\delta_0}\sin\delta_0}{(e^{i\delta_0}\sin\delta_0)^{(1)}}, \quad T=\begin{pmatrix} 0 \\ 2 \end{pmatrix} \tag{6.9}$$

[see (D1)]. Hence ψ and χ, up to constant factors, may be identified with y of (B1), under the interpretation of Secs. B, (ii) and B, (iii), respectively. Taking successive derivatives of (B1), and noting that some powers of k are missing in (C9) and (C10), we obtain, at $k=0$, from (B1),

$$\begin{aligned} y' &= Ay^2, \\ y'' &= A'y^2 + (2A^2+B')y^3, \\ y''' &= A''y^2 + 6AA'y^3 + (6A^3+8AB')y^4, \\ y^{\text{IV}} &= A'''y^2 + (8AA''+6A'^2+B''')y^3 \\ &\quad + (36A^2A'+15A'B')y^4 \\ &\quad + (24A^4+58A^2B'+9B'^2)y^5. \end{aligned} \tag{6.10}$$

We note that only the first three leading terms of A, and the leading term of B, contribute to y', y'', and y''' in (6.10). Comparison of these terms in (C9) and (C10) immediately shows that all three Eqs. (6.6) are automatically and *exactly* satisfied if (6.5) is.

Turning next to the evaluation of y^{IV} from (6.10), we find in a similar way that the coefficients of y^4 and y^5 match automatically between ψ and χ. The remaining terms, inserted in (6.7), yield for $T=0$, 2

$$\begin{aligned} a_2^0 &= \frac{4(a_0^0)^2}{375\pi}\left[1-\left(44\pi-\frac{271}{\pi}\right)\frac{a_0^0}{45}\right], \\ a_2^2 &= \frac{(a_0^2)^2}{25\pi}\left[1-\left(56\pi-\frac{1007}{2\pi}\right)\frac{a_0^2}{27}\right]. \end{aligned} \tag{6.11}$$

Taking our estimates of $a_0^{0,2}$ from (3.6), we find

$$a_2^0 = 0.0032, \quad a_2^2 = 0.0026. \tag{6.12}$$

Together with (6.2) and (5.5), this implies a $T=0$, D-wave resonance at $k^2\approx 49$ (energy width ≈ 2.5), and a $T=2$, D-wave resonance at $k^2\approx 29$ (energy width ≈ 1.4).

7. SUMMARY AND DISCUSSION

A systematic third-order investigation of the two-pion system in the $\lambda\phi^4$ theory, using the new improved-convergence scheme, has been conducted in the case of S- and D-wave scattering. The first important result that has been obtained consists of an excellent confirmation of second-order calculations where these had *a priori* been considered reliable: The coupling constant and S-wave scattering lengths are—by strong interaction standards—almost unmodified in going to higher order. In particular, the repulsive nature of these low-energy S-wave interactions in the $\lambda\phi^4$ model would now seem hard to get rid of.

The S-wave phase shifts themselves, as shown by the lower curves in Figs. 1 and 2, give no sign of becoming attractive at any moderate energy. They do, however, produce very broad peaks in the cross section, namely upon crossing $-\frac{1}{2}\pi$ on the way down, at (760 ± 15) MeV for $T=0$ and at (630 ± 15) MeV for $T=2$. (The errors reflect the over-all inaccuracy of the numerical manipulations which connect these results to the input, i.e., to the rho mass.) Insofar as one can speak of widths, they are of the order of 500 MeV for both $T=0$ and $T=2$. These "peaks" would therefore hardly be seen as such in any cross-section measurement. Also, being due to a repulsion, they surely cannot be called resonances, let alone particles. For some recent experimental data connected with those phase shifts, see Refs. 6–9, as well as the bibliography quoted therein. Most authors seem to consider seriously only the possibility of attractive S waves.

