attributable to ω^0 production. Taking into account the tails of the ω^0 peak in the two control regions the $\omega\pi\pi$ and $\omega\rho$ fractions can be obtained by the expression:

2.34× (distribution d, first part) − 0.67
× (distribution b, second part
+ distribution b, third part).

The result is given in the fourth part of Table I. In Figs. 4(a), 4(b), and 4(c) the energy distributions calculated from the best fits chosen above are shown as solid lines.

IV. CONCLUSIONS

We wish to draw the following conclusions[7]:

1. The channel $\omega^0+\pi^++\pi^-$ contains both nonresonant pions and pions resonating as ρ. The nonresonant state dominates. Production of ρ mesons accounts for (15±6)% of the channel. We recall that $\omega^0\rho^0$ production must be from the ^1S state.

2. The smallness of the $(\mathbf{p}_+\times\mathbf{p}_-)^2$ term shows that the nonresonant production is dominantly from the ^3S state.

3. The fraction of all $\bar{p}p$ annihilations which are attributable to these reactions is $\bar{p}p\rightarrow\omega^0+\pi^++\pi^-$ (nonresonant): 0.039±0.005 of all annihilations, $\bar{p}p\rightarrow\omega^0+\rho$: 0.007±0.003 of all annihilations.

[7] Our conclusions 1 and 3 are in good agreement with the results of M. Cresti, A. Grigoletto, S. Limentani, A. Loria, L. Peruzzo, R. Santangelo, B. Chadwick, W. T. Davies, M. Derrick, C. J. B. Hawkins, P. M. D. Gray, J. H. Mulvey, P. B. Jones, D. Radojicic, and C. A. Wilkinson, in *Proceedings of the Sienna International Conference on Elementary Particles, 1963*, edited by G. Bernadini and G. D. Puppi (Società Italiana di Fisica, Bologna, 1963), p. 263.

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Shmushkevich's Method for a Charge-Independent Theory*

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The Shmushkevich method for deducing consequences of charge independence is explained and discussed. This method, which avoids entirely the use of Clebsch-Gordan coefficients, generates linear equalities and inequalities among cross sections using only a simple counting procedure. A comprehensive list of such relations, applying to most elementary-particle reactions of interest which involve at least one pair of isospin-$\frac{1}{2}$ particles, is presented. A discussion of the various uses of these relations is given.

I. INTRODUCTION

By assuming that a set of elementary-particle reactions exhibits invariance under a given symmetry group, we are enabled to deduce consequences of this invariance in the form of relations among cross sections. One class of relations, which is particularly easy to deduce is the class of relations linear in the cross sections. Linear relations, because of their simplicity, are also of greater use. In what follows, consideration will be restricted to consequences of charge independence[1–5]; similar methods[6–9] may be applied for reactions which display invariance under other compact groups.

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II. CONSEQUENCES OF CHARGE INDEPENDENCE

The standard method of deriving the consequences of charge independence[10–12] requires one to express all relevant transition amplitudes in terms of the set of relevant charge-independent transition amplitudes and appropriate Clebsch-Gordan coefficients. Taking the absolute square of the amplitude for each reaction gives the differential cross section for that reaction, except for purely kinematic factors which we omit in the relations below. All cross sections are thus expressed as linear combinations of terms bilinear in the charge-independent amplitudes. If this set of linear combinations of bilinear quantities is linearly independent, then there are no linear equalities among the cross sections; if the set is linearly dependent, elimination of dependent terms gives the desired linear equalities among the cross sections.

When more than four particles are involved in the reaction, this procedure becomes unwieldy. While the problem of expanding transition amplitudes involves mere tedium, that of finding linear dependence and eliminating dependent terms becomes a formidable task. It was Shmushkevich[1] who first realized that linear relations may be extracted without resorting to this procedure.

We now proceed to demonstrate that there exist linear relations which do not depend on our knowledge of the numerical values of Clebsch-Gordan coefficients, but only on orthogonality relations satisfied by these coefficients. It will then be evident that the second part of the standard procedure just serves to repair the damage done by the explicit introduction of Clebsch-Gordan coefficients in the first part. It suffices to illustrate these arguments by outlining the procedure for a reaction involving five particles having arbitrary isospins,

$$I_1+I_2\to I_3+I_4+I_5,$$

with third components of isospins, ν_i. We seek relations among quantities of the form $|\mathcal{T}|^2$, where $\mathcal{T}$ is the transition amplitude

$$\mathcal{T}\equiv\mathcal{T}(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5)=\langle I_1\nu_1I_2\nu_2|T|I_3\nu_3I_4\nu_4I_5\nu_5\rangle. \quad (1)$$

This transition amplitude, which is defined for all values of the ν_i, vanishes unless charge is conserved, that is, unless

$$\nu_1+\nu_2=\nu_3+\nu_4+\nu_5.$$

First, we expand initial and final states in terms of eigenstates of the total isospin,

$$|I_1\nu_1I_2\nu_2\rangle=\sum_{J,\mu}C(I_1I_2J;\nu_1\nu_2\mu)|J\mu\rangle,$$
$$|I_3\nu_3I_4\nu_4I_5\nu_5\rangle=\sum_{\substack{I_{34},I,\\ \lambda_{34},\lambda}}C(I_3I_4I_{34};\nu_3\nu_4\lambda_{34})C(I_{34}I_5I;\lambda_{34}\nu_5\lambda)|I_{34},I\lambda\rangle. \quad (2)$$

Our coefficients $C(I_aI_bI_c;\nu_a\nu_b\nu_c)$ are defined for all values of their arguments; they vanish unless $\nu_a+\nu_b=\nu_c$ and unless $\mathbf{I}_a$, $\mathbf{I}_b$, $\mathbf{I}_c$ can form a triangle. Then

$$\mathcal{T}=\sum_{\substack{J,I,I_{34},\\ \mu,\lambda,\lambda_{34}}}C(I_1I_2J;\nu_1\nu_2\mu)C(I_3I_4I_{34};\nu_3\nu_4\lambda_{34})\times C(I_{34}I_5I;\lambda_{34}\nu_5\lambda)\langle J\mu|T|I_{34},I\lambda\rangle. \quad (3)$$

Now, imposing conservation of isospin, we have

$$\langle J\mu|T|I_{34}I\lambda\rangle=\delta(I,J)\delta(\mu,\lambda)\langle I\lambda|T|I_{34},I\lambda\rangle.$$

We denote the charge-independent amplitude $\langle I\lambda|T|\times I_{34},I\lambda\rangle$ by $T(I_{34},I)$ to make explicit the lack of dependence on charge labels. We then have

$$|\mathcal{T}|^2=\sum_{\substack{I_{34},I,\\ \lambda_{34},\lambda}}\sum_{\substack{I_{34}',I',\\ \lambda_{34}',\lambda'}}C(I_1I_2I;\nu_1\nu_2\lambda)C(I_1I_2I';\nu_1\nu_2\lambda')\times C(I_3I_4I_{34};\nu_3\nu_4\lambda_{34})C(I_3I_4I_{34}';\nu_3\nu_4\lambda_{34}')\times C(I_{34}I_5I;\lambda_{34}\nu_5\lambda)C(I_{34}'I_5I';\lambda_{34}'\nu_5\lambda')\times T(I_{34},I)T^*(I_{34}',I'). \quad (4)$$

There are five free charge labels. Summing this expression over all charge labels gives the total cross section for all possible charge complexions. Rather than sum over all labels, we may leave one charge label unsummed and use the following properties[13] of Clebsch-Gordan coefficients:

(1) Orthogonality:

$$\sum_{\nu_a,\nu_b}C(I_aI_bI_c;\nu_a\nu_b\nu_c)C(I_aI_bI_c';\nu_a\nu_b\nu_c')=\delta(I_c,I_c')\delta(\nu_c,\nu_c'). \quad (5)$$

