Ground State of an Electron Gas in a Magnetic Field*

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The ground state of an electron gas in a uniform magnetic field is found to be not the customary uniform state, but rather one in which a spin-density wave exists, directed along the field. This conclusion is reached through what is essentially a Hartree-Fock calculation with a repulsive interaction, but in which no restrictive assumptions are made about either the strength *or the range* of the exchange interaction. Thus static screening does *not* eliminate the spin-density wave in the presence of a magnetic field, as it does in the electron gas when no magnetic field is present. The temperature at which the transition to a spin-density-wave state occurs approaches zero as the field vanishes. The pertinent question is therefore not the nature of the ground state, but whether there is a range of field strengths and electronic densities for which the transition temperature is observably high. It is found that spin-density-wave formation is most favorable when only a few Landau levels are occupied, corresponding to large field strengths and low electronic densities. A rough calculation indicates that in InSb a transition temperature as high as 10 millidegrees can be realized.

I. INTRODUCTION

THE ground state of a gas composed of electrons in a uniform magnetic field is usually considered to be a spatially uniform state with a net magnetization directed along the field, resulting from the combined effects of Pauli spin paramagnetism and Landau diamagnetism. We wish to raise the possibility that this picture is incorrect, and that the true ground state is a spatially nonuniform one in which the net spin magnetic moment has a small component perpendicular to the magnetic field, which is circularly polarized along the direction of the field. This is in no sense a rigorous conclusion, since we shall be working in what is essentially a Hartree-Fock approximation. However we shall make no assumptions about the form of the exchange interaction (other than its sign). Thus our analysis is not susceptible to the objection that has been raised against arguments that such a spin-density wave should exist in the absence of a magnetic field, namely, that the apparent effect is entirely due to the artificial use of a long-range Coulomb interaction and disappears when the interaction is properly screened. We shall show that in any magnetic field (as long as it is not so strong that the electron spins are completely aligned) there is always (within the Hartree-Fock approximation) a transition to a spin-density wave state at sufficiently low temperatures, for any repulsive-static-exchange interaction, regardless of its magnitude or range.

We hasten to add that this discovery will not revolutionize solid state physics. The crucial question is how low the transition temperature will be. Since the transition occurs in arbitrarily weak fields even when there is no such transition in the absence of a field, the transition temperature must go to zero for small fields. Evidently the extraordinary state need only concern physicists if the transition temperature is observably high.[1] Calculation of the transition temperature is a very delicate matter and the estimates we shall give, based on rough calculations, are in no sense conclusive. However at present it appears possible that under very restrictive conditions (high fields, low densities, small effective mass, high purity) the transition may occur at temperatures in the vicinity of 10^{-2} degrees Kelvin.

We call the extraordinary state a spin-density wave because of its close similarity to the spin-density waves investigated theoretically by Cloizeaux[2] and Overhauser[3] in the absence of magnetic fields. Overhauser's first communication on the subject pointed out that for spin-$\frac{1}{2}$ fermions in one dimension a spin-density wave state has a lower Hartree-Fock ground-state energy than the conventional state for arbitrarily weak short-range repulsive interactions. Kohn and Nettel[4] quickly

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[1] The possibility of extraordinary nonuniform states of lower energy than the conventional one in the presence of magnetic fields has been known to us for some time [N. D. Mermin and V. Celli, Phys. Rev. **136**, A364 (1964)]. Such a state occurs even if exchange interactions are neglected as a result of the very weak magnetic current-current interaction. Its existence results in an instability in the helicon dispersion relation, if this is calculated in a fully quantum-mechanical way, using Landau levels for the one-electron states. However the transition temperature for this state is unobservably low under nonrelativistic conditions and the state is of interest only to the more theoretical of theoreticians, as an explanation for the rather annoying fact that Newcomb's classical magnetohydrodynamic stability theorem cannot be generalized to the quantum case. We mention this rather bizarre case to emphasize that the mere existence of such states, though amusing, need not be of any practical interest unless the transition temperature can be shown to lie in an accessible range.

[2] J. des Cloizeaux, J. Phys. Radium **20**, 606, 751 (1959).

[3] A. W. Overhauser, Phys. Rev. Letters **4**, 462 (1960); Phys. Rev. **128**, 1437 (1962); J. Appl. Phys. **34**, 1019 (1963).

[4] W. Kohn and S. J. Nettel, Phys. Rev. Letters **5**, 8 (1960).

pointed out that his argument cannot be correct for general arbitrarily weak-repulsive interactions in three dimensions, but Overhauser was able to show that the spin-density wave will still exist in three dimensions, provided the exchange interaction, though as weak as one likes, has a long-range Coulombic form. The counter to this argument is, of course, that one expects the exchange interaction to be screened, and if the calculation is done using a static Fermi-Thomas screening for the exchange, the spin-density wave is again ruled out.[5] This is about as far as the argument can be carried without running up against horrendous mathematical difficulties. The spin-density wave enthusiast is down but not out, for he can claim with some justification that since the crucial coupling in spin-density-wave formation involves electrons rather far apart from one another in momentum space, the Fermi-Thomas exchange interaction need not be reliable for such large momentum transfers; his hope is that a treatment using the full dynamical frequency and momentum depending screening, will again rescue the spin-density wave.

We must emphasize that this controversy concerns only the question of whether spin-density waves can exist under electron gas conditions, or, in applications to metals, when the Fermi surface is close to being spherical as in the alkali metals. Complicated band structure effects can, if they lead to a Fermi surface with large relatively flat portions, permit spin-density-wave formation, essentially by making things look one-dimensional near the flat parts of the Fermi surface. There does not seem to be any theoretical objection to the view that the antiferromagnetism of chromium is due to a spin-density wave, permitted by just this kind of peculiarity in the band structure.[6]

However, the existence of a spin-density wave without assistance from the band structure, and, in particular, under electron gas conditions, strikes us as being most improbable, although as we have said, theorists are not unanimous on this point. Overhauser, for instance, not only maintains that a spin-density wave can exist in potassium, but gives it so large an amplitude that the transition temperature would appear to be on the order of the melting point or higher.[7] Our point of view is quite different from his. We assume that spin-density waves do not exist in an electron gas in the absence of a magnetic field, note that a strong magnetic field enhances the possibility of their formation at accessible temperatures, and then, under very favorable circumstances, estimate a transition temperature in the neighborhood of 10 millidegrees.

The reason for such an enormous disparity in opinion is first, of course, that an accurate microscopic calculation of the transition temperature (or even of whether the transition exists at all) presents formidable difficulties, but, more importantly, that in a weak-coupling theory T_c depends exponentially on the inverse of the coupling constant, so that small changes in the strength of the interaction can result in order of magnitude changes of T_c. Since nobody is terribly sure what the coupling should be, it is hard to settle the issue by such a calculation. Because of this uncertainity we shall take a rather conservative point of view, assuming that the exchange interaction is weak and of short range, so that the formation of a spin-density wave is out of the question in the absence of a magnetic field.

We stress once more that our conclusion that a spin-density wave state always arises at sufficiently low temperatures is reached only within what is essentially the Hartree-Fock approximation, except that we do not make any assumptions (which are crucial to the existence of the effect in no field) about the strength or range of the exchange interaction. Because of the calculational complexities that a magnetic field introduces even in a Hartree-Fock calculation, we would like to give a rough qualitative indication of the basically simple reasons why even a very weak magnetic field can cause the formation of a spin-density wave at very low temperatures.

We start from Overhauser's original observation that the Hartree-Fock ground state of a one-dimensional electron gas has a spin-density wave (for an appropriate range of densities) due to the exchange interaction resulting from any repulsive two-body potential. If one forgot about the spin-density-wave states, one would think that the ground state would either be ferromagnetic or without any spin alignment, depending on the balance between the exchange interaction (favoring ferromagnetic alignment) and the kinetic energy (favoring equal numbers of electrons with each spin). However in the spin-density-wave state a compromise is reached which does better than either of these two possibilities. Single electron states are taken to be superpositions of states of opposite spin on opposite sides of the Fermi surface. Because only states near the Fermi surface differ from the conventional single-particle states, the cost in kinetic energy is not nearly as great as it is in a ferromagnetic state. Furthermore by taking such superpositions one can produce a spatially varying local spin alignment, which lowers the potential energy. It is important, however, for the lowering of the potential energy that all pairs of states making up the new single particle states have the same separation in momentum space. If this were not so, then the many paired states would not add up to a coherent spin-density wave, and there would be no gain in potential energy. When, however, all the paired states are separated by the same momentum p, then they do produce a net local spin alignment, the direction of which varies with a wave vector $p/\hbar$. Spin-density waves are therefore likely to be encountered when there is a set of pairs of states of opposite spin such that all of the states are near the Fermi surface, and such that the displacement in

[5] P. A. Fedders, P. C. Martin, and H. Ehrenreich, Phys. Rev. Letters (to be published).
[6] See, for instance, Ref. 5 and the third of Refs. 3.
[7] A. W. Overhauser, Phys. Rev. Letters **13**, 190 (1964).

momentum space of the states within each pair is the same for all the pairs. Evidently a one-dimensional electron gas is just the thing one wants, insofar as one can think of it as a three-dimensional gas in which the Fermi surface is two parallel planes. Chromium is also a good candidate for a spin-density wave, because of the large flat portions of the Fermi surface. However, in an electron gas (or, more generally, if the Fermi surface has no large flat portions) a coherent spin-density wave requires occupation of one-electron states rather far above the Fermi surface, and the gain in reduced potential energy will only be large enough to compensate for this if the exchange interaction is stronger or of longer range than it is usually thought to be.

However, a magnetic field, by quantizing motion in the plane perpendicular to the field, greatly increases the number of pairs of electrons at opposite ends of the Fermi surface, separated by the same momentum (parallel to the field). To take, for instance, an extreme case, if the field is so strong that only the $n=0$ Landau levels are occupied, then one can regard the Fermi surface as a cylinder with two parallel flat ends perpendicular to the field. In any magnetic field, the density of states, as a function of p_z, the momentum parallel to the field, looks like a sum of one-dimensional densities of states (one such term for each n) and this is enough to give a spin-density-wave ground state for arbitrarily weak repulsive interactions. As the magnetic field gets weaker, the weight of each one-dimensional density of states in the total density of states gets smaller, as does the effective coupling within each "one-dimensional" system. Nevertheless, because the one-dimensional spin-density wave exists for arbitrarily weak coupling, any magnetic field at all is enough to give a spin-density-wave ground state.[8]

Because the value of the transition temperature is the crucial question, and because calculation of the ground-state properties is very difficult, even in a Hartree-Fock type of approximation, we shall primarily be working at temperatures close to T_c. In Sec. II we first derive what the normal Hartree-Fock thermal equilibrium state is in a magnetic field. In Sec. III we show that such a state is always unstable (i.e., is a stationary point, but not a local minimum of the free energy) at sufficiently low temperatures. The stability analysis gives us a way of calculating the transition temperature, and an explicit expression for T_c is found. The manner in which the normal state becomes unstable suggests what the spin-density-wave state should look like just below T_c. Guided by the stability analysis, in Sec. IV we construct an explicit density matrix for the spin-density-wave state, derive a "gap" equation, and show that it begins to have solutions at the same T_c for which the normal state becomes unstable. We also prove that below T_c the spin-density-wave state has lower free energy than the normal state. In Sec. V we make some rough estimates of what the transition temperature might be expected to be in various substances.