One may now ask how well Wigner's inequality

$$d\delta/dk \gtrsim -r-1/2k\,, \qquad (7.1)$$

for an interaction of range r, is satisfied. It turns out that, for $T=0$, the minimum range r compatible with (7.1) is largest near $k^2\approx4$ and is $r\approx0.35$. On the other hand, for $T=2$, the corresponding figures are $k^2\approx6.5$ and $r\approx3.5$. This freakishly large value seems possible only because Wigner's theorem is not strictly applicable except in potential scattering and with a *total* cutoff at r. This rapid decrease of the phase shift has its origin in the coefficient $B(T=2)$ of the differential equation which governs its behavior. The role of B is to damp the decrease of δ. However (see Fig. 5), owing to a remarkable cancellation, especially near $k^2\approx1.3$, B is almost inoperative up to $k^2\approx4$. As a result, the second-order curve (dashed line) is very good up to that point.

The range in which our results for the S-wave phase shifts are believed reliable does not go beyond $k^2\approx30$ for $T=0$ and $k^2\approx11$ for $T=2$. This is based on the observation that no two solutions of the exact (or, for that matter, approximate) improved-convergence series of Ref. 2, can possibly cross[10]: the slope is uniquely defined if the series exists. Therefore, if the physically correct curves do cross (and hence, in general, if their family possesses an envelope), then the exact differential equation can be satisfied only by a function consisting of two pieces: the correct physical result up to the envelope, and from then on the envelope itself. The numerical results shown in Figs. 1 and 2, as well as the nonperturbative nature of the asymptote, are extremely suggestive of such a behavior, the suspected envelope being reached near the stated values of k^2.

As to the D-wave calculations, apart from illustrating the new technique of threshold matching, their main interest is qualitative. The presence of a $T=0$ resonance that can be identified with the f_0 is encouraging, as well as the fact that it lies much higher than the ρ. The parameters found are, for $T=0$, $E\approx1950$ MeV and $\Gamma\approx350$ MeV; for $T=2$, $E\approx1520$ MeV and $\Gamma\approx200$ MeV. Although the latter $T=2$ resonance (the f_2?) would have to be taken seriously if one could trust the $\lambda\phi^4$ model at this energy, it is conceivable that baryonic or strange processes would suppress it. The neglect of such processes is also undoubtedly responsible in part for the substantially wrong position of the f_0. Also, at such high energies it is likely that higher order diagrams must be included even within the $\lambda\phi^4$ context.

In conclusion, we wish to remark on two qualitative features apparent in this paper and in (I): the increase in any resonating mass as the coupling decreases, and the simultaneous broadening of the resonance width. The latter phenomenon is perhaps to be expected from the increase in available phase space.

ACKNOWLEDGMENTS

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APPENDIX A: PARTIAL-WAVE PROJECTIONS

Given a function $F(s)$ of a single variable s, we require its partial-wave projections $F_l(k^2)$ $(l=0,1,2,\cdots)$, defined by

$$F_l(k^2)=\int_{-1}^{1}P_l(z)F(2k^2(z-1))dz\,, \qquad (A1)$$

where P_l is the Legendre polynomial of order l. In this paper the function F can be a linear combination of a, a^2, and b, given by (I.3.11) and (I.3.12). After changing variables according to (I.3.10) and writing the P_l

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explicitly, we obtain the recursion relations

$$F_0=\tfrac{1}{2}k^{-2}\int_\alpha^1 (1-\nu^2)\nu^{-2}F(s)d\nu\,,$$

$$F_1=F_0-(1/4)k^{-}\int_\alpha^1 (1+\nu)(1-\nu)^3\nu^{-3}F(s)d\nu\,, \qquad \text{(A2)}$$

$$F_2=-2F_0+3F_1+(3/16)k^{-6}\times\int_\alpha^1 (1+\nu)(1-\nu)^5\nu^{-4}F(s)d\nu\,,$$

etc., where

$$s=-(1-\nu)^2/\nu \qquad \text{(A3)}$$

and

$$\alpha=2k^2+1-2(k^4+k^2)^{1/2}. \qquad \text{(A4)}$$

In this way the following results are obtained.