(2) Modified orthogonality:

$$\sum_{\nu_b,\nu_c}C(I_aI_bI_c;\nu_a\nu_b\nu_c)C(I_a'I_bI_c;\nu_a'\nu_b\nu_c)=\frac{2I_c+1}{2I_a+1}\delta(I_a,I_a')\delta(\nu_a,\nu_a'). \quad (6)$$

If we sum over ν_2, ν_3, ν_4, ν_5, we are left with a function of ν_1,

$$\sigma_1(\nu_1)=\sum_{I_{34},I}\frac{2I+1}{2I_1+1}|T(I_{34},I)|^2.$$

By taking the appropriate sums, we find that a like

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result holds for each particle,

$$\sigma_i(\nu_i)=\sum_{I_{34},I}\frac{2I+1}{2I_i+1}|T(I_{34},I)|^2; \tag{7}$$

and observe also that

$$\sigma_i(\nu_i)=\frac{2I_j+1}{2I_i+1}\sigma_j(\nu_j).$$

Before discussing the implications of Eq. (7), we first establish the structure of the remaining sums which follow from Eq. (4). Performing the sums over ν_3, ν_4, ν_5 and the sum over ν_1 and ν_2 with the constraint $\nu_1+\nu_2=\nu_{12}$ yields a function of ν_{12},

$$\sigma_{12}(\nu_{12})=\sum_{I\geq|\nu_{12}|}\sum_{I_{34}}|T(I_{34},I)|^2. \tag{8}$$

Summing over ν_1, ν_2, ν_5 and over ν_3 and ν_4 with the constraint $\nu_3+\nu_4=\nu_{34}$ gives a function of ν_{34},

$$\sigma_{34}(\nu_{34})=\sum_I\sum_{I_{34}\geq|\nu_{34}|}\frac{2I+1}{2I_{34}+1}|T(I_{34},I)|^2. \tag{9}$$

If the final sum is performed for Eqs. (7), (8), and (9) the final result is, as expected,

$$\sigma=\sum_I\sum_{I_{34}}(2I+1)|T(I_{34},I)|^2. \tag{10}$$

The extension to the N-particle case is obvious. The Clebsch-Gordan expansion, Eq. (3), involves $(N-2)$ coefficients with N free charge labels. In order to leave one label unsummed, we may sum over $(N-1)$ labels, obtaining a relation resembling Eq. (7). Alternatively, we may sum over k labels and sum over the remaining $(N-k)$ labels with a single constraint, finding relations similar to Eqs. (8) and (9). In either case, all Clebsch-Gordan coefficients are eliminated in the summation process.

The result (7) is particularly useful because $\sigma_i(\nu_i)$ is, in fact, independent of ν_i, i.e., the weight corresponding to finding a particle in one charge state is equal to the weight for finding it in any other charge state. Shmushkevich recognized this result, namely, that the sum of all cross sections corresponding to any particle having a given charge is the same for each possible charge of the particle, as the most direct and intuitive consequence of charge independence. This yields $2I_i$ relations

$$\sigma_i(I_i)=\sigma_i(I_i-1)=\cdots=\sigma_i(-I_i). \tag{11}$$

Of these relations, the relations of the form $\sigma_i(\nu_i)=\sigma_i(-\nu_i)$ are just statements of charge symmetry. If I_i is integral, there are I_i charge-symmetry relations and I_i useful new relations. If I_i is half-integral, there are $I_i+\frac{1}{2}$ and $I_i-\frac{1}{2}$ relations of each type, respectively. We note that a particle will not contribute a relation which goes beyond charge symmetry unless it has $I\geq 1$.

Results (8) and (9) are different from (7) in that the functions $\sigma_{12}(\nu_{12})$ and $\sigma_{34}(\nu_{34})$ do depend explicitly on their arguments. The reason for this is apparent, since if the system (I_3+I_4), e.g., has a charge label ν_{34}, we may only sum in Eq. (9) over isospin states for which this system has isospin $I_{34}\geq\nu_{34}$. In this case, charge independence tells us that the contribution to the weight corresponding to finding a pair of particles with a given total charge label ν_{34}, coming from states with a particular value of I_{34}, is the same as the weight corresponding to finding the pair with charge label ν_{34}' coming from states with that value of I_{34}, as long as $I_{34}\geq\nu_{34}$ and $I_{34}\geq\nu_{34}'$. It would seem at first that the only equality we can extract is the charge-symmetry statement, $\sigma_{34}(\nu_{34})=\sigma_{34}(-\nu_{34})$. However, if I_{34} is unable to assume values smaller than some value $I_{34}^{\min}$, then the sum in Eq. (9) is the same for $\nu_{34}=I_{34}^{\min}$ as it is for ν_{34} equal to any value less than $I_{34}^{\min}$. We can therefore write the nontrivial relations

$$\sigma_{34}(I_{34}^{\min})=\sigma_{34}(I_{34}^{\min}-1)=\cdots=\sigma_{34}(\tfrac{1}{2}\text{ or }0). \tag{12}$$

For values of ν_{34} greater than $I_{34}^{\min}$, we have inequalities relating the $\sigma_{34}(\nu_{34})$, since they differ by positive multiples of the absolute squares of amplitudes.

$$\sigma_{34}(I_{34}^{\max})\leq\sigma_{34}(I_{34}^{\max}-1)\leq\cdots\leq\sigma_{34}(I_{34}^{\min}). \tag{13}$$

The isospin I_{34} has an apparent minimum value given by $|I_3-I_4|$ but, in fact, cannot take on a value less than the minimum value to which the remaining isospins I_1, I_2, I_5 can couple. Therefore

$$I_{34}^{\min}=\max\{|I_3-I_4|,\min(\mathbf{I}_1+\mathbf{I}_2+\mathbf{I}_5)\}.$$

Similarly,

$$I_{34}^{\max}=\min\{I_3+I_4,I_1+I_2+I_5\}.$$

Similar results corresponding to other ways of coupling the isospins follow for any pair of particles. Relations among cross sections are unaffected by transferring any number of particles from final to initial state or vice versa while changing particles to antiparticles. We can therefore find relations coming from $\sigma_{13}(\nu_{13})$, where $\nu_{13}=\nu_1-\nu_3$, if we regard particle three as an antiparticle in the initial state.

Generalizing to the N-particle case, we can consider any subset $\{k\}$ of k particles along with the complementary subset $\{N-k\}$ of $N-k$ particles. Define $I_{\{k\}}^{\min}$ and $I_{\{k\}}^{\max}$ as the extremum isospin values that these subsets of particles can assume. Then

$$I_{\{k\}}^{\min}=\max\{\min(\sum_{i\epsilon\{k\}}\mathbf{I}_i),\min(\sum_{i\epsilon\{N-k\}}\mathbf{I}_i)\}, \tag{14a}$$

$$I_{\{k\}}^{\max}=\min\{\sum_{i\epsilon\{k\}}I_i,\sum_{i\epsilon\{N-k\}}I_i\}, \tag{14b}$$

and the previous results can be summarized by the relation

$$\sigma_{\{k\}}(I_{\{k\}}^{\max})\leq\cdots\leq\sigma_{\{k\}}(I_{\{k\}}^{\min})=\cdots=\sigma_{\{k\}}(\tfrac{1}{2}\text{ or }0). \tag{15}$$

TABLE I. Numbers of cross sections and numbers of charge-independent amplitudes for various numbers of particles.

n	W	W'	ρ	$\frac{1}{2}\rho(\rho+1)$	$W'-\frac{1}{2}\rho(\rho+1)$
		(A) n Isospin 1			
3	7	4	1	1	3
4	19	10	3	6	4
5	51	26	6	21	5
6	141	71	15	120	...
7	393	197	36	666	...
8	1107	554	91	4186	...
		(B) 2 Isospin $\frac{1}{2}$, n Isospin 1			
1	4	2	1	1	1
2	10	5	2	3	2
3	26	13	4	10	3
4	70	35	9	45	...
5	192	96	21	231	...
6	534	267	51	1326	...
		(C) 4 Isospin $\frac{1}{2}$, n Isospin 1			
0	6	3	2	3	0
1	14	7	3	6	1
2	36	18	6	21	...
3	96	48	13	91	...
4	262	131	30	465	...
5	726	363	72	2628	...

