Superconductivity in One and Two Dimensions*†

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The effect of thermodynamic fluctuations on off-diagonal long-range order (ODLRO) is discussed. The two-particle Yang correlation function is calculated using the Ginzburg-Landau theory allowing such fluctuations. The behavior of the correlation function at large separation is examined and it is found to be consistent with Yang's criterion for ODLRO only in three dimensions. In one and two dimensions Yang's criterion is not satisfied.

1. INTRODUCTION

THERE has been considerable interest recently in the possibility of superconductivity in one- and two-dimensional systems. Little[1] has proposed a model for a one-dimensional superconducting organic molecule with a high-transition temperature. Ginzburg[2,3] has suggested that superconductivity could occur in the neighborhood of a surface while the bulk material was in normal state. We shall not enter into a detailed discussion of these models or into an evaluation of the recent experimental evidence concerning two-dimensional superconductivity.[4–7] Instead we shall concentrate on the theoretical question of whether a one- or two-dimensional system of fermions interacting with attractive forces can exhibit off-diagonal long-range order (ODLRO) at low temperature. There is a general theorem that any one-dimensional system with short-range forces will not undergo a phase transition at any finite temperature.[8] The usual criterion for superconductivity is that the BCS equation have a nontrivial solution.[1,2] However, for attractive interaction the BCS equations have a solution in any number of dimensions. Further it has been shown by Bogoliubov, Zubarev, and Tserkovnikov[9] that a system with the model BCS Hamiltonian is soluble to within terms of order $(1/\Omega)$, where Ω is the volume of the system, and that a phase transition occurs. Their proof is independent of the dimensionality of the system. The interparticle forces in the model BCS Hamiltonian are long range and so this result does not contradict the theorem quoted above. The absence of a phase transition in one dimension is caused by the fact that the free energy can always be lowered by breaking up a very long section of one phase into subsections with alternate sections of the other phase. Thus, to treat this problem correctly, it is necessary to start with a Hamiltonian with short-range attractive forces and to allow for thermodynamic fluctuation.

We shall show that if we start with the assumption that below a critical temperature there is an ordered phase with a Ginzburg-Landau order parameter, then because of thermodynamic fluctuations the two-particle correlation function for one- and two-dimensional systems will not be consistent with Yang's general criterion for ODLRO.[10] This result is in agreement with the general theorem on one-dimensional systems quoted above. The general theorem does not apply to two-dimensional systems but our result strongly suggests that there will be no ODLRO in two-dimensional systems either.

Ferrell[11] has also considered the possibility of superconductivity in a one-dimensional system and he found no ODLRO. His treatment is based on the presence of low-lying compressional modes in a one-dimensional system and the relationship of his approach to the general theorem on one-dimensional systems is not clear. In the present approach the thermodynamic fluctuations in the phase of the order parameter are seen unambiguously to be responsible for the absence of ODLRO.

2. THE TWO-PARTICLE CORRELATION FUNCTION

We start by assuming that below a critical temperature there will be ODLRO present in the system and that the system can be characterized by a Ginzburg-Landau order parameter $\Psi(\mathbf{x})$. We further assume the free energy of the system will be given by $F[\Psi(\mathbf{x})]$,

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where[12]

$$F[\Psi(\mathbf{x})]=a\int d\mathbf{x}|\Psi(\mathbf{x})|^2+(b/2)\int d\mathbf{x}|\Psi(\mathbf{x})|^4+c\int d\mathbf{x}|\nabla\Psi(\mathbf{x})|^2. \quad (1)$$

The critical temperature T_c is determined by the condition that $a(T_c)=0$ and for $T<T_c$, $a(T)<0$. The minimum value of $F[\Psi(\mathbf{x})]$, F_0 for $T<T_c$ is given by choosing $\Psi(\mathbf{x})=\Psi_0 e^{i\varphi_0}$ and $\Psi_0^2=-a/b$ and φ_0 is an arbitrary constant. Yang has shown that the criterion for the existence of ODLRO is that the off-diagonal elements of the two-particle correlation function are finite in the limit as the separation goes to infinity, i.e., $G(\mathbf{x}_1,\mathbf{x}_2)\to$ a finite constant as $|\mathbf{x}_1-\mathbf{x}_2|\to\infty$ where

$$G(\mathbf{x}_1,\mathbf{x}_2)=\langle\psi_\uparrow(\mathbf{x}_1)\psi_\downarrow(\mathbf{x}_2)\psi_\downarrow^\dagger(\mathbf{x}_2)\psi_\uparrow^\dagger(\mathbf{x}_2)\rangle \quad (2)$$

$$=\mathrm{const}\langle\Psi(\mathbf{x}_1)\Psi^*(\mathbf{x}_2)\rangle\,. \quad (3)$$