S waves:

$$a_0=\tfrac{1}{4}k^{-2}\ln^2\alpha-(1+k^{-2})^{1/2}\ln\alpha-1\,,$$

$$(a^2)_0=-a_0+k^{-2}\left[\int_\alpha^1 (1-\nu)^{-1}\ln^2\nu d\nu-(1/6)\ln^3\alpha\right]+\tfrac{1}{2}(1+k^{-2})\ln^2\alpha\,, \qquad \text{(A5)}$$

$$b_0=-a_0-(1/48)k^{-2}\ln^4\alpha+[\tfrac{1}{2}+(\tfrac{1}{2}-\pi^2/24)k^{-2}]\ln^2\alpha.$$

P waves [N. B. The function a_1 is called P in (I)]:

$$a_1=[\tfrac{1}{4}k^{-2}+\tfrac{1}{8}k^{-4}]\ln^2\alpha+\tfrac{1}{2}k^{-2}(1+k^{-2})^{1/2}\ln\alpha+(-\tfrac{1}{2}+\tfrac{1}{2}k^{-2})\,,$$

$$(a^2)_1=-\tfrac{5}{2}a_1+k^{-2}\left[\int_\alpha^1 (1-\nu)^{-1}\ln^2\nu d\nu-\tfrac{1}{6}\ln^3\alpha\right]-\tfrac{1}{8}k^{-2}\ln^2\alpha+\tfrac{1}{2}(1+k^{-2})^{1/2}\ln\alpha-\tfrac{1}{2}\,,$$

$$b_1=-(\tfrac{1}{2}+\pi^2/3)a_1-(1/48)(k^{-2}+k^{-4})\ln^4\alpha-\tfrac{1}{6}k^{-2}(1+k^{-2})^{1/2}\ln^3\alpha+(\pi^2/24-\tfrac{1}{8})k^{-2}\ln^2\alpha+\tfrac{1}{2}(1+k^{-2})^{1/2}\ln\alpha-(\tfrac{1}{6}\pi^2-\tfrac{1}{2}). \qquad \text{(A6)}$$

D waves (N.B. The function a_2 is also called D in Sec. 5):

$$a_2=[\tfrac{1}{4}k^{-2}+\tfrac{3}{8}k^{-4}+\tfrac{3}{16}k^{-6}]\ln^2\alpha+[k^{-2}+\tfrac{3}{4}k^{-4}](1+k^{-2})^{1/2}\ln\alpha+[-\tfrac{1}{6}+(5/4)k^{-2}+\tfrac{3}{4}k^{-4}]\,,$$

$$(a^2)_2=-\tfrac{4}{3}a_2+k^{-2}\left[\int_\alpha^1 (1-\nu)^{-1}\ln^2\nu d\nu-\tfrac{1}{6}\ln^3\alpha\right]-[(11/12)k^{-2}+(7/16)k^{-4}]\ln^2\alpha+[\tfrac{1}{6}-(7/4)k^{-2}](1+k^{-2})^{1/2}\ln\alpha+[\tfrac{1}{4}-(7/4)k^{-2}]\,,$$

$$b_2=-(25/12+\tfrac{1}{2}\pi^2)a_2-[(1/48)k^{-2}+\tfrac{1}{16}k^{-4}+(3/64)k^{-6}]\ln^4\alpha-[\tfrac{1}{4}k^{-2}+\tfrac{3}{8}k^{-4}](1+k^{-2})^{1/2}\ln^3\alpha+[(\tfrac{1}{12}\pi^2-17/48)k^{-2}+(\tfrac{1}{16}\pi^2-17/32)k^{-4}]\ln^2\alpha+[\tfrac{1}{6}+(\tfrac{7}{8}+\tfrac{1}{4}\pi^2)k^{-2}](1+k^{-2})^{1/2}\ln\alpha+[-(\pi^2/12-\tfrac{1}{8})+(13/8+\tfrac{1}{4}\pi^2)k^{-2}]. \qquad \text{(A7)}$$

The behavior of all these functions for small k is given in Appendix C.