For a charge-independent reaction, we can make the following assertion: If experiment were to show that for an $I_{\{k\}}\geq I_{\{k\}}^{\min}$, the equality $\sigma_{\{k\}}(I_{\{k\}}+1)=\sigma_{\{k\}}(I_{\{k\}})$ held, this would mean that all amplitudes for which the subset $\{k\}$ had isospin $I_{\{k\}}$ vanish.

We now emphasize the following important point: The relations which exist among the cross sections, as well as the number of differential cross sections and the number of charge-independent amplitudes which occur for a given configuration of isospins, are "channel invariants," that is, they depend only on the set of all isospins present, $(I_1,\cdots,I_N)$. It is immaterial to the problem at hand whether any particular particle is in in the initial state or its antiparticle is in the final state. We may therefore find relations for any k-particle subset of the N particles, but we note that the relations so found are identical with the relations found by considering the complementary subset of $N-k$ particles. It follows that there are $2^{N-1}-1$ independent subsets to consider.

In the application of the foregoing, it is convenient to impose the requirement of charge symmetry from the beginning and ignore the charge symmetry relations rather than to consider all cross sections as independent and make use of charge symmetry relations. It is worthwhile to look more closely at the number of cross sections and the number of charge-independent amplitudes, considered as functions of the isospins, which we denote by $W_N(I_1,\cdots,I_N)$ and $\rho_N(I_1,\cdots,I_N)$, respectively.

If all the particles do not have integral isospin, that is, if there is at least one pair of half-integral isospin particles, the number of cross sections W_N is even, since no reaction is related to itself by charge symmetry. Therefore, by making use of charge symmetry there are $W_N'=W_N/2$ independent cross sections. If all the particles do have integral isospin, one reaction, namely, that for which each particle is in the $\nu_i=0$ state is charge symmetric with itself, so that the number of cross sections W_N is odd. Using charge symmetry, there are then $W_N'=(W_N+1)/2$ independent cross sections.

We now compare, for some simple cases, the number of bilinear terms to which the cross sections are related and the number of cross sections. If ρ_N is the number of amplitudes which describes the process, there may be $\rho_N(\rho_N+1)/2$ bilinear terms to which the W_N' cross sections are equated. From Table I, we observe that for reactions involving a small number of isospin-$\frac{1}{2}$ and

TABLE II. Reactions having the same isotopic structure.

(A) $[\frac{1}{2},\frac{1}{2}',n(1)]$

Prototype reaction

$\bar{N}N\to n\pi$

Equivalent reactions

(1) $\bar{K}N\to\Lambda+n\pi$
(2) $\bar{K}d\to\Lambda N+n\pi$
(3) $\bar{N}d\to N+n\pi$
(4) $\bar{N}d\to\Lambda K+n\pi$
(5)[a] $NN'\to d+n\pi$
(6) $\bar{\Lambda}N\to K+n\pi$

(B) $[\frac{1}{2},\frac{1}{2}',1',n(1)]$

Prototype reaction

$\bar{N}N\to\rho+n\pi$

Equivalent reactions

(1) $\bar{N}N\to\bar{\Lambda}\Sigma+n\pi$
(2) $\bar{N}N\to\Lambda\bar{\Sigma}+n\pi$
(3) $\bar{N}d\to N\rho+n\pi$
(4) $\bar{N}d\to\Lambda K\rho+n\pi$
(5) $\bar{K}N\to\Sigma+n\pi$
(6) $\bar{K}N\to\Lambda\rho+n\pi$
(7) $\bar{K}d\to\Sigma N+n\pi$
(8) $\bar{K}d\to\Lambda N\rho+n\pi$
(9) $\pi N\to N+n\pi$
(10) $\pi N\to\Lambda K+n\pi$
(11)[a] $\pi d\to NN'+n\pi$
(12) $\Sigma N\to\Lambda N+n\pi$
(13) $\Lambda N\to\Sigma N+n\pi$
(14) $\bar{\Sigma}N\to K+n\pi$

(C) $[\frac{1}{2},\frac{1}{2}',\frac{1}{2}'',\frac{1}{2}''',n(1)]$

Prototype reaction

$\bar{N}N\to\bar{K}K+n\pi$

Equivalent reactions

(1) $\bar{N}N\to\bar{N}N+n\pi$
(2) $\bar{K}N\to\bar{K}N+n\pi$
(3) $KN\to KN+n\pi$
(4)[a] $NN'\to NN'+n\pi$
(5) $\bar{K}N\to\Lambda\bar{K}K+n\pi$
(6) $\bar{N}N\to\Lambda K\bar{N}+n\pi$
(7) $\bar{N}d\to\bar{K}KN+n\pi$
(8)[a] $\bar{K}d\to\bar{K}KN'+n\pi$
(9) $\bar{K}N\to\Xi K+n\pi$
(10) $\bar{K}d\to\Xi KN+n\pi$
(11)[a] $NN'\to\Lambda KN+n\pi$
(12)[a] $\Lambda N\to NN'K+n\pi$
(13) $\pi N\to\bar{K}KN$ $(n=1)$
(14)[a] $NN'\to NK\Sigma$ $(n=1)$
(15) $\bar{K}N\to\Sigma\bar{K}K$ $(n=1)$
(16)[a] $\Sigma N\to NN'K$ $(n=1)$

(D) $[\frac{1}{2},\frac{1}{2}',1',1'',n(1)]$

Prototype reaction

$\bar{N}N\to\bar{\Sigma}\Sigma+n\pi$

Equivalent reactions

(1) $\Sigma N\to\Sigma N+n\pi$
(2) $\bar{\Sigma}\Sigma\to\bar{N}N+n\pi$
(3) $\bar{\Sigma}N\to\bar{\Sigma}N+n\pi$
(4) $\pi N\to\Sigma K+n\pi$
(5) $\pi N\to N\rho+n\pi$
(6) $\pi N\to\Lambda K\rho+n\pi$
(7) $\bar{K}N\to\Sigma\rho+n\pi$

(E) $[\frac{1}{2},\frac{1}{2}',\frac{1}{2}'',\frac{3}{2},n(1)]$

Prototype reaction

[b] $\bar{N}N\to\Delta\bar{N}+n\pi$

Equivalent reactions

(1) $\bar{K}N\to\Delta\bar{K}+n\pi$
(2) $KN\to\Delta K+n\pi$
(3) $\bar{K}d\to\Delta N\bar{K}+n\pi$
(4) $Kd\to\Delta NK+n\pi$
(5)[a] $NN'\to\Delta N+n\pi$

[a] The two nucleons N, N' are to be considered distinguishable.
[b] The symbol Δ is used for an $I=3/2$ nucleon isobar.

isospin-1 particles, the number of linear equalities we expect, $W_N' - \rho_N(\rho_N+1)/2$, is equal to the number of isospin-1 particles present, which is just the number of Shmushkevich equalities. This result holds up to a certain small value of N, beyond which the number of bilinear quantities rapidly increases above the number of cross sections. In this region, we would not know whether to expect any linear equalities without testing for linear independence of the bilinear quantities. However, we do know there are still Shmushkevich equalities, one for each isospin-1 particle.