We have concerned ourselves only with the question of the existence of the spin-density-wave state in this paper, reserving a general discussion of its properties for a future publication. However one can show that some of its thermodynamic properties should be similar to those of the superconducting state. (Overhauser first pointed this out, and his arguments appear to be qualitatively unaltered by the magnetic field.) For example, there should be a specific heat discontinuity at the transition temperature of very much the same form as is found at the superconducting transition.

We conclude this introduction with a summary of the notation to be used in subsequent sections. We take the magnetic field $\mathbf{B}$ to be directed along the z axis and use the Landau gauge,

$$A_x=A_z=0, \quad A_y=Bx.$$

The Hamiltonian for a single electron in the field $\mathbf{B}$ is then

$$H_0=(2m)^{-1}[\mathbf{p}-(e/c)\mathbf{A}]^2-\omega_0 s_z, \tag{1.1}$$

where

$$\omega_0=g^*eB/2m_0c, \quad e=-|e|,$$

and we allow for the possibility that the mass m appearing in (1.1) may differ from the free-electron mass m_0. The effective Lande factor g^* takes all band effects into account, and s_z has eigenvalues $s=+\frac{1}{2}$ or $s=-\frac{1}{2}$ for spin parallel or antiparallel to $\mathbf{B}$. (We use units in which $\hbar=1$.) The eigenstates of the spatial part of H_0 are specified by the three quantum numbers q, p, and n, which we denote collectively by α; the eigenstates are

$$\psi_\alpha(\mathbf{r})=\frac{1}{L}e^{i(qy+pz)}\phi_n(x+q/m\omega_c),$$

where $\phi_n(x)$ are the orthonormal eigenstates of the one-dimensional harmonic oscillator Hamiltonian

$$\frac{p^2}{2m}+\tfrac{1}{2}m\omega_c^2x^2, \quad \omega_c=|e|B/mc.$$

(We work in a cubical box of side L with faces perpendicular to the coordinate axes.) The corresponding

[8] These categorical statements are always to be understood to mean within the Hartree-Fock approximation (but without assumptions on the form of the repulsive interaction).

Note added in proof. The question of whether our conclusions are valid beyond the Hartree-Fock approximation is, of course, a very difficult one. Experimental evidence (e.g., the de Haas effect) leaves no doubt that correlation effects do not destroy the essential features of the quantized Landau levels, so we would expect the favorable "one-dimensionality" brought about by the magnetic field to persist even when correlation effects are taken into account. The really critical question, then, is whether correlation effects can remove the spin density wave even when the "one-dimensionalness" of the situation still favors them. This is a question that is pertinent to the study of spin density waves, both in and out of magnetic fields. A convincing theoretical answer is probably beyond computational ability. However the experimentally established existence of spin density waves in chromium is evidence that correlations need not eliminate spin density waves.

eigenvalues are

$$E_\alpha = p^2/2m + (n+\tfrac{1}{2})\omega_c .$$

We shall use Latin indices to denote the three Landau quantum numbers and the spin-quantum number, (q,p,n,s), so that the eigenvalues of the full single-particle Hamiltonian (1.1) are

$$\epsilon_i^0 = p^2/2m + (n+\tfrac{1}{2})\omega_c - \omega_0 s .$$

We shall take the electrons to interact through a potential

$$v(\mathbf{r}-\mathbf{r}') = \frac{1}{L^3}\sum_{\mathbf{k}} \exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]v(k^2), \qquad (1.2)$$

about which we shall make no restrictive assumptions in establishing the existence of spin-density waves, except for the requirement that certain matrix elements be positive. Thus v could be of very short range, and/or very weak.

II. NORMAL HARTREE-FOCK STATE

By the normal Hartree-Fock state we mean a distribution of electrons in Landau states with self-consistently determined single particle energies. We stress the fact that the Landau states *do* give a solution to the Hartree-Fock equations, since this point is essential to our argument. It is not, however, immediately obvious, as can be seen by comparing the matrix element for scattering between Landau states with the corresponding matrix element between plane wave states in the absence of a magnetic field. In the latter case the total momentum of the scattered pair is conserved: $\langle \mathbf{p}_1\mathbf{p}_2|v|\mathbf{p}_3\mathbf{p}_4\rangle \propto \delta_{\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4}$, and it is the presence of this delta function in the exchange term that permits one to solve the Hartree-Fock equation with plane-wave states. Landau states would trivially be solutions in a magnetic field if scattering conserved the sum of each of the three quantum numbers n, q, and p, of the initial and final pairs, but this is not so. In fact $\langle\alpha\beta|v|\gamma\delta\rangle$ is proportional to $\delta_{p_\alpha+p_\beta-p_\gamma-p_\delta}\delta_{q_\alpha+q_\beta-q_\gamma-q_\delta}$, but it fails to vanish when $n_\alpha+n_\beta \neq n_\gamma+n_\delta$ [see Appendix A, Eq. (A7)]. Nevertheless Landau states do solve the Hartree-Fock problem. This is because the energy of a Landau level does not depend on the quantum number q. If we therefore occupy Landau levels having the same n and p with the same statistical weight for all q, then the exchange potential will enter into the problem only in the form

$$\sum_{q_\beta}\langle\alpha\beta|v|\gamma\delta\rangle|_{q_\gamma=q_\beta},$$

and in this form it does vanish unless $n_\alpha+n_\beta=n_\gamma+n_\delta$ [see Eq. (A8)]. One can then verify that the resulting single-particle energy levels are still independent of q, so that the initial ansatz for the occupation of states differing only in q is self consistent.

The only nontrivial part of the Hartree-Fock calculation is the proof of (A8), which is derived in Appendix A. Given this it is easily shown that the Landau states solve the Hartree-Fock equation with single particle energies given by solutions to

$$\epsilon_s(np) = \epsilon_s^0(np) - \sum_{n'}\int \frac{dp'}{2\pi} g_{nn'}(p-p')f(\epsilon_s(n'p')-\mu), \qquad (2.1)$$

$$f(x) = 1/(e^{\beta x}+1), \quad \beta = 1/K_BT,$$

where

$$g_{nn'}(p) = \int_0^\infty \frac{kdk}{2\pi} v(p^2+k^2)|\langle n|e^{ikx}|n'\rangle|^2,$$
$$\langle n|e^{ikx}|n'\rangle = \int_{-\infty}^{\infty} dx\phi_{n'}(x)e^{ikx}\phi_n(x). \qquad (2.2)$$

[As in the absence of a magnetic field, the direct term gives an energy $\int d\mathbf{r}v(\mathbf{r})n_0$ which is cancelled by the uniform background of positive charge.]

III. STABILITY OF THE NORMAL HARTREE-FOCK STATE

The Hartree-Fock equations are just the condition that the free energy

$$F = \sum_{ij}(H_0)_{ij}\varphi_{ji} + \tfrac{1}{2}\sum_{ijmn}\varphi_{ji}\varphi_{mn}(\langle in|v|jm\rangle - \langle in|v|mj\rangle) + \frac{1}{\beta}\sum_{ij}\varphi_{ij}(\ln\varphi)_{ji} + (1-\varphi)_{ij}(\ln(1-\varphi))_{ji} \qquad (3.1)$$

be stationary under variations of the single-particle density matrix φ. (Here Latin indices stand for spin- and spatial-quantum numbers, i.e., $i \leftrightarrow \alpha$, s, $s=\pm\frac{1}{2}$.) The stability condition—that the stationary point be a local minimum—requires that the second variation[9]:

$$\delta^2F = \sum_{ij}\frac{\epsilon_i-\epsilon_j}{f_j-f_i}|\delta\varphi_{ij}|^2 + \sum_{ijmn}\delta\varphi_{ji}\delta\varphi_{mn}(\langle in|v|jm\rangle - \langle in|v|mj\rangle), \qquad (3.2)$$

be positive for all variations $\delta\varphi$ of the density matrix, where ϵ_i is the single-particle energy (2.1), and $f_i = f(\epsilon_i-\mu)$. Equation (3.2) is put in a more useful form by substituting

$$\delta\varphi_{ij} = -[(f_i-f_j)/(\epsilon_j-\epsilon_i)]\Delta_{ij}, \qquad (3.3)$$

[9] Condition (3.2) is derived in N. D. Mermin, Ann. Phys. (N. Y.) **21**, 99 (1963). It also arises from requiring that all spin-wave oscillations about the normal state have real frequencies, provided one makes some plausible assumptions about the onset of the spin-wave instability.

equivalent to lowest order in Δ to letting

$$\delta\varphi_{ij}=[f(\epsilon+\Delta-\mu)-f(\epsilon-\mu)]_{ij}. \tag{3.4}$$

Evidently Δ is the variation in the single-particle energy; in terms of Δ the stability condition is

$$\sum_{ij}|\Delta_{ij}|^2\frac{f_i-f_j}{\epsilon_j-\epsilon_i}+\sum_{ij,mn}\Delta_{ji}\frac{f_i-f_j}{\epsilon_j-\epsilon_i}\times(\langle in|v|jm\rangle-\langle in|v|mj\rangle)\Delta_{mn}\frac{f_m-f_n}{\epsilon_n-\epsilon_m}\geqslant 0. \tag{3.5}$$

We wish to examine the stability of the normal state against variations in Δ that are off-diagonal in spin. For a general variation of this kind the stability condition is quite complicated; we consider the general case in Appendix B, where we argue that at least for weak coupling (which is almost certain to be the case in any real situation) the instability will first occur for Δ diagonal in n and q, and independent of q. The physical content of this observation is that the transition is to a state in which the spin-density wave is directed along the magnetic field. We therefore consider in (3.5) variations of the form

$$\Delta_{ij}=\tfrac{1}{2}(\delta_{s_i\uparrow}\delta_{s_j\downarrow}+\delta_{s_i\downarrow}\delta_{s_j\uparrow})\delta_{q_iq_j}\delta_{n_in_j}\Delta_{n_i}(p_i,p_j), \tag{3.6}$$

$$\uparrow=+\tfrac{1}{2},\quad \downarrow=-\tfrac{1}{2}.$$

If we define

$$\Delta_n(p+\tfrac{1}{2}P,p-\tfrac{1}{2}P)=\Delta_n{}^P(p), \tag{3.7}$$

$$G_n{}^P(p)=\frac{f_{n\uparrow}(p+\frac{1}{2}P)-f_{n\downarrow}(p-\frac{1}{2}P)}{\epsilon_{n\downarrow}(p-\frac{1}{2}P)-\epsilon_{n\uparrow}(p+\frac{1}{2}P)}, \tag{3.8}$$

$$f_{ns}(p)=f(\epsilon_{ns}(p)-\mu), \tag{3.9}$$

then the stability condition breaks up into a sum of independent terms for each value of P. Using (A8) and converting momentum sums to integrals, we find that the normal state will be unstable provided

$$0>\sum_n\int\frac{dp}{2\pi}\Delta_n{}^P(p)^2G_n{}^P(p)-\sum_{nn'}\int\frac{dpdp'}{(2\pi)^2}\Delta_n{}^P(p)G_n{}^P(p)\times g_{nn'}(p-p')\Delta_{n'}{}^P(p')G_{n'}{}^P(p'), \tag{3.10}$$

for some P.