APPENDIX B: COEFFICIENTS OF THE DIFFERENTIAL EQUATIONS

Let $y=y(x)$ be an unknown physical quantity, x standing for some given dynamical variable related to the center-of-mass energy of the two pions. In this paper, any such y which pertains to an S wave is computed by means of a differential equation of the form

$$y'=Ay^2+By^3\,, \qquad \text{(B1)}$$

where the prime indicates differentiation with respect to x, and where A and B are known functions of x. This Appendix lists the various y of interest, specifies the corresponding variable x and coefficients A, B, and finally records the analytic expressions and numerical results for such A and B as are needed explicitly.

We emphasize that (B1) is always of such a form that choosing a given variable x is entirely equivalent to choosing any differentiable function of x as a variable. [Thus, whether we use k^2, k, or $(k^2+1)^{1/2}$ as a variable is only a matter of convenience.] In the following, we only deal with isospins $T=0$, 2.

(i) Full Amplitudes below Threshold

We choose

$$y=(2\pi)^{-2}\gamma_T/\gamma_T^{(1)}\,, \qquad \text{(B2)}$$

where γ_T and $\gamma_T^{(n)}$ are taken from (I.2.4) and from (I.3.13)–(I.3.15), respectively. Here and in what follows the factor $(2\pi)^{-2}$ is included for convenience. We reduce the number of variables to one by taking

$$t=u=2-\tfrac{1}{2}s \qquad \text{(B3)}$$

and then choose $x=s$. The prime in (B1) thus corresponds to ∂_{11} of (I.5.15). We are interested in the range

$$\tfrac{4}{3}\leqslant s<4. \qquad \text{(B4)}$$

The coefficients are the real functions

$$A=(2\pi)^2(\gamma_T^{(2)}/\gamma_T^{(1)})'\,, \qquad \text{(B5)}$$

$$B=(2\pi)^4[\gamma_T^{(3)}/\gamma_T^{(1)}-(\gamma_T^{(2)}/\gamma_T^{(1)})^2]'. \qquad \text{(B6)}$$

An explicit calculation gives, at the symmetry point

$s=\frac{4}{3}$, and for $T=0$,

$$A=a'(\tfrac{4}{3})\approx -0.115\,,$$

$$B=2[-a'(\tfrac{4}{3})+b'(\tfrac{4}{3})-2a(\tfrac{4}{3})a'(\tfrac{4}{3})]\approx -0.180\,. \quad \text{(B7)}$$

(ii) Full Amplitudes above Threshold

Here again, we set $t=u$. It is simplest to take

$$x=k\equiv(\tfrac{1}{4}s-1)^{1/2}. \quad \text{(B8)}$$

Equations (B2), (B5), and (B6) can be taken over, if we keep in mind the new meaning of the prime. In the present case A and B are complex functions. Their behavior close to threshold is given in Appendix C.

(iii) S-Wave Amplitudes above Threshold

Let $\mathcal{F}_l$ be the partial-wave amplitude

$$\mathcal{F}_l=e^{i\delta_l}\sin\delta_l \quad \text{(B9)}$$

corresponding to an angular momentum l. (We suppress the isospin index.) Then, here,

$$y=(2\pi)^{-2}\mathcal{F}_0/\mathcal{F}_0{}^{(1)}. \quad \text{(B10)}$$

The superscript (n) refers, as always, to the nth-order Born term with respect to the bare coupling constant g_0 [see (I.3.1)]. The variable x is

$$x=k\,, \quad \text{(B11)}$$

the center-of-mass momentum. The coefficients are

$$A=(2\pi)^2(\mathcal{F}_0{}^{(2)}/\mathcal{F}_0{}^{(1)})'\,, \quad \text{(B12)}$$

$$B=(2\pi)^4[\mathcal{F}_0{}^{(3)}/\mathcal{F}_0{}^{(1)}-(\mathcal{F}_0{}^{(2)}/\mathcal{F}_0{}^{(1)})^2]'. \quad \text{(B13)}$$

The functions $A(T=0,\,2)$ are complex here. However, it is easily seen that, owing to a peculiar cancellation, the $B(T=0,\,2)$ are exactly real. (In verifying this, one uses the fact that δ_0 is real to third order in the coupling constant.) The threshold expansions of these A and B are given in Appendix C.