III. PROCEDURE

The task of producing relations is essentially one of tabulation. Enumerate the W_N reactions and find the weights $\sigma(\nu_{\{k\}})$ by adding the cross sections for all reactions in which the set $\{k\}$ of particles has charge label $\nu_{\{k\}}$. Alternatively, we can shorten our list by writing only the W_N' independent reactions. Then to find $\sigma(\nu_{\{k\}})$, add the cross sections for all reactions in which the set $\{k\}$ has charge label either $\nu_{\{k\}}$ or $-\nu_{\{k\}}$, if $\nu_{\{k\}} \neq 0$. For $\sigma(\nu_{\{k\}}=0)$, count reactions which have $\nu_{\{k\}}=0$ twice if they are not self-charge-symmetric, once otherwise.

The meaning of the relations obtained is sufficiently clear if the N particles are all distinguishable. The cross sections which appear can be regarded either as differential cross sections, that is, referring to fixed values of all variables not involving the isospins, or as partial (total) cross sections summed over any (all) of the other variables. Thus if the ith particle is a π and the jth particle is a Σ, there is no danger of confusing a reaction containing a π^+ and Σ^- with one containing a π^- and Σ^+. However, if the ith and jth particles are both π's, one must be reminded to distinguish differential cross sections in which a π^+ has momentum p_i and π^- has momentum p_j from a cross section in which a π^- has momentum p_i and π^+ has momentum p_j. If one does not measure momentum intervals in which particles are produced, but only total rates of production, then these two differential cross sections contribute to the same total cross section, namely, the cross section for production of a π^+ and π^- irrespective of other labels. Indeed, in the present context our distinction between the terms "differential" and "total" cross sections is actually a distinction with respect to one point only. If we are given n momentum intervals, with one particle produced in each interval, we ask which particle has which momentum for the differential cross sections while we do not ask this question for the total cross sections. In a discussion of total production rates, only total cross sections are relevant.

In processes involving identical particles, restricting consideration to total cross sections greatly reduces the number of quantities being related. For instance, for nucleon antinucleon annihilation into 8 pions, there are 2123 independent differential cross sections, but only 9

TABLE III. Relations following from charge independence for the reaction $\bar{N}N \to n\pi$.

(A)		$\bar{N}N \to \pi$
	σ_1	$\bar{p}p \to \pi^0$
	σ_2	$\bar{p}n \to \pi^-$
		$\sigma_2 = 2\sigma_1$
(B)		$\bar{N}N \to 2\pi$
	σ_1	$\bar{p}p \to \pi^+\pi^-$
	σ_2	$\bar{p}p \to 2\pi^0$
	σ_3	$\bar{p}n \to \pi^-\pi^0$
		$2\sigma_1 = 4\sigma_2 + \sigma_3$
(C)		$\bar{N}N \to 3\pi$
	σ_1	$\bar{p}p \to \pi^+\pi^-\pi^0$
	σ_2	$\bar{p}p \to 3\pi^0$
	σ_3	$\bar{p}n \to \pi^+,2\pi^-$
	σ_4	$\bar{p}n \to \pi^-,2\pi^0$
	π	$\sigma_3 = \sigma_4 + 2\sigma_2$
	$(\bar{N}N)_0 \geq 1$	$\sigma_1 \geq \sigma_4$
	$(2\pi)_1 \geq 2$	$2(\sigma_1 + \sigma_4) \geq \sigma_3$
(D)		$\bar{N}N \to 4\pi$
	σ_1	$\bar{p}p \to 2\pi^+,2\pi^-$
	σ_2	$\bar{p}p \to \pi^+,\pi^-,2\pi^0$
	σ_3	$\bar{p}p \to 4\pi^0$
	σ_4	$\bar{p}n \to \pi^+,2\pi^-,\pi^0$
	σ_5	$\bar{p}n \to \pi^-,3\pi^0$
	π	$4\sigma_1 + \sigma_4 = 2\sigma_2 + 8\sigma_3 + 5\sigma_5$
	$(\bar{N}N)_0 \geq 1$	$\sigma_1 \geq \sigma_3 + \sigma_5$
	$(3\pi)_0 \geq 1$	$\sigma_4 \geq (3/2)\sigma_5$
	$(2\pi)_0 \geq 1$	$4\sigma_2 + 2\sigma_4 + 3\sigma_5 \geq 2\sigma_1$
(E)		$\bar{N}N \to 5\pi$
	σ_1	$\bar{p}p \to 2\pi^+,2\pi^-,\pi^0$
	σ_2	$\bar{p}p \to \pi^+,\pi^-,3\pi^0$
	σ_3	$\bar{p}p \to 5\pi^0$
	σ_4	$\bar{p}n \to 2\pi^+,3\pi^-$
	σ_5	$\bar{p}n \to \pi^+,2\pi^-,2\pi^0$
	σ_6	$\bar{p}n \to \pi^-,4\pi^0$
	π	$2\sigma_1 + 5\sigma_4 = 4\sigma_2 + 10\sigma_3 + \sigma_5 + 7\sigma_6$
	$(\bar{N}N)_0 \geq 1$	$2\sigma_1 + \sigma_2 \geq \sigma_5 + 2\sigma_6$
	$(3\pi)_2 \geq 3$	$2\sigma_1 + 2\sigma_5 \geq \sigma_4$
	$(3\pi)_0 \geq 1$	$8\sigma_1 + \sigma_4 + 2\sigma_5 \geq 6\sigma_2 + 12\sigma_6$
	$(4\pi)_0 \geq 1$	$11\sigma_4 + \sigma_5 \geq 9\sigma_6$
	$(4\pi)_1 \geq 2$	$4\sigma_1 + 2\sigma_2 + \sigma_5 + 4\sigma_6 \geq 2\sigma_4$
(F)		$\bar{N}N \to 6\pi$
	σ_1	$\bar{p}p \to 3\pi^+,3\pi^-$
	σ_2	$\bar{p}p \to 2\pi^+,2\pi^-,2\pi^0$
	σ_3	$\bar{p}p \to \pi^+,\pi^-,4\pi^0$
	σ_4	$\bar{p}p \to 6\pi^0$
	σ_5	$\bar{p}n \to 2\pi^+,3\pi^-,\pi^0$
	σ_6	$\bar{p}n \to \pi^+,2\pi^-,3\pi^0$
	σ_7	$\bar{p}n \to \pi^-,5\pi^0$
	π	$2\sigma_1 + \sigma_5 = 2\sigma_3 + 4\sigma_4 + \sigma_6 + 3\sigma_7$
	$(5\pi)_0 \geq 1$	$4\sigma_5 \geq \sigma_6 + 6\sigma_7$
	$(\bar{N}N)_0 \geq 1$	$2\sigma_1 + \sigma_2 \geq \sigma_4 + \sigma_6 + 2\sigma_7$
	$(2\pi)_1 \geq 2$	$6\sigma_2 + 8\sigma_3 + \sigma_5 + 8\sigma_6 + 5\sigma_7 \geq 6\sigma_1$
	$(3\pi)_2 \geq 3$	$4\sigma_2 + 3\sigma_5 + 3\sigma_6 \geq 2\sigma_1$
	$(3\pi)_0 \geq 1$	$2\sigma_1 + 8\sigma_2 + 13\sigma_5 \geq 16\sigma_3 + 6\sigma_6 + 20\sigma_7$
	$(4\pi)_1 \geq 2$	$6\sigma_2 + 2\sigma_3 + 10\sigma_5 + \sigma_7 \geq 12\sigma_4$
	$(4\pi)_0 \geq 1$	$18\sigma_1 + 2\sigma_2 + 6\sigma_3 + 30\sigma_4 + 6\sigma_6 \geq 3\sigma_5$

TABLE IV. Relations following from charge independence for the reaction $\bar{N}N \to \rho+n\pi$.