We define a Fermi momentum for each Landau cylinder and spin by

$$\epsilon_{n\uparrow}(p_{n\uparrow})=\epsilon_{n\downarrow}(p_{n\downarrow})=\mu. \tag{3.11}$$

The behavior of Eq. (3.10) is very much like that encountered in the analysis of the Overhauser instability of the one-dimensional Fermi gas. At zero temperature when $P=p_{n\uparrow}+p_{n\downarrow}$, $G_n{}^P(p)$ is singular at $p=(p_{n\uparrow}-p_{n\downarrow})/2$, which leads to a logarithmic divergence of the (one-dimensional) integral, $\int dpG_n{}^P(p)$. Thus, if $g_{nn}(p)$ is positive, any choice of Δ_n which is positive and nonvanishing at $p=\frac{1}{2}(p_{n\uparrow}-p_{n\downarrow})$ will lead to the product of two logarithmic divergences in the second (negative) term of (3.10), against only one in the first (positive) term. Thus the choice

$$P=P_n=p_{n\uparrow}+p_{n\downarrow} \tag{3.12}$$

for any n leads to the value $-\infty$ for (3.10) at $T=0$, i.e., the normal ground state is unstable for arbitrarily weak repulsive interactions in an arbitrarily weak magnetic field. Since we know[4,5] that there is no instability for arbitrarily weak forces in the absence of a magnetic field, the transition temperature for the onset of the instability must approach zero as $B\to 0$, and the physically pertinent question is how high the transition temperature can be for attainable field strengths. To answer this question we anticipate that T_c will in fact be quite low (probably on the order of ω_c or less under the most favorable conditions), and calculate the stability condition for a particular choice $n=\nu$ in (3.12) under the assumption that T_c is much less than the range of occupied energies in either of the two νth Landau cylinders.

It is first useful to show that in this limit

$$\int\frac{dp}{2\pi}G_\nu{}^{P_\nu}(p)=A_\nu\ln\beta+B_\nu+\text{terms that vanish as } T\to 0. \tag{3.13}$$

Evidently T_c will be a very sensitive function of A, which must therefore be evaluated with as much accuracy as possible. We shall have to be content with a more crude estimate of B, but errors in B have a considerably less drastic effect on T_c.

To calculate A note that

$$A=\lim_{\beta\to\infty}\beta\frac{\partial}{\partial\beta}\int\frac{dp}{2\pi}G(p)=\lim_{\beta\to\infty}\int\frac{dp}{2\pi}\frac{(\epsilon_\uparrow(p+p_\uparrow)-\mu)\dfrac{\partial}{\partial\epsilon_\uparrow}f_\uparrow(p+p_\uparrow)-(\epsilon_\downarrow(p-p_\downarrow)-\mu)\dfrac{\partial}{\partial\epsilon_\downarrow}f_\downarrow(p-p_\downarrow)}{\epsilon_\downarrow(p-p_\downarrow)-\epsilon_\uparrow(p+p_\uparrow)}. \tag{3.14}$$

(Throughout the next two paragraphs we drop the index ν and P_ν to simplify the formulas.) The derivative of $f_\uparrow$ is nonvanishing only in the neighborhood of $p=0$ and $p=-2p_\uparrow$. At both of these points $\epsilon(p+p_\uparrow)-\mu=0$, but at the point $p=0$ the denominator of the integrand also vanishes, whereas it is nonvanishing at $p=-2p_\uparrow$; thus the first term contributes to the integral only in the neighborhood of $p=0$. The same can be seen to be true

of the term in $f_\downarrow$, and therefore the whole integral can be evaluated as $T\to 0$ by expanding the ϵ's explicitly appearing in (3.14) about $p=0$:

$$\epsilon_\uparrow(p+p_\uparrow)=\mu+pv_\uparrow\,,\quad \epsilon_\downarrow(p-p_\downarrow)=\mu-pv_\uparrow\,,\qquad v_{ns}=(\partial/\partial p)\epsilon_{ns}(p)\,|_{p=p_{ns}}\,. \tag{3.15}$$

Inserting (3.15) in (3.14), transforming the ϵ derivatives to p derivatives, and taking the limit $T\to 0$, one finds

$$A_\nu=1/\pi(v_{\nu\uparrow}+v_{\nu\downarrow})\,. \tag{3.16}$$

To calculate B it is necessary in principle to know the detailed p dependence of the single-particle energies. Since T_c is not nearly as sensitive to B as it is to A, we shall simply approximate these by a quadratic law chosen to agree with (3.15) near $p=\pm p_\uparrow$, $\pm p_\downarrow$:

$$\epsilon_s(p)=\frac{p^2}{2m_s}+\left(\mu-\frac{p_s^2}{2m_s}\right),\quad m_s=p_s/v_s\,. \tag{3.17}$$

This gives

$$\int\frac{dp}{2\pi}G(p)=\frac{1}{v_\uparrow+v_\downarrow}\int\frac{dp}{2\pi}\frac{f_\uparrow(p+p_\uparrow)-f_\downarrow(p-p_\downarrow)}{\dfrac{p^2}{p_0}-p}, \tag{3.18}$$

$$p_0=(m_\uparrow p_\downarrow+m_\downarrow p_\uparrow)/(m_\uparrow-m_\downarrow)\,.$$

Integrating by parts, we can write (3.18)

$$\int\frac{dp}{2\pi}G(p)=\frac{1}{v_\uparrow+v_\downarrow}\int\frac{dp}{2\pi}\left[\ln\left|\frac{p+p_\downarrow-p_0}{p+p_\downarrow}\right|\frac{\partial}{\partial p}f_\downarrow(p)-\ln\left|\frac{p-p_\uparrow-p_0}{p-p_\uparrow}\right|\frac{\partial}{\partial p}f_\uparrow(p)\right]. \tag{3.19}$$

If we replace the p derivatives of f by δ functions everwhere that this leads to finite expressions, we find

$$\int\frac{dp}{2\pi}G(p)=\frac{1}{2\pi(v_\uparrow+v_\downarrow)}\left[\ln\left|\frac{4p_\uparrow p_\downarrow}{(1-2p_\uparrow/p_0)(1-2p_\downarrow/p_0)}\right|+\int_0^\infty dp\left(\ln(p-p_\uparrow)\frac{\partial}{\partial p}f_\uparrow(p)+\ln(p-p_\downarrow)\frac{\partial}{\partial p}f_\downarrow(p)\right)\right]. \tag{3.20}$$

The last two integrals in (3.20) can be replaced to the same degree of accuracy by

$$\ln(v_\uparrow v_\downarrow)+2\ln\beta+\sum_s\int_{-\beta p_s^2/2m_s}^\infty dt\ln|t|\frac{\partial}{\partial t}\frac{1}{e^t+1}\,. \tag{3.21}$$

Provided $\beta p_s^2/2m_s\gg 1$, we can replace the lower limits of both integrals by $-\infty$, and using the result

$$\int_{-\infty}^\infty dt\ln|t|\frac{\partial}{\partial t}\frac{1}{e^t+1}=\gamma-\ln\frac{\pi}{2}\,,$$

where γ is Euler's constant, we find

$$\int\frac{dp}{2\pi}G_\nu^{P_\nu}(p)=\frac{1}{\pi(v_{\nu\uparrow}+v_{\nu\downarrow})}\times\ln\left(\frac{8C}{\pi}\beta\left[\frac{\mu_{\nu\uparrow}\mu_{\nu\downarrow}}{(1-2p_{\nu\uparrow}/p_0)(1-2p_{\nu\downarrow}/p_0)}\right]^{1/2}\right),\qquad \mu_{\nu s}=p_{\nu s}^2/2m_{\nu s}\,,\quad e^\gamma=C\,. \tag{3.22}$$

Having established the low-temperature form (3.13) we now evaluate the stability condition at temperatures low enough that (3.13) can be used. If we define

$$F(\Delta)=\frac{\displaystyle\sum_{nn'}\int\frac{dpdp'}{(2\pi)^2}\Delta_n^P(p)G_n^P(p)g_{nn'}(p-p')G_{n'}^P(p')\Delta_{n'}^P(p')}{\displaystyle\sum_n\int\frac{dp}{2\pi}\Delta_n^P(p)^2G_n^P(p)}\,, \tag{3.23}$$

then the normal state will be unstable at temperatures low enough that for some choice of Δ,

$$F(\Delta)>1\,. \tag{3.24}$$

We continue using $P=P_\nu$ for some fixed ν and drop explicit reference to P from subsequent formulas. We evaluate (3.23) by retaining only the nonvanishing terms as $T\to 0$. This requires that we evaluate the denominator to order $A\ln\beta+B$, and the numerator to order $A(\ln\beta)^2+B\ln\beta$. We define p_ν to be $\frac{1}{2}(p_{\nu\uparrow}-p_{\nu\downarrow})$, the value of p at which G_ν has its pole. We can write the denominator of (3.23) in the form

$$\Delta_\nu^2(p_\nu)\int\frac{dp}{2\pi}G_\nu(p)+\int\frac{dp}{2\pi}(\Delta_\nu^2(p)-\Delta_\nu^2(p_\nu))G_\nu(p)+\sum_{n\neq\nu}\int\frac{dp}{2\pi}\Delta_n^2(p)G_n(p)\,. \tag{3.25}$$

The second term in (3.25) approaches a constant as $T\to 0$, since $G_\nu(p)$ is multiplied by a term that cancels its pole at $p=p_\nu$. The third term in (3.25) is also well behaved as $T\to 0$, since the value $n=\nu$ is excluded from

the sum. Thus the numerator of (3.23) can be written

$$A_\nu \ln\beta+B_\nu+\int\frac{dp}{2\pi}(\Delta_\nu^2(p)-1)G_\nu(p)+\sum_{n\neq\nu}\int\frac{dp}{2\pi}\Delta_n^2(p)G_n(p), \quad (3.26)$$

where we have normalized Δ so that $\Delta_\nu(p_\nu)=1$, and the last two integrals in (3.26) are to be evaluated in the limit $T\to 0$.