(iv) S-Wave Phase Shifts above Threshold

Here we set

$$y=(2\pi)^{-2}\delta_0/\delta_0{}^{(1)}. \quad \text{(B14)}$$

Taking as a variable

$$x=k^2\,, \quad \text{(B15)}$$

we define

$$A=(2\pi)^2(\delta_0{}^{(2)}/\delta_0{}^{(1)})'\,, \quad \text{(B16)}$$

$$B=(2\pi)^4[\delta_0{}^{(3)}/\delta_0{}^{(1)}-(\delta_0{}^{(2)}/\delta_0{}^{(1)})^2]'. \quad \text{(B17)}$$

These coefficients are real. Their explicit form, although somewhat unwieldy, is useful for numerical work. We find, with use of the auxiliary variable α [see (A4)], and by making use of the perturbative terms listed in Appendix D,

$$A(T=0)=\frac{2\alpha^2}{1-\alpha^2}\left[-\frac{3(1+\alpha)}{(1-\alpha)^3}\ln^2\alpha-\frac{5}{(1+\alpha)^2}\ln\alpha+\left(\frac{11}{2\alpha}-\frac{5}{1+\alpha}+\frac{6}{1-\alpha}\right)\right], \quad \text{(B18)}$$

$$\begin{aligned}B(T=0)=\frac{-2\alpha^2}{1-\alpha^2}\Bigg\{&\frac{22(1+\alpha)}{(1-\alpha)^3}\int_\alpha^1\frac{\ln^2\nu}{1-\nu}d\nu-\frac{(1+\alpha)(11+86\alpha+11\alpha^2)}{12(1-\alpha)^5}\ln^4\alpha\\&+\left[\frac{-22}{3(1-\alpha)^3}+\frac{9(1+\alpha)^2}{(1-\alpha)^4}+\frac{5(1+\alpha^2)}{(1-\alpha^2)^2}\right]\ln^3\alpha\\&+\left[\frac{15}{1-\alpha^2}-\left(\frac{11\pi^2}{6}-\frac{25}{2}\right)\frac{1+\alpha}{(1-\alpha)^3}\right]\ln^2\alpha\\&+\left[-\left(\frac{11\pi^2}{3}-30\right)\frac{1}{(1-\alpha)^2}+\frac{5\pi^2(1+\alpha^2)}{(1-\alpha^2)^2}\right]\ln\alpha\\&+\left[\frac{5\pi^2}{1-\alpha^2}+\frac{37(1+\alpha)}{2\alpha(1-\alpha)}+\frac{25\pi^2(1-\alpha)}{6(1+\alpha)^3}\right]\Bigg\}, \quad \text{(B19)}\end{aligned}$$

$$A(T=2)=\frac{2\alpha^2}{1-\alpha^2}\left[-\frac{9(1+\alpha)}{2(1-\alpha)^3}\ln^2\alpha-\frac{2}{(1+\alpha)^2}\ln\alpha+\left(\frac{11}{2\alpha}-\frac{2}{1+\alpha}+\frac{9}{1-\alpha}\right)\right], \quad \text{(B20)}$$