(A)	$\bar{N}N \to \rho\pi$
σ_1	$\bar{p}p \to \rho^+\pi^-$
σ_2	$\bar{p}p \to \rho^0\pi^0$
σ_3	$\bar{p}p \to \rho^-\pi^+$
σ_4	$\bar{p}n \to \rho^-\pi^0$
σ_5	$\bar{p}n \to \rho^0\pi^-$
	$\sigma_4=\sigma_5$
	$\sigma_1+\sigma_3=2\sigma_2+\sigma_4$
$(N^T\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+3\sigma_3\geq 3\sigma_2$
$(N^T\rho)_{\frac{1}{2}\geq\frac{3}{2}}$	$3\sigma_1+\sigma_3\geq 3\sigma_2$
(B)	$\bar{N}N \to \rho,2\pi$
σ_1	$\bar{p}p \to \rho^+\pi^-\pi^0$
σ_2	$\bar{p}p \to \rho^0\pi^+\pi^-$
σ_3	$\bar{p}p \to \rho^0,2\pi^0$
σ_4	$\bar{p}p \to \rho^-\pi^+\pi^0$
σ_5	$\bar{p}n \to \rho^-\pi^+\pi^-$
σ_6	$\bar{p}n \to \rho^-,2\pi^0$
σ_7	$\bar{p}n \to \rho^0\pi^-\pi^0$
σ_8	$\bar{p}n \to \rho^+,2\pi^-$
ρ	$\sigma_1+\sigma_4+\sigma_5+\sigma_6+\sigma_8=2(\sigma_2+\sigma_3+\sigma_7)$
π	$\sigma_1+\sigma_4+\sigma_7+4(\sigma_3+\sigma_6)=2(\sigma_2+\sigma_5+\sigma_8)$
$(\bar{N}N)_{0\geq 1}$	$2(\sigma_1+\sigma_2+\sigma_3+\sigma_4)\geq\sigma_5+\sigma_6+\sigma_7+\sigma_8$
$(2\pi)_{0\geq 1\geq 2}$	$2(\sigma_2+\sigma_3+\sigma_5+\sigma_6)\geq\sigma_1+\sigma_4+\sigma_7\geq\sigma_8$
$(\rho\pi)_{0\geq 1}$	$\sigma_1+\sigma_4+\sigma_7+2\sigma_5+4(\sigma_3+\sigma_8)\geq 2(\sigma_2+\sigma_6)$
$(\rho\pi)_{1\geq 2}$	$\sigma_1+\sigma_4+\sigma_7+2(\sigma_2+\sigma_6)\geq\sigma_5$
$(N^T\rho)_{\frac{1}{2}\geq\frac{3}{2}}$	$3(\sigma_1+\sigma_5+\sigma_6)\geq\sigma_4+\sigma_8$
$(\bar{N}^T\rho)_{\frac{1}{2}\geq\frac{3}{2}}$	$3(\sigma_4+\sigma_5+\sigma_6)\geq\sigma_1+\sigma_8$
(C)	$\bar{N}N \to \rho,3\pi$
σ_1	$\bar{p}p \to \rho^+\pi^+,2\pi^-$
σ_2	$\bar{p}p \to \rho^+\pi^-,2\pi^0$
σ_3	$\bar{p}p \to \rho^0,\pi^+\pi^-\pi^0$
σ_4	$\bar{p}p \to \rho^0,3\pi^0$
σ_5	$\bar{p}p \to \rho^-,2\pi^+\pi^-$
σ_6	$\bar{p}p \to \rho^-\pi^+,2\pi^0$
σ_7	$\bar{p}n \to \rho^+,2\pi^-\pi^0$
σ_8	$\bar{p}n \to \rho^0\pi^+,2\pi^-$
σ_9	$\bar{p}n \to \rho^0\pi^-,2\pi^0$
σ_{10}	$\bar{p}n \to \rho^-\pi^+\pi^-\pi^0$
σ_{11}	$\bar{p}n \to \rho^-,3\pi^0$
ρ	$\sigma_1+\sigma_2+\sigma_5+\sigma_6+\sigma_7+\sigma_{10}+\sigma_{11}=2(\sigma_3+\sigma_4+\sigma_8+\sigma_9)$
π	$\sigma_1+\sigma_5+\sigma_8=\sigma_2+\sigma_6+\sigma_9+2(\sigma_4+\sigma_{11})$
$(2\pi)_{1\geq 2}$	$2(\sigma_2+\sigma_3+\sigma_6+\sigma_9+\sigma_{10})+\sigma_7\geq\sigma_1+\sigma_5+\sigma_8$
$(2\pi)_{0\geq 1}$	$2(\sigma_1+\sigma_5+\sigma_8)+3(\sigma_4+\sigma_{11})\geq\sigma_7$
$(3\pi)_{0\geq 1\geq 2}$	$2(\sigma_3+\sigma_4+\sigma_{10}+\sigma_{11})\geq\sigma_1+\sigma_2+\sigma_5+\sigma_6+\sigma_8+\sigma_9\geq\sigma_7$
$(\bar{N}N)_{0\geq 1}$	$2(\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6)\geq\sigma_7+\sigma_8+\sigma_9+\sigma_{10}+\sigma_{11}$
$(\rho\pi)_{1\geq 2}$	$\sigma_7+\sigma_9+2(\sigma_2+\sigma_3+\sigma_6)+3(\sigma_8+\sigma_{11})\geq\sigma_1+\sigma_5$
$(\rho\pi)_{0\geq 1}$	$4(\sigma_1+\sigma_5)+6\sigma_4+3(\sigma_7+\sigma_9)+\sigma_{10}\geq 3(\sigma_8+\sigma_{11})$
$(\rho,2\pi)_{1\geq 2}$	$3(\sigma_1+\sigma_5+\sigma_{11})+2(\sigma_3+\sigma_9)+\sigma_2+\sigma_6+\sigma_7\geq\sigma_8$
$(\rho,2\pi)_{0\geq 1}$	$3(\sigma_2+\sigma_6+\sigma_7)+6\sigma_4+4\sigma_8+\sigma_{10}\geq 3(\sigma_1+\sigma_5+\sigma_{11})$
$(\bar{N}^T\rho)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_8+\sigma_9+\sigma_{10}+\sigma_{11}\geq\sigma_1+\sigma_2+\sigma_7$
$(N^T\rho)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_8+\sigma_9+\sigma_{10}+\sigma_{11}\geq\sigma_5+\sigma_6+\sigma_7$
$(\bar{N}^T\rho\pi)_{\frac{3}{2}\geq\frac{5}{2}}$	$\sigma_3+\sigma_5+\sigma_7+\sigma_8+\sigma_{10}+2\sigma_2\geq\sigma_1$
$(N^T\rho\pi)_{\frac{3}{2}\geq\frac{5}{2}}$	$\sigma_1+\sigma_3+\sigma_7+\sigma_8+\sigma_{10}+2\sigma_6\geq\sigma_5$

independent total cross sections. The prospects of measuring a sufficient number of the differential cross sections to verify the differential relations are remote. One can arrive at the total cross-section relations directly, rather than by the tedious process of adding differential relations. In tabulating $\sigma(\nu_{\{k\}})$, regard identical particles as unordered; each cross section is then counted as many times as the subset $\{k\}$ has charge label $\nu_{\{k\}}$. For example, the cross section for the reaction $\bar{p}n \to \pi^+\pi^-\pi^-\pi^0\pi^0\pi^0$ is counted six times in the weight $\sigma(\nu_{\{2\pi\}}=-1)$, since with two π^- and three π^0 there are six ways to form a two-pion system with charge -1.

The number of total cross sections, and therefore the relations which exist among them, are not channel invariants, since two particles which we might consider indistinguishable if they were both final-state particles would be distinguishable if one were in the final and one in the initial state. For instance, for the reaction $\bar{N}N \to 2\pi$, regarding the pions as indistinguishable, there are three independent total cross sections, while for $\pi+N \to \pi+N$ all particles are distinguishable, since we know which is the incident and which the final pion, and there are five independent total cross sections. On the other hand, the reaction $\bar{N}N \to \rho+2\pi$ and the reaction $\pi+N \to N+2\pi$ both contain a pair of distinguishable isospin-$\frac{1}{2}$ particles, two indistinguishable isospin-1 particles, and a single isospin-1 particle which is distinguishable from the others. Therefore the two reactions have the same number of total cross sections, with a unique correspondence between the sets of cross sections for each reaction. The same set of relations is obeyed by each set of cross sections.