In a similar way the denominator of (3.23) can be rearranged to give

$$g_{\nu\nu}(0)(A_\nu\ln\beta+B_\nu)^2+2(A_\nu\ln\beta+B_\nu)\times\left(\int\frac{dp}{2\pi}G_\nu(p)[\Delta_\nu(p)g_{\nu\nu}(p-p_\nu)-g_{\nu\nu}(0)]+\sum_{n\neq\nu}\int\frac{dp}{2\pi}G_n(p)[\Delta_n(p)g_{\nu n}(p_\nu-p)]\right) \quad (3.27)$$

plus terms that approach constants as $T\to 0$.

Placing (3.27) and (3.26) in (3.23) and evaluating $F(\Delta)$ to order $A\ln\beta+B$, we find

$$F(\Delta)=g_{\nu\nu}(0)(A_\nu\ln\beta+B_\nu)+2\left\{\int\frac{dp}{2\pi}G_\nu(p)[\Delta_\nu(p)g_{\nu\nu}(p_\nu-p)-g_{\nu\nu}(0)]+\sum_{n\neq\nu}\int\frac{dp}{2\pi}g_{\nu n}(p_\nu-p)\Delta_n(p)G_n(p)\right\}-g_{\nu\nu}(0)\times\left\{\int\frac{dp}{2\pi}[(\Delta_\nu(p)^2-1)G_\nu(p)+\sum_{n\neq\nu}\Delta_n(p)^2G_n(p)]\right\}. \quad (3.28)$$

Minimizing this with respect to Δ gives

$$\Delta_n(p)=g_{\nu n}(p_\nu-p)/g_{\nu\nu}(0). \quad (3.29)$$

If we place this value of Δ in (3.28) and use the explicit forms (3.22) for the coefficients A_ν and B_ν we find for the stability condition

$$\frac{1}{g_{\nu\nu}(0)}\geqslant\frac{1}{\pi(v_\uparrow+v_\downarrow)}\times\ln\left(\frac{8\beta C}{\pi}\left[\frac{\mu_\uparrow\mu_\downarrow}{(1-2p_\uparrow/p_0)(1-2p_\downarrow/p_0)}\right]^{1/2}\right)-\int\frac{dp}{2\pi}G_\nu{}^{P_\nu}(p)[1-(g_{\nu\nu}(p_\nu-p)/g_{\nu\nu}(0))^2]+\sum_{n\neq\nu}\int\frac{dp}{2\pi}G_n{}^{P_\nu}(p)(g_{\nu n}(p_\nu-p)/g_{\nu\nu}(0))^2, \quad (3.30)$$

where $\mu_\uparrow$, $\mu_\downarrow$, $v_\uparrow$, $v_\downarrow$, $p_\uparrow$, $p_\downarrow$, and p_0 [see (3.18)] are all the values for the νth Landau cylinder.

IV. SPIN-DENSITY WAVE STATE NEAR T_c

We have seen that the normal state is unstable, below T_c, for variations in the one-particle density matrix connecting states of opposite spin with the same n and q, and values of p at opposite points on the Fermi surface. Guided by this, we re-examine the Hartree-Fock problem starting with a set of trial single particle states more general than the Landau levels, chosen to allow for the possibility of the kind of pairing for which the Hartree-Fock problem of Sec. II becomes unstable. We choose the complete orthonormal set:

$$\psi_\alpha{}^{(1)}(\mathbf{r})=a_\alpha\psi_{\alpha+}(\mathbf{r})+b_\alpha\psi_{\alpha-}(r),$$
$$\psi_\alpha{}^{(2)}(\mathbf{r})=b_\alpha{}^*\psi_{\alpha+}(\mathbf{r})-a_\alpha{}^*\psi_{\alpha-}(r), \quad (4.1)$$
$$|a_\alpha|^2+|b_\alpha|^2=1,$$

where

$$\alpha+=n, q, p+\tfrac{1}{2}P, \uparrow$$
$$\alpha-=n, q, p-\tfrac{1}{2}P, \downarrow$$

for every n, q, and p. Note that if a and b are allowed only the values 0 or 1, the set (4.1) reduces to a set of Landau levels, so that the original Hartree-Fock problem is contained within this one. We expect that near T_c a and b will have values appreciably different from 1 or 0 only over a very small range of p.

We define a set of statistical weights for the set (4.1)

$$w_\alpha{}^{(\lambda)}, \quad \lambda=1,2, \quad (4.2)$$

so that the single-particle density matrix is

$$\langle r|\varphi|r'\rangle=\sum_{\lambda\alpha}w_\alpha{}^{(\lambda)}\psi_\alpha{}^{(\lambda)}(r)\psi_\alpha{}^{(\lambda)}(r')^*. \quad (4.3)$$

Guided again by the kind of instability found in Sec. III, we consider only the case where the a's, b's, and w's are independent of the quantum number q. Given (4.2) and (4.1) we can calculate the mean density and mean spin density at any point in space. We find:

$$\langle n(\mathbf{r})\rangle=\frac{m\omega_c}{(2\pi)^2}\sum_n\int dpW_{np},$$
$$\langle s_z(r)\rangle=\frac{1}{2}\frac{m\omega_c}{(2\pi)^2}\sum_n\int dpw_{np}(|a_{np}|^2-|b_{np}|^2),$$
$$\langle s_+(r)\rangle=\frac{m\omega_c}{(2\pi)^2}\sum_n\int dpw_{np}a_{np}{}^*b_{np}e^{-iPz}, \quad (4.4)$$
$$\langle s_-(r)\rangle=\langle s_+(r)\rangle^*,$$
$$s_\pm=s_x\pm is_y,$$

where

$$w_\alpha=w_\alpha{}^{(1)}-w_\alpha{}^{(2)}, \quad W_\alpha=w_\alpha{}^{(1)}+w_\alpha{}^{(2)}. \quad (4.5)$$

In the normal state either a or b is zero for any value of n and p, so there is no component of spin perpendicular to **B**. However if there are values of n and p for which both a and b are nonvanishing, then there will be a nonvanishing spin density perpendicular to the field, the direction of which is constant in the x-y plane for a given value of z, but which varies in the z direction. This variation can be thought of as a superposition of many spin-density waves, one for each set of Landau levels with the same n, all with wave vector P.[10] Note that if we replace the a's and b's in (4.1) by the set

$$a_\alpha' = e^{i\varphi} a_\alpha, \quad b_\alpha' = e^{i\varphi} b_\alpha, \tag{4.6}$$

where φ is independent of q, p, n, then all that results is a shift in phase of the spin-density wave. This degeneracy will be reflected in the variational calculation, from which we can conclude that the best choice of a's and b's is one in which they all have the same phase (so that in particular they might all be real). However there is nothing in the variational calculation to determine this phase. [Thus, if one is bothered by the fact that the equilibrium states we are working with are not eigenstates at $T=0$ of the total spin or total z momentum (which is reflected in the existence of a nonvanishing spin current along the direction of **B**), one can work with linear combinations of (4.3) for different phases, to arrive at states that satisfy the conservation laws.]

We now evaluate the Hartree-Fock free-energy using the states (4.1) and statistical weights (4.2), and then minimize the result with respect to the a's, b's, and w's.

The single-particle energy can be written in the form[11]

$$\langle H_0\rangle = \frac{m\omega_c}{2\pi} L^3 \times \sum_n \int \frac{dp}{2\pi} \left\{ W_{np} \left[\frac{1}{2m}(p^2 + \tfrac{1}{4}P^2) + (n+\tfrac{1}{2})\omega_c - \mu \right] - w_{np} \left(\frac{pP}{2m} - \tfrac{1}{2}\omega_0 \right) (|b_{np}|^2 - |a_{np}|^2) \right\}. \tag{4.7}$$

The direct part of the interaction energy is just a constant, cancelled by the uniform background of positive charge. The exchange energy can be evaluated with the results of Appendix A, using the fact that the a's, b's, and w's are independent of q. The result is[11]

$$-\frac{m\omega_c}{2\pi}\frac{1}{4} L^3 \int \frac{dp\,dp'}{(2\pi)^2} \sum_{nn'} g_{nn'}(p-p') \{ W_{np} W_{n'p'} + w_{np} w_{n'p'} [(|b_{np}|^2 - |a_{np}|^2)(|b_{n'p'}|^2 - |a_{n'p'}|^2) + 4\,\mathrm{Re}(a_{np}^* b_{np} a_{n'p'} b_{n'p'}^*)] \}, \tag{4.8}$$

with g given by (2.2). For a repulsive interaction we wildo best by choosing the a's and b's to be real and positive, so we will restrict ourselves to that case [bearing in mind, however, that the transformation (4.6) leaves (4.7) and (4.8) invariant, and thus provides us with an equally good choice].

If we add to (4.7) and (4.8) the Hartree-Fock expression for the entropy, the free energy becomes (we have absorbed μN into E)

$$F = E - TS = \frac{m\omega_c L^3}{2\pi} \Big\{ \sum_n \int \frac{dp}{2\pi} \Big(W_{np} \Big[\frac{1}{2m}(p^2 + \tfrac{1}{4}P^2) + (n+\tfrac{1}{2})\omega_c - \mu \Big] - w_{np} \Big[\frac{pP}{2m} - \tfrac{1}{2}\omega_0 \Big] [b_{np}^2 - a_{np}^2] \Big)$$
$$- \frac{1}{4} \sum_{nn'} \int \frac{dp\,dp'}{(2\pi)^2} g_{nn'}(p-p') (W_{np} W_{n'p'} + w_{np} w_{n'p'} [(b_{np}^2 - a_{np}^2)(b_{n'p'}^2 - a_{n'p'}^2) + 4\gamma_0 a_{np} b_{np} a_{n'p'} b_{n'p'}])$$
$$+ T \sum_n \int \frac{dp}{2\pi} \sum_\lambda [w_{np}^{(\lambda)} \ln w_{np}^{(\lambda)} + (1 - w_{np}^{(\lambda)}) \ln(1 - w_{np}^{(\lambda)})] \Big\}. \tag{4.9}$$

The constant γ_0 appearing in (4.9) is numerically equal to one, but it is convenient, in proving that the spin-density-wave state has lower free energy than the normal state, to allow it to assume values between zero and one. The physical case, however, is always $\gamma_0 = 1$.