$$\begin{aligned}B(T=2)=\frac{-2\alpha^2}{1-\alpha^2}\Bigg\{&\frac{43(1+\alpha)}{(1-\alpha)^3}\int_\alpha^1\frac{\ln^2\nu}{1-\nu}d\nu-\frac{(1+\alpha)(14+215\alpha+14\alpha^2)}{12(1-\alpha)^5}\ln^4\alpha\\&+\left[-\frac{43}{3(1-\alpha)^3}+\frac{81(1+\alpha)^2}{4(1-\alpha)^4}+\frac{11+10\alpha+11\alpha^2}{2(1-\alpha^2)^2}\right]\ln^3\alpha\\&+\left[\frac{9}{1-\alpha^2}+\left(\frac{119}{4}-\frac{7\pi^2}{3}\right)\frac{1+\alpha}{(1-\alpha)^3}\right]\ln^2\alpha\\&+\left[-\left(\frac{14\pi^2}{3}-18\right)\frac{1}{(1-\alpha)^2}+\frac{3\pi^2(1+\alpha^2)}{(1-\alpha^2)^2}\right]\ln\alpha\\&+\left[\frac{3\pi^2}{1-\alpha^2}+\frac{37(1+\alpha)}{2\alpha(1-\alpha)}+\frac{2\pi^2(1-\alpha)}{3(1+\alpha)^3}\right]\Bigg\}. \quad \text{(B21)}\end{aligned}$$

The functions A and $-B$ are plotted against k^2 in Figs. 4 and 5, respectively. Unless k is very small or very

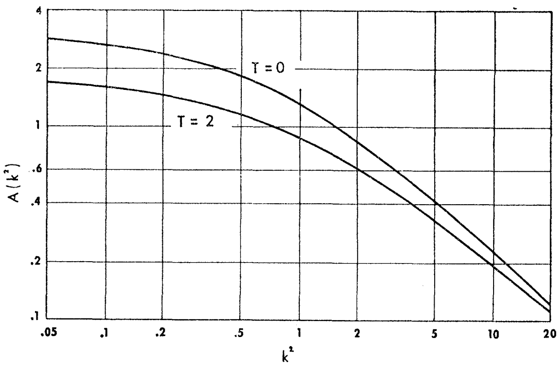

FIG. 4. The coefficients A as functions of k^2 [see Eqs. (B18) and (B20)]. The horizontal and vertical scales are logarithmic. To the left, the curves approach the asymptotes 3 and 7/4; to the right, $11/4k^2$ is the common asymptote.

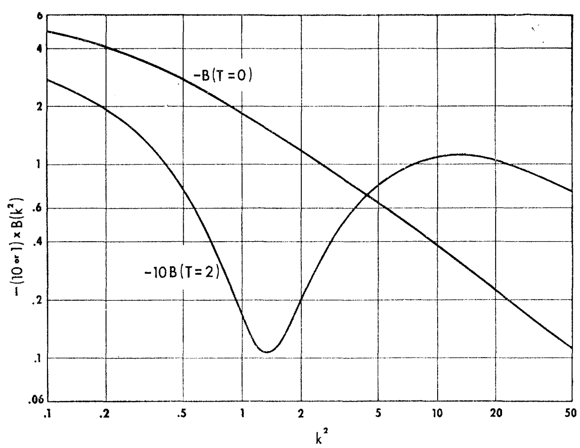

FIG. 5. Log-log plot of the coefficients $-B$ against k^2 [see Eqs. (B19) and (B21)]. The asymptotic values of $-B$ are $71/6-89\pi^2/144\approx 5.73$ and $25/3-29\pi^2/36\approx 0.383$ to the left and $37/4k^2$ to the right.

large, a machine calculation of B is almost unavoidable. Individual terms must be computed to high accuracy, owing to the fact that they nearly cancel each other in the final result, especially for $T=2$.

APPENDIX C: THRESHOLD EXPANSIONS

In what follows we set down the behavior, near $k=0$, of several functions described in the previous Appendices. A knowledge of this behavior is necessary for two reasons: First, these functions are, in many instances, difficult to calculate for low k from the explicit formulas, owing to the near cancellation of large terms. The threshold expansions provide an adequate means of doing this calculation, and at the same time serve as a check against serious numerical mistakes in higher regions of k. The second and main reason for wanting the small-k information is its use in matching the phase shifts, calculated above threshold, with the full amplitudes, calculated below threshold. This matching across the threshold, as discussed in Sec. 6, is an important feature of the approximation method we are using.