Reactions which contain particles with the same set of isospins, with corresponding members of each set being considered indistinguishable, may be said to have the same "isotopic structure." The above pair of equivalent reactions has the isotopic structure $(\frac{1}{2},\frac{1}{2}',1,1,1')$. The presence of any number of isospin-zero particles obviously leaves the isotopic structure unchanged.

IV. DISCUSSION

In Tables III–VIII, detailed results of the application of the Shmushkevich method are presented. Not a single Clebsch-Gordan coefficient has been used in writing down these results. Although the relations are written in terms of cross sections for nucleon-antinucleon reactions, they hold equally well for all reactions with the same isotopic structure. A partial list of such equivalent reactions is given in Table II.

For each type of reaction, the independent cross sections are listed. The charge symmetric partner of each reaction, which has the same cross section as the given one, is not written out. When identical multiplets appear, they are treated as indistinguishable so that, in these cases, the relations are relations among total cross sections. If no indistinguishable particles are

TABLE V. Relations following from charge independence for the reaction $\bar{N}N \to K\bar{K}+n\pi$.

(A)	$\bar{N}N \to K\bar{K}$
σ_1	$\bar{p}p \to K^+K^-$
σ_2	$\bar{p}p \to K^0\bar{K}^0$
σ_3	$\bar{p}n \to K^0K^-$
$(\bar{N}N)_{0\geq 1}$	$2(\sigma_1+\sigma_2)\geq\sigma_3$
$(\bar{N}^TK)_{0\geq 1}$	$2(\sigma_2+\sigma_3)\geq\sigma_1$
$(N^TK)_{0\geq 1}$	$2(\sigma_1+\sigma_3)\geq\sigma_2$
(B)	$\bar{N}N \to K\bar{K}\pi$
σ_1	$\bar{p}p \to K^+K^-\pi^0$
σ_2	$\bar{p}p \to K^0\bar{K}^0\pi^0$
σ_3	$\bar{p}p \to K^+\bar{K}^0\pi^-$
σ_4	$\bar{p}p \to K^0K^-\pi^+$
σ_5	$\bar{p}n \to K^+K^-\pi^-$
σ_6	$\bar{p}n \to K^0K^-\pi^0$
σ_7	$\bar{p}n \to K^0\bar{K}^0\pi^-$
π	$\sigma_3+\sigma_4+\sigma_5+\sigma_7=2(\sigma_1+\sigma_2+\sigma_6)$
$(\bar{N}N)_{0\geq 1}$	$\sigma_3+\sigma_4\geq\sigma_6$
$(K\bar{K})_{0\geq 1}$	$\sigma_5+\sigma_7\geq\sigma_6$
$(\bar{N}^TK)_{0\geq 1}$	$\sigma_4+\sigma_7\geq\sigma_1$
$(N^T\bar{K})_{0\geq 1}$	$\sigma_3+\sigma_5\geq\sigma_1$
$(N^TK)_{0\geq 1}$	$\sigma_3+\sigma_7\geq\sigma_2$
$(\bar{N}^T\bar{K})_{0\geq 1}$	$\sigma_4+\sigma_5\geq\sigma_2$
$(N^T\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_2+\sigma_6\geq\frac{2}{3}\sigma_3$
$(\bar{N}^T\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_2+\sigma_6\geq\frac{2}{3}\sigma_4$
$(\bar{K}\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_2+\sigma_6\geq\frac{2}{3}\sigma_5$
$(K\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_2+\sigma_6\geq\frac{2}{3}\sigma_7$
(C)	$\bar{N}N \to K\bar{K},2\pi$
σ_1	$\bar{p}p \to K^+K^-\pi^+\pi^-$
σ_2	$\bar{p}p \to K^+K^-,2\pi^0$
σ_3	$\bar{p}p \to K^0\bar{K}^0\pi^+\pi^-$
σ_4	$\bar{p}p \to K^0\bar{K}^0,2\pi^0$
σ_5	$\bar{p}p \to K^+\bar{K}^0\pi^-\pi^0$
σ_6	$\bar{p}p \to K^0K^-\pi^+\pi^0$
σ_7	$\bar{p}n \to K^+K^-\pi^-\pi^0$
σ_8	$\bar{p}n \to K^0K^-\pi^+\pi^-$
σ_9	$\bar{p}n \to K^0K^-,2\pi^0$
σ_{10}	$\bar{p}n \to K^0\bar{K}^0\pi^-\pi^0$
σ_{11}	$\bar{p}n \to K^+\bar{K}^0,2\pi^-$
π	$2(\sigma_1+\sigma_3+\sigma_8+\sigma_{11})=\sigma_5+\sigma_6+\sigma_7+\sigma_{10}+4(\sigma_2+\sigma_4+\sigma_9)$
$(\bar{N}N)_{0\geq 1}$	$2(\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6)\geq\sigma_7+\sigma_8+\sigma_9+\sigma_{10}+\sigma_{11}$
$(K\bar{K})_{0\geq 1}$	$2(\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_7+\sigma_{10})\geq\sigma_5+\sigma_6+\sigma_8+\sigma_9+\sigma_{11}$
$(N^TK)_{0\geq 1}$	$2(\sigma_1+\sigma_2+\sigma_5+\sigma_8+\sigma_9+\sigma_{10})\geq\sigma_3+\sigma_4+\sigma_6+\sigma_7+\sigma_{11}$
$(N^T\bar{K})_{0\geq 1}$	$2(\sigma_3+\sigma_4+\sigma_5+\sigma_7+\sigma_8+\sigma_9)\geq\sigma_1+\sigma_2+\sigma_6+\sigma_{10}+\sigma_{11}$
$(2\pi)_{0\geq 1\geq 2}$	$2(\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_8)\geq\sigma_5+\sigma_6+\sigma_7+\sigma_{10}\geq\sigma_{11}$
$(\bar{N}NK^T)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_8+\sigma_9+\sigma_{10}\geq\sigma_7+\sigma_{11}$
$(\bar{N}N\bar{K}^T)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6+\sigma_7+\sigma_8+\sigma_9\geq\sigma_{10}+\sigma_{11}$
$(N^TK\bar{K})_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_7+\sigma_8+\sigma_9+\sigma_{10}\geq\sigma_6+\sigma_{11}$
$(\bar{N}^TK\bar{K})_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_6+\sigma_7+\sigma_8+\sigma_9+\sigma_{10}\geq\sigma_5+\sigma_{11}$
$(\bar{N}N\pi^T)_{1\geq 2}$	$\sigma_5+\sigma_6+\sigma_7+\sigma_{10}+2(\sigma_1+\sigma_3+\sigma_9)\geq\sigma_8$
$(\bar{N}N\pi^T)_{0\geq 1}$	$\sigma_5+\sigma_6+\sigma_7+\sigma_{10}+2\sigma_8+4(\sigma_2+\sigma_4+\sigma_{11})\geq 2(\sigma_1+\sigma_3+\sigma_9)$
$(\bar{N}^TK\pi)_{1\geq 2}$	$\sigma_5+\sigma_6+\sigma_7+\sigma_{10}+2(\sigma_2+\sigma_3+\sigma_8)\geq\sigma_1$
$(\bar{N}^TK\pi)_{0\geq 1}$	$\sigma_5+\sigma_6+\sigma_7+\sigma_{10}+2\sigma_1+4(\sigma_4+\sigma_9+\sigma_{11})\geq 2(\sigma_2+\sigma_3+\sigma_8)$
$(\bar{N}^T\bar{K}\pi)_{1\geq 2}$	$\sigma_5+\sigma_6+\sigma_7+\sigma_{10}+2(\sigma_1+\sigma_4+\sigma_8)\geq\sigma_3$
$(\bar{N}^T\bar{K}\pi)_{0\geq 1}$	$\sigma_5+\sigma_6+\sigma_7+\sigma_{10}+2\sigma_3+4(\sigma_2+\sigma_9+\sigma_{11})\geq 2(\sigma_{11}+\sigma_4+\sigma_8)$

TABLE VI. Relations following from charge independence for the reaction $\bar{N}N \to \Sigma\bar{\Sigma}\pi$.