In performing the variational calculation it is convenient to define several new quantities. We introduce an "energy gap," Δ_{np} by

$$\Delta_{np} = \gamma_0 \sum_{n'} \int \frac{dp'}{2\pi} g_{nn'}(p-p') w_{n'p'} a_{n'p'} b_{n'p'}. \tag{4.10}$$

[Comparing this with the third of equations (4.4) we see that Δ is a measure of the magnitude of the self-consistent exchange field due to the spin-density wave.] We also define ω_{np} by

$$\omega_{np} = -\sum_{n'} \int \frac{dp'}{2\pi} g_{nn'}(p-p') w_{n'p'} (b_{n'p'}^2 - a_{n'p'}^2). \tag{4.11}$$

[Comparing this with the second of equations (4.4),

[10] One could also consider different P for each Landau n-level, but it is only when each level has the same P that the spin-density waves from different levels combine coherently. The distinction between the two possibilities is, however, probably academic since, as it turns out, in either case only the spin-density wave in the last occupied set of n levels has any appreciable amplitude.

[11] If one allows for the possibility of different P for different n levels, then if P has the value $P(n)$ for the nth level, (4.7) is modified by replacing P by $P(n)$, while (4.8) is unchanged except that $a_{np}^* b_{np} a_{n'p'} b_{n'p'}^*$ must be multiplied by $\delta_{P(n),P(n')}$.

one can interpret $\frac{1}{2}\omega_{np}$ as the exchange energy due to interaction of a given electron with the self-consistent exchange field arising from the component of spin along the magnetic field.] Finally, we introduce the total self-energy difference, $2\Omega_{np}$, between the states $\psi_{\alpha+}$ and $\psi_{\alpha-}$,

$$\Omega_{np}=\tfrac{1}{2}(pP/m-\omega_0-\omega_{np}). \tag{4.12}$$

If we minimize (4.9) with respect to the a's and b's (remembering that $a_{np}^2+b_{np}^2=1$), we find that

$$a_{np}b_{np}=\tfrac{1}{2}\Delta_{np}/(\Delta_{np}^2+\Omega_{np}^2)^{1/2}, \tag{4.13}$$

or, equivalently,

$$b_{np}^2-a_{np}^2=\Omega_{np}/(\Delta_{np}^2+\Omega_{np}^2)^{1/2}. \tag{4.14}$$

Minimizing with respect to the $w_{np}^{(\lambda)}$ gives

$$w_{np}^{(\lambda)}=f(\epsilon_{np}^{(\lambda)}-\mu), \tag{4.15}$$

where the quasiparticle energies, $\epsilon_{np}^{(\lambda)}$, are given by ($\lambda=1, 2$)

$$\epsilon_{np}^{(\lambda)}=\frac{1}{2m}(p^2+\tfrac{1}{4}P^2+(n+\tfrac{1}{2}\omega_c)) -\frac{1}{2}\int\frac{dp'}{2\pi}\sum_{n'}g_{nn'}(p-p')W_{n'p'} +(-1)^{\lambda}(\Delta_{np}^2+\Omega_{np}^2)^{1/2}. \tag{4.16}$$

Equations (4.12)–(4.16), when inserted in (4.10) and (4.11), give a pair a pair of nonlinear coupled equations which determine Δ and Ω.

Evidently if one is to go any further, considerable additional approximations are necessary. Since in this paper we are only concerned with the *existence* of spin-density-wave states and the value of the transition temperature, we shall limit ourselves to showing, first, that the gap equation begins to have solutions at just the temperature determined by the stability analysis of Sec. III, and second, that whenever such solutions exist they give a lower free energy than the normal state.

First note that quite generally we can write (4.10) and (4.11) as

$$\Delta_{np}=\gamma_0\sum_{n'}\int\frac{dp'}{2\pi}g_{nn'}(p-p') \times\frac{w_{n'p'}^{(1)}-w_{n'p'}^{(2)}}{\epsilon_{n'p'}^{(2)}-\epsilon_{n'p'}^{(1)}}\Delta_{n'p'}, \tag{4.17}$$

and

$$\omega_{np}=-\sum_{n'}\int\frac{dp'}{2\pi}g_{nn'}(p-p') \times\frac{w_{n'p'}^{(1)}-w_{n'p'}^{(2)}}{\epsilon_{n'p'}^{(2)}-\epsilon_{n'p'}^{(1)}}2\Omega_{n'p'}. \tag{4.18}$$

Just below the transition temperature the a's and b's are very close to zero or one for almost all values of n and p [i.e., from (4.4), the spin-density wave has a very small amplitude] so that $\Delta_{np}\approx 0$ near T_c. As Δ vanishes, (4.18) reduces to

$$\omega_{np}=-\sum_{n'}\int\frac{dp'}{2\pi}g_{nn'}(p-p') \times(w_{n'p'}^{(1)}-w_{n'p'}^{(2)})\,\mathrm{sgn}(\Omega_{n'p'}). \tag{4.19}$$

We define

$$\epsilon_{n,p+\frac{1}{2}P,\uparrow}=\frac{1}{2m}(p+\tfrac{1}{2}P)^2+(n+\tfrac{1}{2})\omega_c-\tfrac{1}{2}\omega_0-\tfrac{1}{2}\omega_{np} -\frac{1}{2}\sum_{n'}\int\frac{dp'}{2\pi}g_{nn'}(p-p')W_{n'p'}, \tag{4.20}$$

and

$$\epsilon_{n,p-\frac{1}{2}P,\downarrow}=\frac{1}{2m}(p-\tfrac{1}{2}P)^2+(n+\tfrac{1}{2})\omega_c+\tfrac{1}{2}\omega_0+\tfrac{1}{2}\omega_{np} -\frac{1}{2}\sum_{n'}\int\frac{dp'}{2\pi}g_{nn'}(p-p')W_{n'p'}, \tag{4.21}$$

[where the notation is the same as in (2.1) in anticipation of the fact that (4.20) and (4.21) reduce to (2.1) as $\Delta\to 0$]. From (4.16) it follows that as $\Delta\to 0$, we have

$$\begin{aligned}\epsilon_{np}^{(1)}&=\epsilon_{n,p-\frac{1}{2}P,\downarrow}\\ \epsilon_{np}^{(2)}&=\epsilon_{n,p+\frac{1}{2}P,\uparrow}\end{aligned}\qquad \Omega_{np}>0, \tag{4.22}$$

and

$$\begin{aligned}\epsilon_{np}^{(1)}&=\epsilon_{n,p+\frac{1}{2}P,\uparrow}\\ \epsilon_{np}^{(2)}&=\epsilon_{n,p-\frac{1}{2}P,\downarrow}\end{aligned}\qquad \Omega_{np}<0. \tag{4.23}$$

Therefore, one can write (4.19) as

$$\omega_{np}=\sum_{n'}\int\frac{dp'}{2\pi}g_{nn'}(p-p') \times[f(\epsilon_{n,p'+\frac{1}{2}P,\uparrow}-\mu)-f(\epsilon_{n',p'-\frac{1}{2}P,\downarrow}-\mu)]. \tag{4.24}$$

Placing (4.24) in (4.20) and (4.21) we find

$$\epsilon_{np\uparrow}=\frac{p^2}{2m}+(n+\tfrac{1}{2})\omega_c-\tfrac{1}{2}\omega_0 -\sum_{n'}\int\frac{dp'}{2\pi}g_{nn'}(p-p')f(\epsilon_{n'p'\uparrow}-\mu), \tag{4.25}$$

and

$$\epsilon_{np\downarrow}=\frac{p^2}{2m}+(n+\tfrac{1}{2})\omega_c+\tfrac{1}{2}\omega_0 -\sum_{n'}\int\frac{dp'}{2\pi}g_{nn'}(p-p')f(\epsilon_{n'p'\downarrow}-\mu), \tag{4.26}$$

so that (4.20) and (4.21) reduce to the normal Hartree-Fock quasiparticle energies as $\Delta\to 0$.

If one retains only terms linear in Δ in (4.17), one

finds, using (4.22) and (4.23),

$$\Delta_{np}=\gamma_0\sum_{n'}\int\frac{dp'}{2\pi}g_{nn'}(p-p')G_{n'}{}^{P}(p')\Delta_{n'p'}, \quad (4.27)$$

where G is given by (3.8). Now the normal state becomes unstable at that value of T for which (3.10) can be made to vanish for some $P=P_0$, and $\Delta_{np}{}^{P_0}=\Delta_{np}{}^0$. Since the Hermitian form (3.10) assumes (at T_c) its *minimum* value for $P=P_0$, $\Delta_{np}=\Delta_{np}{}^0$, it must also be true that

$$0=\Delta_{np}{}^0-\sum_{n'}\int\frac{dp'}{2\pi}g_{nn'}(p-p')G_{n'}{}^{P_0}(p')\Delta_{n'p'}{}^0. \quad (4.28)$$

A solution to (4.27) (when $\gamma_0=1$) at T_c is therefore given by taking $P=P_0$ and $\Delta_{np}=\Delta_{np}{}^0$. Thus the linearized gap equation will have solutions at the T_c calculated in Sec. II.