Functions $a(s)$, $b(s)$ near $s=0$

From Eqs. (I.3.11) and (I.3.12) near $s=0^+$ we obtain

$$a(s)=1-s/12-s^2/120-s^3/840-s^4/5040+\cdots, \quad \text{(C1)}$$

$$b(s)=-(1/12)[\pi^2+(2+\pi^2/6)s+\pi^2s^2/30+(\pi^2/140-1/40)s^3+(\pi^2/630-31/3780)s^4+\cdots]. \quad \text{(C2)}$$

Functions $a(s)$, $b(s)$ near $s=4^+$

Again using Eqs. (I.3.11) and (I.3.12), this time near $s=4^-$, and performing an infinitesimal analytic continuation across the threshold with

$$k=(\tfrac{1}{4}s-1)^{1/2}, \quad \text{(C3)}$$

we obtain

$$a(s)=(k^2-2k^4/3+8k^6/15-16k^8/35+128k^{10}/315+\cdots)-(i\pi/2)(k-k^3/2+3k^5/8-5k^7/16+35k^9/128+\cdots), \quad \text{(C4)}$$

$$b(s)=[-5\pi^2/12+(\tfrac{4}{3}+\pi^2/9)k^2-(\tfrac{2}{3}+4\pi^2/45)k^4+(22/45+8\pi^2/105)k^6-(1144/2835+64\pi^2/945)k^8+\cdots]-(i\pi/2)(3k-7k^3/6+313k^5/360-3679k^7/5040+129719k^9/201600+\cdots). \quad \text{(C5)}$$

Partial-Wave Projections of a, a^2, and b

These threshold series are most simply obtained by expanding the integrand of (A1) before carrying out the integration, rather than by expanding the explicit results.

S waves:

$$a_0=2+k^2/3-4k^4/45+4k^6/105-32k^8/1575+\cdots,$$

$$(a^2)_0=2+2k^2/3-14k^4/135+2k^6/63-104k^8/7875+\cdots,$$

$$b_0=-\tfrac{1}{6}\pi^2+(2+\tfrac{1}{6}\pi^2)k^2/3-4\pi^2k^4/135+(2\pi^2/7-1)k^6/15-32(2\pi^2-31/3)k^8/4725+\cdots. \quad \text{(C6)}$$

P waves:

$$a_1=-k^2/9+2k^4/45-4k^6/175+64k^8/4725+\cdots,$$

$$(a^2)_1=-2k^2/9+7k^4/135-2k^6/105+208k^8/23625+\cdots,$$

$$b_1=-(2+\pi^2/6)k^2/9+2\pi^2k^4/135-(2\pi^2/7-1)k^6/25+64(2\pi^2-31/3)k^8/14175+\cdots. \quad \text{(C7)}$$

D waves:

$$a_2 = -2k^4/225 + 4k^6/525 - 64k^8/11025 + \cdots,$$
$$(a^2)_2 = -7k^4/675 + 2k^6/315 - 208k^8/55125 + \cdots,$$
$$b_2 = -2\pi^2k^4/675 + (2\pi^2/7-1)k^6/75 - 64(2\pi^2 - 31/3)k^8/33075 + \cdots. \quad \text{(C8)}$$

Coefficients A, B of the Differential Equations

Using the preceding results of this Appendix, we obtain the following expansions.