	$\bar{N}N \to \Sigma\bar{\Sigma}\pi$
σ_1	$\bar{p}p \to \Sigma^+\bar{\Sigma}^-\pi^0$
σ_2	$\bar{p}p \to \Sigma^-\bar{\Sigma}^+\pi^0$
σ_3	$\bar{p}p \to \Sigma^+\bar{\Sigma}^0\pi^-$
σ_4	$\bar{p}p \to \Sigma^0\bar{\Sigma}^+\pi^-$
σ_5	$\bar{p}p \to \Sigma^-\bar{\Sigma}^0\pi^+$
σ_6	$\bar{p}p \to \Sigma^0\bar{\Sigma}^-\pi^+$
σ_7	$\bar{p}p \to \Sigma^0\bar{\Sigma}^0\pi^0$
σ_8	$\bar{p}n \to \Sigma^-\bar{\Sigma}^-\pi^+$
σ_9	$\bar{p}n \to \Sigma^+\bar{\Sigma}^-\pi^-$
σ_{10}	$\bar{p}n \to \Sigma^-\bar{\Sigma}^+\pi^-$
σ_{11}	$\bar{p}n \to \Sigma^-\bar{\Sigma}^0\pi^0$
σ_{12}	$\bar{p}n \to \Sigma^0\bar{\Sigma}^-\pi^0$
σ_{13}	$\bar{p}n \to \Sigma^0\bar{\Sigma}^0\pi^-$
	$\sigma_8+\sigma_9+\sigma_{10}=2\sigma_7+\sigma_{11}+\sigma_{12}+\sigma_{13}$
	$\sigma_3+\sigma_5+\sigma_{11}=\sigma_4+\sigma_6+\sigma_{12}$
	$\sigma_3+\sigma_5+\sigma_{13}=\sigma_1+\sigma_2+\sigma_{12}$
$(\bar{N}N)_{0\geq 1}$	$\sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_6\geq\sigma_{11}+\sigma_{12}+\sigma_{13}$
$(\bar{\Sigma}\Sigma)_{0\geq 1\geq 2}$	$\sigma_1+\sigma_2+\sigma_7+\sigma_9+\sigma_{10}+\sigma_{13}\geq\sigma_3+\sigma_5+\sigma_{11}\geq\sigma_8/2$
$(\Sigma\pi)_{0\geq 1\geq 2}$	$\sigma_3+\sigma_5+\sigma_7+\sigma_8+\sigma_9+\sigma_{12}\geq\sigma_4+\sigma_6+\sigma_{13}\geq\sigma_{10}/2$
$(\bar{\Sigma}\pi)_{0\geq 1\geq 2}$	$\sigma_4+\sigma_6+\sigma_7+\sigma_8+\sigma_{10}+\sigma_{11}\geq\sigma_1+\sigma_2+\sigma_{12}\geq\sigma_9/2$
$(N^T\Sigma)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_7+\sigma_8+\sigma_{10}+\sigma_{13}+2(\sigma_3+\sigma_{11})\geq\sigma_2+\sigma_9$
$(N^T\bar{\Sigma})_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_2+\sigma_7+\sigma_8+\sigma_9+\sigma_{13}+2(\sigma_4+\sigma_{12})\geq\sigma_1+\sigma_{10}$
$(N^T\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_6+\sigma_7+\sigma_9+\sigma_{10}+\sigma_{11}+2(\sigma_5+\sigma_{13})\geq\sigma_4+\sigma_8$
$(\bar{N}^T\Sigma)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_5+\sigma_7+\sigma_8+\sigma_{10}+\sigma_{12}+2(\sigma_2+\sigma_{11})\geq\sigma_3+\sigma_9$
$(\bar{N}^T\bar{\Sigma})_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_6+\sigma_7+\sigma_8+\sigma_9+\sigma_{11}+2(\sigma_1+\sigma_{12})\geq\sigma_4+\sigma_{10}$
$(\bar{N}^T\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_3+\sigma_7+\sigma_9+\sigma_{10}+\sigma_{12}+2(\sigma_4+\sigma_{13})\geq\sigma_5+\sigma_8$

present, the relations can be thought of either as differential or as total cross section relations.

Where there is more than one equality for a given reaction, the "raw" equalities which follow from Shmushkevich have sometimes been combined with each other to simplify them. The "raw" inequalities have often been reduced to a more useful form, but only by combining them with equalities, never with other inequalities. We remark that the relations for $\bar{N}N \to K\bar{K}+2\pi$ have been left in their raw form. Next to each inequality is a notation indicating which subset of particles it arises from and what values of the charge labels are involved. A notation $\{3\pi\}_{0\geq 1}$ means that an inequality comes from

$$\sigma(\nu_{\{3\pi\}}=1)\leqslant\sigma(\nu_{\{3\pi\}}=0).$$

A superscript "T" appears on the symbol for a particle when the particles in a subset are taken partly from the initial and partly from the final state. For example, the notation $(\bar{N}^TK\bar{K})_{\frac{1}{2}\geq\frac{3}{2}}$ indicates that the subset contains an antinucleon selected from one side of the reaction and a $K\bar{K}$ pair from the other side. It will be noticed that inequalities corresponding to some particle subsets and to some values of the charge labels have been omitted. This has been done when an inequality is either satisfied identically or follows from an equality or another, stronger inequality.

One particularly apparent inequality may be written for each set of reactions, namely,

$$\sigma(\nu_{\{\bar{N}N\}}=1)\leqslant\sigma(\nu_{\{\bar{N}N\}}=0),$$

giving

$$\sigma[\bar{p}p]+\sigma[\bar{n}n]=2\sigma[\bar{p}p]\geqslant\sigma[\bar{p}n].$$

The equality holds only if all isospin-zero amplitudes vanish. Even such an obvious inequality is capable of yielding an unexpectedly useful result. For example, when combined with the equality for the reaction $\bar{N}N \to 4\pi$, it gives

$$\sigma(\bar{p}p \to 2\pi^+,2\pi^-)\geqslant\sigma(\bar{p}p \to 4\pi^0)+\sigma(\bar{p}n \to \pi^-,3\pi^0).$$

Depending on the level of certainty with which relevant cross sections have been determined experimentally, there are various uses for which these results

TABLE VII. Relations following from charge independence for the reaction $\bar{N}N \to \Delta\bar{N}+n\pi$.