We next prove that when nontrivial solutions to (4.17) and (4.18) exist, they always give a lower value to the free energy than the corresponding normal solution, $\Delta\equiv 0$. To see this, consider minimizing the free energy (4.9) for *each* value of γ_0 between zero and one. Because the a's, b's, and w's are determined by making F stationary, $\partial F_{\gamma_0}{}^{\min}/\partial\gamma_0$ is given by differentiating only the explicit occurence of γ_0, so that

$$\begin{aligned}\frac{\partial}{\partial\gamma_0}F_{\gamma_0}{}^{\min}&=-\sum_{nn'}\int\frac{dpdp'}{(2\pi)^2}g_{nn'}(p-p')\\&\quad\times w_{np}w_{n'p'}a_{np}b_{np}a_{n'p'}b_{n'p'}\\&=-\frac{1}{2\gamma_0}\sum_{n}\int\frac{dp}{2\pi}w_{np}\Delta_{np}{}^2/(\Omega_{np}{}^2+\Delta_{np}{}^2)^{1/2},\end{aligned} \quad (4.29)$$

[from (4.13) and (4.10)]. Now when $\gamma_0=0$, (4.17) requires that $\Delta=0$, so the solution to the self-consistent equations minimizing F is just the normal state. If we are below T_c, then for some value of γ_0 between zero and one, nontrivial solutions to (4.17) will appear, and at $\gamma_0=1$, the solution is the actual spin-density-wave state. Hence

$$\begin{aligned}\Delta F=F_{\rm SDW}-F_{\rm normal}&=\int_0^1 d\gamma_0\frac{\partial}{\partial\gamma_0}F^{\min}\\&=-\frac{1}{2}\int_0^1\frac{d\gamma_0}{\gamma_0}\int\frac{dp}{2\pi}\frac{w_{np}\Delta_{np}{}^2}{(\Omega_{np}{}^2+\Delta_{np}{}^2)^{1/2}},\end{aligned}$$

where ΔF is just the free-energy difference between the normal and spin-density-wave states for the given temperature and chemical potential. Since $\epsilon_{np}{}^{(1)}<\epsilon_{np}{}^{(2)}$, $w_{np}=f(\epsilon_{np}{}^{(1)}-\mu)-f(\epsilon_{np}{}^{(2)}-\mu)$ is always positive, so $\Delta F<0$.[12]

[12] At zero temperature it is easy to carry out the same argument at fixed density, leading to the conclusion that the spin-density wave ground state has lower energy than the normal one at a given density.

Before concluding this section we should point out that the best values of P can also be determined by differentiating with respect to the explicit occurence of P in (4.9); this gives

$$P\int\frac{dp}{2\pi}\sum_n W_{np}=\int\frac{dp}{2\pi}\sum_n w_{np}(b_{np}{}^2-a_{np}{}^2). \quad (4.30)$$

Comparing this with the first two of equations (4.4), we see that P is just minus the spin-current density divided by the mass density. (Spin-density wave states have the peculiarity of having a net nonvanishing spin-current density—i.e., spin-up electrons have a net current, as do the spin down ones. The two currents are equal and opposite so there is no net current.) To get the best spin-density-wave state one should also investigate the solutions to (4.30) along with (4.10) and (4.11). This is a difficult matter. Even near T_c the linearized equation turns out to be satisfied identically for any P. Since P is closely related to the wavelength of the spin-density wave, it is a parameter of some physical importance, and (4.30) deserves further study.

V. ESTIMATE OF T_c

We conclude with a rough estimate of the transition temperature as determined by (3.30). One should not give too much quantitative significance to the results of this section. The calculation is crude, and our aim is to get some idea of the circumstances under which a spin-density wave might be observable, rather than to compute T_c accurately. Thus at best we are estimating the order of magnitude of T_c (and at worst, the order of magnitude of its order of magnitude).

To get a preliminary orientation, we first consider the case in which many Landau levels are occupied, and the interaction is of the static screened Fermi-Thomas form:

$$v(p)=4\pi e^2/(p^2+k_{FT}{}^2), \quad k_{FT}{}^2=(4/\pi)me^2p_F. \quad (5.1)$$

(If many Landau levels are occupied the screening should be quite close to its value in the absence of a magnetic field; in the case in which only a few Landau levels are occupied, which we shall examine later, we will have to take into account the magnetic field dependence of the screening.) We make the further approximation of replacing $v(p)$ by its value at $p=0$:

$$v(p)\equiv v(0)=\pi^2/mp_F. \quad (5.2)$$

We suspect that since this has the effect of increasing the size of v everywhere in momentum space, it will lead to an overestimate of T_c (although we have not verified this in any detail).

When v is given by (5.2), $g_{nn'}$ is most easily computed

by using (A3) to rewrite (2.2) in the form:

$$g_{nn'}(p)=v(0)\int\frac{dk_xdk_y}{(2\pi)^2}\int dxdx'\times e^{ik_x(x-x')}\phi_n\left(x+\frac{k_y}{2m\omega_c}\right)\phi_{n'}\left(x-\frac{k_y}{2m\omega_c}\right)\times\phi_n\left(x'+\frac{k_y}{2m\omega_c}\right)\phi_{n'}\left(x'-\frac{k_y}{2m\omega_c}\right). \quad (5.3)$$

It follows from (5.3) that

$$g_{nn'}(p)\equiv(m\omega_c/2\pi)v(0), \quad (5.4)$$

independent of p, n, or n'.

We can certainly ignore the effective mass difference between up and down spins in the same Landau n level; placing (5.4) in (3.30) we find that the transition temperature for an instability in the νth Landau level is

$$T_c=\frac{4e^\gamma}{\pi}\frac{p_{\nu\uparrow}p_{\nu\downarrow}}{m}\exp\left[-\frac{2p_F}{m\omega_c}(p_{n\uparrow}+p_{n\downarrow})\times\left(1-m\omega_c\sum_{n\neq\nu}\int\frac{dp}{(2\pi)^2}G_n^{P_\nu}(p)\Big/\frac{mp_F}{\pi^2}\right)\right]. \quad (5.5)$$

If the highest occupied Landau levels have quantum number $n_{\max}$ ($p_f^2/2m=n_{\max}\omega_c$) then letting

$$p_{\nu\uparrow}^2/2m\approx p_{\nu\downarrow}^2/2m\approx(n_{\max}-\nu)\omega_c, \quad (5.6)$$

we can write (5.5) as

$$T_c\approx(8e^\gamma/\pi)\omega_c(n_{\max}-\nu)\exp\left\{-8[n_{\max}(n_{\max}-\nu)]^{1/2}\times\left(1-\sum_{n\neq\nu}\frac{\pi^2\omega_c}{p_F}\int\frac{dp}{(2\pi)^2}G_n^{P_\nu}(p)\right)\right\}. \quad (5.7)$$

We first show that at metallic densities (5.7) implies that

$$T_c<(8e^\gamma/\pi)\omega_c(n_{\max}-\nu)e^{-4[n_{\max}(n_{\max}-\nu)]^{1/2}}. \quad (5.8)$$

To establish (5.8) we must show that

$$1-\frac{\pi^2\omega_c}{p_F}\sum_{n\neq\nu}\int\frac{dp}{(2\pi)^2}G_n^{P_\nu}(p)>\tfrac{1}{2}. \quad (5.9)$$

Since $T\ll\omega_c$ only the term with $n=\nu$, which is excluded from the sum in (5.9) has an appreciable singularity. Therefore when many Landau levels are occupied (5.9) is very well approximated by converting the sum over n into an integral. (The mathematical argument here is very much like that used in handling Bose-Einstein condensation.) This amounts precisely to taking the zero-field limit for the orbital electronic motion. If we assume a parabolic spectrum, then we can replace (5.9) by

$$\tfrac{1}{2}>\frac{\pi^2}{mp_F}I,$$

$$I=\int\frac{d\mathbf{p}}{(2\pi)^3}\frac{f\left(\frac{1}{2m}(\mathbf{p}+\frac{1}{2}\mathbf{P})^2-\frac{1}{2}\omega_0-\mu\right)-f\left(\frac{1}{2m}(\mathbf{p}-\frac{1}{2}\mathbf{P})^2+\frac{1}{2}\omega_0-\mu\right)}{\omega_0-\mathbf{p}\cdot\mathbf{P}/m}, \quad (5.10)$$

$$\mathbf{P}=\hat{\mathbf{z}}P_\nu.$$

We next perform the following sequence of operations on the integral occurring in (5.10):

$$\begin{aligned}I&=-\int_{-1}^{1}\frac{d\lambda}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}f'\left(\frac{1}{2m}(\mathbf{p}^2+\lambda\mathbf{p}\cdot\mathbf{P}+\tfrac{1}{4}\mathbf{P}^2)-\frac{\lambda}{2}\omega_0-\mu\right)\\&=-\int_{-1}^{1}\frac{d\lambda}{2}\int\frac{d\mathbf{p}}{(2\pi)^3}f'\left(\frac{1}{2m}p^2+\frac{1}{8m}P^2(1-\lambda^2)-\frac{\lambda}{2}\omega_0-\mu\right)\\&=-\int_{-1}^{1}\frac{d\lambda}{2}\int_0^\infty\frac{p^2dp}{2\pi^2}\frac{\partial}{\partial(p^2/2m)}f\left(\frac{1}{2m}p^2+\frac{1}{8m}P^2(1-\lambda^2)-\frac{\lambda}{2}\omega_0-\mu\right)\\&=\int_{-1}^{1}\frac{d\lambda}{2}\frac{m}{2\pi^2}\int_0^\infty dpf\left(\frac{1}{2m}p^2+\frac{1}{8m}P^2(1-\lambda^2)-\frac{\lambda}{2}\omega_0-\mu\right).\end{aligned} \quad (5.11)$$

The last form occurring in (5.11) demonstrates that the integral I is maximum at $P=0$. Since $\omega_0\ll\mu$, the term in ω_0 can be ignored, and, evaluating the integral at $P=0$, we find that it is bounded above by $mp_F/2\pi^2$, which completes the proof of (5.9).

For metallic densities of electrons even at fields as

large as 10 kG, $n_{\max}$ will be on the order of a thousand; ω_c at such fields could be as high as a few degrees K, so (5.8) requires the transition temperature to be hopelessly low due to the large argument of the exponential. It increases quite rapidly as n gets closer to $n_{\max}$, but even when we consider the last few occupied levels, the tiny factor

$$e^{-a(n_{\max})^{1/2}}, \tag{5.12}$$

where a is of order unity, eliminates any hope of observing the transition at metallic densities.

We therefore consider next the case in which **B** is so large and the density so low that only a single Landau cylinder is occupied at $T=0$. In this case (3.30) leads to a T_c given by

$$T_c=\frac{4e^\gamma}{\pi}\frac{p_\uparrow p_\downarrow}{m}e^{-\pi(p_\uparrow+p_\downarrow)/mg_{00}(0)} \times\exp\left[-\frac{1}{2}\int_0^{2p_\uparrow}\frac{dp}{p}\left(1-\frac{g_{00}(p)^2}{g_{00}(0)^2}\right) -\frac{1}{2}\int_0^{2p_\downarrow}\frac{dp}{p}\left(1-\frac{g_{00}(p)^2}{g_{00}(0)^2}\right)\right]. \tag{5.13}$$

(We again neglect the effective mass difference at the spin-up and spin-down Fermi surfaces, and assume a parabolic spectrum.) We suspect that the size of T_c is primarily determined by the first exponential in (5.13). By neglecting the second we overestimate T_c (since the argument of the second is negative), so requiring the T_c determined by

$$T_c=\frac{4e^\gamma}{\pi}\frac{p_\uparrow p_\downarrow}{m}e^{-\pi(p_\uparrow+p_\downarrow)/mg_{00}(0)} \tag{5.14}$$

to be observably high gives a necessary condition that the transition be observable. If we can achieve this it will then be necessary to go back and examine the second exponential to make sure that it does not depress T_c too much.