Full amplitudes above threshold [see Appendix B, (ii)]:

$$A(T=0) = (6k - 106k^3/15 + \cdots) - (i\pi/2)(\tfrac{5}{2} - 15k^2/4 + \cdots),$$
$$B(T=0) = -(71 - 41\pi^2/6)k/3 + (2881/2 - 142\pi^2)k^3/45 + \cdots,$$
$$A(T=2) = (7k/2 - 49k^3/15 + \cdots) - (i\pi/2)(1 - 3k^2/2 + \cdots),$$
$$B(T=2) = -(50 - 16\pi^2/3)k/3 + (1921/2 - 100\pi^2)k^3/45 + \cdots. \quad \text{(C9)}$$

It is interesting that, to this order in k, $B(T=0, 2)$ is real. (This feature is not expected to persist to indefinitely high order, however.) As a direct consequence of this, the threshold unitarity condition (I.5.22), which was exactly valid within the two-term scheme, is now exactly valid also within the three-term scheme.

S-wave amplitudes above threshold [see Appendix B, (iii)]:

$$A(T=0) = (6k - 36k^3/5 + \cdots) - (i\pi/2)(\tfrac{5}{2} - 15k^2/4 + \cdots),$$
$$B(T=0) = -(71 - 41\pi^2/6)k/3 + (4457 - 448\pi^2)k^3/135 + \cdots,$$
$$A(T=2) = (7k/2 - 52k^3/15 + \cdots) - (i\pi/2)(1 - 3k^2/2 + \cdots),$$
$$B(T=2) = -(50 - 16\pi^2/3)k/3 + (12533/4 - 328\pi^2)k^3/135 + \cdots. \quad \text{(C10)}$$

APPENDIX D: PERTURBATIVE RESULTS FOR S WAVES

From (I.3.13), (I.3.14), and (I.3.15) we obtain, after substituting (I.2.6) and taking S-wave projections, the following formulas. (Throughout this Appendix, δ refers to S waves.)

First order:

$$(e^{i\delta}\sin\delta)^{(1)} = -\begin{pmatrix} 5/16\pi \\ 1/8\pi \end{pmatrix} \frac{k}{(k^2+1)^{1/2}}, \quad T = \begin{pmatrix} 0 \\ 2 \end{pmatrix}. \quad \text{(D1)}$$

Second order:

$$(e^{i\delta}\sin\delta)^{(2)} = \frac{-5ik}{16\pi(k^2+1)^{1/2}}[5I(s) + 3I_0(k^2)] \quad (T=0),$$
$$(e^{i\delta}\sin\delta)^{(2)} = \frac{-ik}{16\pi(k^2+1)^{1/2}}[4I(s) + 9I_0(k^2)] \quad (T=2). \quad \text{(D2)}$$

Third order:

$$(e^{i\delta}\sin\delta)^{(3)} = \frac{-5k}{16\pi(k^2+1)^{1/2}}[60iJ(s) - 25I^2(s) + 44iJ_0(k^2) - 11(I^2)_0(k^2)] \quad (T=0),$$
$$(e^{i\delta}\sin\delta)^{(3)} = \frac{-k}{16\pi(k^2+1)^{1/2}}[72iJ(s) - 8I^2(s) + 112iJ_0(k^2) - 43(I^2)_0(k^2)] \quad (T=2). \quad \text{(D3)}$$

In all these equations, $s = 4(k^2+1)$; the functions I and J are therefore taken above threshold; in using (I.3.11) and (I.3.12), one can use

$$\bar{\nu} = \alpha \quad \text{(D4)}$$

[see (A4)]; the subscripts zero refer to the notation (A1).

The perturbative results for δ can be obtained from the above by taking

$$\delta^{(1)} = (e^{i\delta}\sin\delta)^{(1)},$$
$$\delta^{(2)} = \mathrm{Re}(e^{i\delta}\sin\delta)^{(2)}, \quad \text{(D5)}$$
$$\delta^{(3)} = \mathrm{Re}(e^{i\delta}\sin\delta)^{(3)} + \tfrac{2}{3}(\delta^{(1)})^3.$$

These relations have of course no universal validity, but are due to the reality of δ to third order in this case.

The automatic cancellation of cutoff-dependent terms which occurs as one constructs the A and B of Appendix B is worth noting; it may be taken as a partial check on the manipulations.