(A)	$\bar{N}N \to \Delta\bar{N}$	
σ_1	$\bar{p}p \to \Delta^+\bar{p}$	
σ_2	$\bar{p}p \to \Delta^0\bar{n}$	
σ_3	$\bar{p}n \to \Delta^-\bar{n}$	
σ_4	$\bar{p}n \to \Delta^0\bar{p}$	
Δ	$\sigma_3=\sigma_1+\sigma_2+\sigma_4$	$\sigma_1=\sigma_2=\sigma_3/3=\sigma_4$
$(\bar{N}N)_{0=1}$	$2(\sigma_1+\sigma_2)=\sigma_3+\sigma_4$	
$(\bar{N}^T\Delta)_{0=1}$	$2(\sigma_2+\sigma_4)=\sigma_1+\sigma_3$	
(B)	$\bar{N}N \to \Delta\bar{N}\pi$	
σ_1	$\bar{p}p \to \Delta^{++}\bar{p}\pi^-$	
σ_2	$\bar{p}p \to \Delta^+\bar{p}\pi^0$	
σ_3	$\bar{p}p \to \Delta^+\bar{n}\pi^-$	
σ_4	$\bar{p}p \to \Delta^0\bar{p}\pi^+$	
σ_5	$\bar{p}p \to \Delta^0\bar{n}\pi^0$	
σ_6	$\bar{p}p \to \Delta^-\bar{n}\pi^+$	
σ_7	$\bar{p}n \to \Delta^-\bar{n}\pi^0$	
σ_8	$\bar{p}n \to \Delta^-\bar{p}\pi^+$	
σ_9	$\bar{p}n \to \Delta^0\bar{n}\pi^-$	
σ_{10}	$\bar{p}n \to \Delta^0\bar{p}\pi^0$	
σ_{11}	$\bar{p}n \to \Delta^+\bar{p}\pi^-$	
	$\sigma_7=(\frac{3}{2})\sigma_9$	
Δ	$\sigma_1+\sigma_6+\sigma_7+\sigma_8=\sigma_2+\sigma_3+\sigma_4+\sigma_5+\sigma_9+\sigma_{10}+\sigma_{11}$	
$(\bar{N}\Delta)_{0=1\geq 2}$	$\sigma_1+\sigma_3+\sigma_4+\sigma_6+\sigma_7+\sigma_{10}=2(\sigma_2+\sigma_5+\sigma_9+\sigma_{11})\geq\sigma_8$	
$(N^T\Delta)_{0=1\geq 2}$	$\sigma_1+\sigma_4+\sigma_5+\sigma_7+\sigma_8+\sigma_{11}=2(\sigma_2+\sigma_3+\sigma_9+\sigma_{10})\geq\sigma_6$	
$(\bar{N}^T\Delta)_{0=1\geq 2}$	$\sigma_2+\sigma_3+\sigma_6+\sigma_7+\sigma_8+\sigma_{11}=2(\sigma_4+\sigma_5+\sigma_9+\sigma_{10})\geq\sigma_1$	
$(\Delta\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_1+\sigma_6+\sigma_8\geq\sigma_9$	
$(\bar{N}\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_7+\sigma_8\geq\sigma_{11}$	
$(N^T\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_7+\sigma_6\geq\sigma_3$	
$(\bar{N}^T\pi)_{\frac{1}{2}\geq\frac{3}{2}}$	$\sigma_7+\sigma_1\geq\sigma_4$	
$(\bar{N}N)_{0\geq 1}$	$\sigma_1+\sigma_3+\sigma_4+\sigma_6\geq\sigma_7+\sigma_{10}$	
$(N^T\bar{N})_{0\geq 1}$	$\sigma_3+\sigma_6+\sigma_8+\sigma_{11}\geq\sigma_7+\sigma_2$	
$(\bar{N}^T\bar{N})_{0\geq 1}$	$\sigma_1+\sigma_4+\sigma_8+\sigma_{11}\geq\sigma_7+\sigma_5$	

TABLE VIII. Relations following from charge independence for the reaction $\bar{N}N \to \Delta\bar{\Delta}$.

	$\bar{N}N \to \Delta\bar{\Delta}$
σ_1	$\bar{p}p \to \Delta^{++}\bar{\Delta}^{--}$
σ_2	$\bar{p}p \to \Delta^{+}\bar{\Delta}^{-}$
σ_3	$\bar{p}p \to \Delta^{0}\bar{\Delta}^{0}$
σ_4	$\bar{p}p \to \Delta^{-}\bar{\Delta}^{+}$
σ_5	$\bar{p}n \to \Delta^{-}\bar{\Delta}^{0}$
σ_6	$\bar{p}n \to \Delta^{0}\bar{\Delta}^{-}$
σ_7	$\bar{p}n \to \Delta^{+}\bar{\Delta}^{--}$
	$\sigma_5=\sigma_7$
	$\sigma_1+\sigma_4=\sigma_2+\sigma_3+\sigma_6$
$(\bar{N}^T\Delta)_{0-1}\geq 2$	$2(\sigma_3+\sigma_6)=\sigma_2+\sigma_4+2\sigma_5\geq\sigma_1$
$(N^T\Delta)_{0-1}\geq 2$	$2(\sigma_2+\sigma_6)=\sigma_1+\sigma_3+2\sigma_5\geq\sigma_4$
$(\bar{N}N)_0\geq 1$	$4\sigma_2+4\sigma_3+\sigma_6\geq 2\sigma_5$

may serve. If a sufficient number of the cross sections has been measured, the relations will serve as a test of charge independence. On the other hand, if certain cross sections are unmeasured or unmeasurable, the relations will often provide upper or lower bounds for them.

Some of the results can be easily compared to the predictions of the statistical model, in which statistical weights[14] for each charge complexion are calculated giving each isospin amplitude equal weight. For example, charge independence requires that

$$\frac{\sigma(\bar{p}n \to \pi^+,\, 2\pi^-,\, \pi^0)}{\sigma(\bar{p}n \to \pi^-,\, 3\pi^0)} \geq \tfrac{3}{2},$$

no matter how the individual amplitudes behave, whereas the statistical weights give the value 4 for this ratio.

Observing the notation $\{3\pi\}_{0\geq 1}$ next to our relation, we see that the inequality becomes an equality only if all amplitudes corresponding to three pions having isospin zero vanish.

We may expect, in general, that inequalities will be easily satisfied; if some inequality is close to being an equality, then the vanishing of a set of amplitudes is indicated. The possession of sufficiently detailed experimental results will therefore permit us to investigate the relative strength of amplitudes and to uncover effects occurring in any channel. Using the usual procedure, such effects could be investigated only by examining every possible coupling scheme.

[14] F. Cerulus, Nuovo Cimento Suppl. **15**, 402 (1960); G. Pinski, Ph.D. thesis, University of Rochester, 1963 (upublished).

V. CONCLUDING REMARKS

We have been able to find relations linear in the cross sections merely by tabulating sums of cross sections which correspond to any subset of $\{k\}$ particles having any charge label $\nu_{\{k\}}$. The linear relations follow from Eq. (15). Each such sum of cross sections is equal to a linear combination of the absolute squares of all the amplitudes, in a given coupling scheme, for which $I_{\{k\}}\geqslant\nu_{\{k\}}$. These combinations may be written at sight, the coefficient of each squared amplitude being $(2I+1)/(2I_{\{k\}}+1)$. When we take such a sum, all phases disappear and with them all information concerning interference effects. However, as is pointed out in the Appendix, the relations which arise from dealing with phases are complicated nonlinear equations and involve explicit Clebsch-Gordan coefficients. These relations, being nonlinear, cannot be added to give relations among total cross sections, when indistinguishable particles are present, and hence are useless for many-particle reactions.

APPENDIX: NONLINEAR RELATIONS

If ρ is the number of amplitudes, there are also ρ phases, one of which is arbitrary. There are therefore $(2\rho-1)$ independent real quantities. If one enumerates the amplitudes as $T_1,\cdots,T_\rho$, then each cross term,

$$|T_i^*T_j+T_j^*T_i|=2|T_i||T_j|\cos(\delta_i-\delta_j),$$

which appears in the expansion of a particular single cross section introduces a phase difference $(\delta_i-\delta_j)\equiv\delta_{ij}$, $i<j$. There may be as many as $\rho(\rho-1)/2$ phase differences appearing, if each cross term appears in the expansion of at least one cross section. Only $\rho-1$ of these phase differences are independent. We may choose as the independent ones the $\delta_{l,l+1}$. The dependent phase differences are then

$$\delta_{ij}=\sum_{l=i}^{j-1}\delta_{l,l+1}.$$

If there are W'-independent cross sections and $2\rho-1$-independent real quantities, there must be a total of $W'-(2\rho-1)$ equalities among the cross sections. We have discussed how linear equalities may be found. Those relations which are nonlinear arise from the elimination of dependent phase differences.