For $v(\mathbf{r})$ we continue to use a statically screened Coulomb potential, but since we are now under extreme quantum conditions, the Fermi-Thomas screening is inadequate, and we must calculate the screening using the dielectric constant in a magnetic field. This calculation has been done by Stephen[13] and by Mermin and Canel.[14] The result is

$$\begin{aligned} &v(k_{\parallel},k_\perp)=4\pi e^2/(k_{\parallel}^2+k_\perp^2+4\pi e^2|L^0(k_\perp,k_{\parallel})|),\\ &L^0=L_\uparrow{}^0+L_\downarrow{}^0,\\ &L_s{}^0=m\omega_c\int\frac{dp}{(2\pi)^2}\sum_{nn'}\left[\frac{f(\epsilon_{ns}(p-\frac{1}{2}k_{\parallel})-\mu)-f(\epsilon_{ns'}(p+\frac{1}{2}k_{\parallel})-\mu)}{\epsilon_{ns}(p-\frac{1}{2}k_{\parallel})-\epsilon_{ns'}(p+\frac{1}{2}k_{\parallel})}|\langle n|e^{ik_\perp x}|n'\rangle|^2\right]. \end{aligned} \tag{5.15}$$

At this point we assume, for simplicity, a parabolic spectrum:

$$\epsilon_{ns}=p^2/2m+(n+\tfrac{1}{2})\omega_c-s\omega_0, \tag{5.16}$$

where m is the effective mass and ω_0 contains the renormalized g factor. Since $f(\epsilon_n-\mu)=0$ unless $n=0$, (5.15) can be considerably simplified. Using the fact that [see Ref. 14, Eq. (A8)]

$$|\langle 0|e^{ik_\perp x}|n\rangle|^2=x^ne^{-x}/n!, \quad x=k_\perp{}^2/2m\omega_c. \tag{5.17}$$

We find that

$$|L^0(k_\perp,0)|=\frac{mn_0}{p_\uparrow p_\downarrow}e^{-x}\left\{1+\frac{2p_\uparrow p_\downarrow}{m\omega_c}\sum_{n=1}^{\infty}\frac{x^n}{nn!}\right\}, \tag{5.18}$$

or

$$v(0,k_\perp{}^2)=\frac{4\pi e^2}{2m\omega_c}\left(x+e^{-x}\left\{x_0+x_1\sum_{n=1}^{\infty}\frac{x^n}{nn!}\right\}\right)^{-1}, \tag{5.19}$$

where

$$x_0=(me^2/\pi)[(p_\uparrow+p_\downarrow)/p_\uparrow p_\downarrow], \tag{5.20}$$

$$x_1=(2e^2/\pi\omega_c)(p_\uparrow+p_\downarrow), \tag{5.21}$$

and we have used the fact that when a single Landau cylinder is occupied

$$n_0=(m\omega_c/2\pi^2)(p_\uparrow+p_\downarrow). \tag{5.22}$$

Using (5.19) we can calculate $g_{00}(0)$. [Note that the argument of Appendix A does not require an isotropic $v(\mathbf{k})$, but goes through unchanged if v has the form $v(\mathbf{k})=v(k_{\parallel}{}^2,k_\perp{}^2)$.] Since

$$|\langle 0|e^{ik_\perp x}|0\rangle|^2=e^{-k_\perp{}^2/2m\omega_c} \tag{5.23}$$

we have

$$g_{00}(0)=e^2\int_0^\infty dx\frac{e^{-x}}{x+e^{-x}\left\{x_0+x_1\sum_{n=1}^{\infty}\frac{x^n}{nn!}\right\}}. \tag{5.24}$$

Inserting this in (5.14) one finds

$$T_c=\frac{4p_\uparrow p_\downarrow}{m\pi}e^\gamma\exp[-(p_\uparrow+p_\downarrow)^2/p_\uparrow p_\downarrow\Gamma(x_0,x_1)], \tag{5.25}$$

where

$$\Gamma(x_0,x_1)=x_0\int_0^\infty dx\frac{e^{-x}}{x+e^{-x}\left\{x_0+x_1\sum_{n=1}^{\infty}\frac{x^n}{nn!}\right\}}. \tag{5.26}$$

[13] M. J. Stephen, Phys. Rev. **129**, 997 (1963).
[14] N. D. Mermin and E. Canel, Ann. Phys. (N. Y.) **26**, 247 (1964).

[If we had replaced $v(0,k_\perp^2)$ by its value at $k_\perp=0$, we would have found $\Gamma(x_0,x_1)=1$. This corresponds to taking a δ-function interaction in real space, with amplitude determined by the screening in a magnetic field. We find, however, that $\Gamma(x_0,x_1)$ is considerably less than 1, so that the approximation of a delta function interaction is quite unjustifiable in the present case.]

We have carried out a preliminary computation of (5.25) for the case of InSb.[15] This material has a small effective mass (1/75 of the free electron mass) which makes it easy to reach the situation in which only one Landau level is occupied. However a small m tends to decrease Γ, which is unfavorable to a high T_c. Furthermore, a small m is correlated with a large static dielectric constant (from interband transitions), which, when taken into account, will further depress Γ. Pending a more careful analysis, we have treated the interband contributions to the dielectric constant by simply replacing e^2 by e^2/ϵ_0, using the value $\epsilon_0=17$ for InSb. We also take for ω_0 the value (negative g factor) $\omega_0=+0.33|\omega_c|$.

Using these values we find that (5.25) leads to the highest T_c at a field of 630 G and an electronic density of $6.4\times10^{13}/\text{cm}^3$. The upper bound for T_c in this case is about 14 millidegrees.

One can verify that for the density and field strength quoted above our assumption that only a single Landau n level is occupied is satisfied. Because of the small effective mass the value of ω_c corresponding to 630 G is $\omega_c=6.5°\text{K}$, so that $T_c\ll\omega_c$, as required for our analysis to be valid. The Fermi momenta for the two spin levels are $p_\downarrow^2/2m=0.09\omega_c=0.6°\text{K}$, and $p_\uparrow^2/2m=0.42\omega_c=2.7°\text{K}$. (Thus, although at this low density the electron gas is not degenerate at 14 millidegrees in the absence of a magnetic field, the presence of the field restores degeneracy.)

It is necessary to verify that the term dropped in going from (5.13) to (5.14) does not drastically reduce T_c. To establish this in detail requires a lengthy numerical calculation which we have not yet performed. Considering how far we have already pushed and doctored the Hartree-Fock approximation, the value of such a calculation is somewhat questionable. One can see that the second exponential in (5.13) should have an effect very much less than the first by the following extremely crude argument. We know that

$$1-g_{00}(p)^2/g_{00}(0)^2 \tag{5.27}$$

vanishes with p as $p\to0$, and is bounded above by 1. If we therefore replace it by

$$ap,\quad 0\le p\le 1/a,\quad 1,\quad 1/a\le p, \tag{5.28}$$

we should get some rough idea of the kind of effect it might have. Now (5.28) gives

$$\int_0^{2p_\uparrow}\frac{dp}{p}(1-g_{00}(p)^2/g_{00}(0)^2)=1+\ln(2ap_\uparrow) \tag{5.29}$$

and since the dependence on a is logarithmic, the correction to (5.14) will at worst affect its order of magnitude [rather than affecting the order of magnitude of its order of magnitude, as does the first integral in (5.13)]. We therefore conjecture that the better formula (5.14) will lead to a T_c lower than the one given by (5.13) by at worst an order of magnitude. However this conclusion has by no means been rigorously established.

Finally, we re-emphasize that the burden of this section has been to show that for appropriate materials the transition temperature might occur in the millidegree range, and not to give a precise prediction of the temperature. We feel that we have pushed the Hartree-Fock approximation about as far as it can fruitfully go; it might be instructive to try another approach with some suitably simplified model Hamiltonian that enabled one to treat the screening dynamically, but this is a rather ambitious project.

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We are grateful to Walter Kohn for pointing out that the magnetic-field effects leading to the "helicon instability" should also enhance the possibility for spin-density-wave formation. We have had several helpful discussions with him during the course of this work.

APPENDIX A

We derive below some useful forms for the matrix element

$$\langle\alpha\gamma|v|\beta\delta\rangle=\int d\mathbf{r}d\mathbf{r}'\psi_\alpha^*(\mathbf{r})\psi_\gamma^*(\mathbf{r}')\psi_\beta(\mathbf{r})\psi_\delta(\mathbf{r}')v(\mathbf{r}-\mathbf{r}'), \tag{A1}$$

where the ψ are the normalized Landau states (1.1). Inserting (1.1) and (1.2) into (A1), one can easily reduce the matrix element to

$$\langle\alpha\gamma|v|\beta\delta\rangle=\delta_{p_\alpha+p_\beta-p_\gamma-p_\delta}\delta_{q_\alpha+q_\beta-q_\gamma-q_\delta}\frac{1}{L^3}\sum_{k_x}v(k_x^2+(q_\alpha-q_\beta)^2+(p_\alpha-p_\beta)^2)\int dx\phi_{n_\alpha}\left(x+\frac{q_\alpha-q_\beta}{2m\omega_c}\right)$$
$$\times\phi_{n_\beta}\left(x-\frac{q_\alpha-q_\beta}{2m\omega_c}\right)e^{ik_xx}\int dx'\phi_{n_\gamma}\left(x'+\frac{q_\gamma-q_\delta}{2m\omega_c}\right)\phi_{n_\delta}\left(x'-\frac{q_\gamma-q_\delta}{2m\omega_c}\right)e^{-ik_xx'}e^{-ik_x(q_\alpha+q_\beta-q_\gamma-q_\delta)/2m\omega_c}. \tag{A2}$$

[15] Data were taken from C. Hilsum and A. C. Rose-Innes, *Semiconducting III-V Compounds* (Pergamon Press, New York, 1961), p. 18.

Equation (A2) can be simplified considerably by using the identity[14]

$$\int dx\phi_n\left(x+\frac{k_y}{2m\omega_c}\right)\phi_{n'}\left(x-\frac{k_y}{2m\omega_c}\right)e^{ik_x x}=\langle n|e^{ikx}|n'\rangle e^{i(n'-n)\varphi}, \tag{A3}$$

where

$$k_x=k\cos\varphi, \quad k_y=k\sin\varphi, \tag{A4}$$

and

$$\langle n|e^{ikx}|n'\rangle=\int dx e^{ikx}\phi_n(x)\phi_{n'}(x). \tag{A5}$$

Thus if we define

$$k^2=k_x{}^2+(q_\alpha-q_\beta)^2, \quad \varphi=\tan^{-1}[(q_\alpha-q_\beta)/k_x] \tag{A6}$$

(A2) reduces to

$$\langle\alpha\gamma|v|\beta\delta\rangle=\delta_{p_\alpha+p_\gamma-p_\beta-p_\delta}\delta_{q_\alpha+q_\gamma-q_\beta-q_\delta}\frac{1}{L^3}\sum_{k_x}v(k^2+(p_\alpha-p_\beta)^2)$$
$$\times\langle n_\alpha|e^{ikx}|n_\beta\rangle\langle n_\gamma|e^{-ikx}|n_\delta\rangle e^{-i(n_\alpha+n_\gamma-n_\beta-n_\delta)\varphi}e^{-ik_x(q_\alpha+q_\beta-q_\gamma-q_\delta)/2m\omega_c}. \tag{A7}$$

For the exchange term in the Hartree-Fock equations one needs the summation of (A7) over q_β with q_γ set equal to q_β. Replacing the sums over k_x and q_β by integrals and going to the polar coordinates k and φ, we have

$$\sum_{q_\beta}\langle\alpha\gamma|v|\beta\delta\rangle|_{q_\beta=q_\gamma}=\delta_{p_\alpha+p_\gamma-p_\beta-p_\delta}\delta_{q_\alpha-q_\delta}\delta_{n_\alpha+n_\gamma-n_\beta-n_\delta}\frac{1}{L}\int_0^\infty\frac{kdk}{2\pi}v(k^2+(p_\alpha-p_\beta)^2)\langle n_\alpha|e^{ikx}|n_\beta\rangle\langle n_\gamma|e^{-ikx}|n_\delta\rangle. \tag{A8}$$

APPENDIX B

To test the stability of the normal state against formation of a spin-density wave we evaluate (3.2) for $\delta\varphi$ which are off diagonal in spin

$$\delta\varphi_{ij}=\delta\varphi_{\alpha\beta}\tfrac{1}{2}(\delta_{s_i\uparrow}\delta_{s_j\downarrow}+\delta_{s_i\downarrow}\delta_{s_j\uparrow}). \tag{B1}$$

This reduces the stability condition to

$$\sum_{\alpha\delta}|\delta\varphi_{\alpha\delta}|^2\frac{\epsilon_{\alpha\uparrow}-\epsilon_{\delta\downarrow}}{f_{\delta\downarrow}-f_{\alpha\uparrow}}-\sum_{\alpha\beta,\gamma\delta}\delta\varphi_{\alpha\delta}{}^*\langle\alpha\gamma|v|\beta\delta\rangle\delta\varphi_{\beta\gamma}\gtrless 0. \tag{B2}$$

It is convenient to make the change of variables

$$\begin{aligned} &p_\alpha=p+\tfrac{1}{2}P, \quad p_\delta=p-\tfrac{1}{2}P, \quad p_\beta=p'+\tfrac{1}{2}P', \quad p_\gamma=p'-\tfrac{1}{2}P', \\ &q_\alpha=q+\tfrac{1}{2}Q, \quad q_\delta=q-\tfrac{1}{2}Q, \quad q_\beta=q'+\tfrac{1}{2}Q', \quad q_\gamma=q'-\tfrac{1}{2}Q', \end{aligned} \tag{B3}$$

and to write

$$\begin{aligned} \delta\varphi_{\alpha\delta}&=\int d\xi\delta\varphi_{n_\alpha n_\delta}{}^{PQ\xi}(p)e^{-iq\xi}, \\ \delta\varphi_{\beta\gamma}&=\int d\xi'\delta\varphi_{n_\beta n_\gamma}{}^{P'Q'\xi'}(p')e^{-iq'\xi'}. \end{aligned} \tag{B4}$$

If one insets (B4) in (B2) and uses (A7) one finds that the term in v reduces to

$$-\sum_{PQ}\int d\xi\sum_{pp'}\int_0^\infty\frac{kdk}{2\pi}v(k^2+(p-p')^2)\delta\varphi_{n_\alpha n_\delta}{}^{PQ\xi}(p)^*\delta\varphi_{n_\beta n_\gamma}{}^{PQ\xi}(p')$$
$$\times\int_0^{2\pi}\frac{d\varphi}{2\pi}\exp[ik(\xi\sin\varphi-Q\cos\varphi/m\omega_c)-i(n_\alpha+n_\gamma-n_\beta-n_\delta)\varphi]\langle n_\alpha|e^{ikx}|n_\beta\rangle\langle n_\gamma|e^{-ikx}|n_\delta\rangle. \tag{B5}$$

The integral over φ gives a phase factor times a Bessel function

$$J_{n_\alpha+n_\gamma-n_\beta-n_\delta}([\xi^2+(Q/m\omega_c)^2]^{1/2})\left(\frac{\xi+iQ/m\omega_c}{[\xi^2+(Q/m\omega_c)^2]^{1/2}}\right)^{n_\beta+n_\delta-n_\alpha-n_\gamma}.$$

If we combine (B5) with the expression for the first term in (B2) the resulting stability condition is

$$0\leqslant\delta^2F=L^2\int d\xi\sum_{PQ}\Bigg\{\sum_{n_\alpha n_\delta}\int\frac{dp}{2\pi}|\,\delta\varphi_{n_\alpha n_\delta}{}^{PQ\xi}(p)\,|^2[\epsilon_{n_\alpha\uparrow}(p+\tfrac{1}{2}P)-\epsilon_{n_\delta\downarrow}(p-\tfrac{1}{2}P)]/[f_{n_\delta\downarrow}(p-\tfrac{1}{2}P)-f_{n_\alpha\uparrow}(p+\tfrac{1}{2}P)]$$

$$-\sum_{n_\alpha n_\gamma n_\beta n_\delta}\int\frac{dpdp'}{(2\pi)^2}\delta\varphi_{n_\alpha n_\delta}{}^{PQ\xi}(p)^*\delta\varphi_{n_\beta n_\gamma}{}^{PQ\xi}(p')\int_0^\infty\frac{kdk}{2\pi}v(k^2+(p-p')^2)\langle n_\alpha|e^{ikx}|n_\beta\rangle\langle n_\gamma|e^{-ikx}|n_\delta\rangle$$

$$\times J_{n_\alpha+n_\gamma-n_\beta-n_\delta}([\xi^2+(Q/m\omega_c)^2]^{1/2})\left(\frac{\xi+iQ/m\omega_c}{[\xi^2+(Q/m\omega_c)^2]^{1/2}}\right)^{n_\beta+n_\delta-n_\alpha-n_\gamma}\Bigg\}. \quad \text{(B6)}$$

For stability the quantity in curly brackets must be positive for all values of P, Q, and ξ.

We have carried things to this rather gruesome stage for general $\delta\varphi$ because we can now make the point that the instability first appears at $Q=\xi=0$, i.e., for $\delta\varphi_{\alpha\delta}$ which is diagonal in q_α and q_δ, and independent of q_δ. Such a $\delta\varphi$ corresponds to the onset of a spin-density wave directed *along* the magnetic field. To see this consider $\delta\varphi$ of the form

$$\delta\varphi_{n_\alpha n_\delta}{}^{PQ\xi}(p)=\frac{f_{n_\delta\downarrow}(p-\frac{1}{2}P)-f_{n_\alpha\uparrow}(p+\frac{1}{2}P)}{\epsilon_{n_\alpha\uparrow}(p+\frac{1}{2}P)-\epsilon_{n_\delta\downarrow}(p-\frac{1}{2}P)}\Delta_{n_\alpha n_\delta}{}^{PQ\xi}(p).$$

If $P=p_{n\uparrow}+p_{n'\downarrow}$ then at $T=0$ the p integral of $\delta\varphi_{nn'}$ will diverge logarithmically at $p=\frac{1}{2}(p_{n\uparrow}-p_{n'\downarrow})$. For arbitrarily weak coupling δ^2F will be negative because the exchange term in (B6) is dominated by the term with $n_\alpha=n_\beta=n$, $n_\gamma=n_\delta=n'$, which is the product of two logarithmically divergent integrals. At nonzero but very low temperatures, the contribution to the first term of (B6) with $n_\alpha=n$, $n_\delta=n'$, will diverge like $\ln\beta$, whereas the dominant contribution to the second will go like $(\ln\beta)^2$. (This can be shown in detail by using arguments almost identical to those of Sec. III.) To get an instability at the highest possible temperature we must maximize the coefficient of this term, which is proportional to $J_0([\xi^2+(Q/m\omega_c)^2]^{1/2})$; therefore the best choices of ξ and Q are $\xi=Q=0$, as asserted above.

With this choice the remaining Bessel functions reduce to $\delta_{n_\alpha+n_\gamma-n_\beta-n_\delta}$, and the stability condition becomes

$$0\leqslant\sum_n\int\frac{dp}{2\pi}|\,\delta\varphi_{n,n+m}(p)\,|^2\frac{\epsilon_{n\uparrow}(p+\frac{1}{2}P)-\epsilon_{n+m\downarrow}(p-\frac{1}{2}P)}{f_{n+m\downarrow}(p-\frac{1}{2}P)-f_{n\uparrow}(p+\frac{1}{2}P)}$$

$$-\sum_{nn'}\int\frac{dpdp'}{(2\pi)^2}\delta\varphi_{n,n+m}(p)^*\delta\varphi_{n',n'+m}(p')\int_0^\infty\frac{kdk}{2\pi}v(k^2+(p-p')^2)\langle n|e^{ikx}|n'\rangle\langle n'+m|e^{-ikx}|n+m\rangle, \quad \text{(B7)}$$

for each value of m. Now the argument of Sec. II when applied to this more general case indicates (at least in the weak-coupling limit) that the transition temperature will be highest when $v_{n\uparrow}+v_{n+m\downarrow}$ is as small as possible. Thus (B7) will first be violated when $m=0$, and $P=p_{n\uparrow}+p_{n\downarrow}$, where n is the quantum number of the highest occupied Landau cylinder.