The terms omitted in (3) are negligible when $|\mathbf{x}_1-\mathbf{x}_2|$ is large and the constant is the square of the factor which relates the Gorkov F function to the Ginzburg-Landau parameter $\Psi(\mathbf{x})$.[13] $\langle\cdots\rangle$ denotes the thermodynamic average which can be written

$$G(\mathbf{x}_1,\mathbf{x}_2)=\mathrm{const}\int\mathfrak{D}\Psi(\mathbf{x})\Psi(\mathbf{x}_1)\Psi^*(\mathbf{x}_2)\times\exp\{-\beta(F[\Psi(\mathbf{x})]-F_0)\}\,, \quad (4)$$

where $\mathfrak{D}\Psi(\mathbf{x})$ denotes the functional integration of $\Psi(\mathbf{x})$ over all function $\Psi(\mathbf{x})$ and the weighting factor is determined by the free energy given in (1) with $\beta=1/k_BT$. We are interested in examining $G(\mathbf{x}_1,\mathbf{x}_2)$ in the limit as $|\mathbf{x}_1-\mathbf{x}_2|\to\infty$.

The functional integral in (4) cannot be evaluated exactly. Instead we shall evaluate it by expanding $\Psi(\mathbf{x})$ about Ψ_0 and keep only quadratic terms. Writing $\Psi(\mathbf{x})$ in terms of its modulus and phase $\Psi(\mathbf{x})=(\Psi_0+\tilde{\Psi}(\mathbf{x}))\times\exp\{i\varphi(\mathbf{x})\}$ we obtain for the change in free energy

$$\Delta F[\Psi(\mathbf{x})]=F[\Psi(\mathbf{x})]-F_0=-2a\int\tilde{\Psi}^2(\mathbf{x})d\mathbf{x}+c\int(\nabla\tilde{\Psi}(\mathbf{x}))^2d\mathbf{x}+c\Psi_0^2\int(\nabla\varphi(\mathbf{x}))^2d\mathbf{x}, \quad (5)$$

where the terms omitted are of higher order in grad φ, $\tilde{\Psi}$. Substituting into (4) gives

$$G(\mathbf{x}_1,\mathbf{x}_2)=\mathrm{const}\int\mathfrak{D}\tilde{\Psi}(\mathbf{x})\mathfrak{D}\varphi(\mathbf{x})\times(\Psi_0+\tilde{\Psi}(\mathbf{x}_1))(\Psi_0+\tilde{\Psi}(\mathbf{x}_2))\times\exp\{i(\varphi(\mathbf{x}_1)-\varphi(\mathbf{x}_2))-\beta\Delta F[\tilde{\Psi}(\mathbf{x}),\varphi(\mathbf{x})]\}\,. \quad (6)$$

The integrations over the phase and modulus can be done separately.

To perform the phase integration we expand $\varphi(\mathbf{x})$ in a Fourier series

$$\varphi(\mathbf{x})=\sum_{\mathbf{k}}\Phi_{\mathbf{k}}\exp\{i\mathbf{k}\cdot\mathbf{x}\}$$

with $\Phi_{\mathbf{k}}=\Phi_{-\mathbf{k}}^*$. We write $\Phi_{\mathbf{k}}=\varphi_{\mathbf{k}}+i\chi_{\mathbf{k}}$ and integrate separately over $\varphi_{\mathbf{k}}$ and $\chi_{\mathbf{k}}$. The phase integration becomes

$$I_\varphi\sim\int\mathfrak{D}\varphi_{\mathbf{k}}\mathfrak{D}\chi_{\mathbf{k}}\times\exp\{i\sum_{\mathbf{k}}[\varphi_{\mathbf{k}}(\cos(\mathbf{k}\cdot\mathbf{x}_1)-\cos(\mathbf{k}\cdot\mathbf{x}_2))+\chi_{\mathbf{k}}(\sin(\mathbf{k}\cdot\mathbf{x}_1)-\sin(\mathbf{k}\cdot\mathbf{x}_2))]-\beta c\Psi_0^2\Omega\sum_{\mathbf{k}}\mathbf{k}^2(\varphi_{\mathbf{k}}^2+\chi_{\mathbf{k}}^2)\}\,, \quad (7)$$

where Ω is the volume of the system. Since we are only interested in the dependence of the integral on $\mathbf{x}_1$, and $\mathbf{x}_2$, we can evaluate it immediately by completing the square on $\varphi_{\mathbf{k}}$ and $\chi_{\mathbf{k}}$ and we obtain

$$I_\varphi\sim\exp\left\{-\frac{1}{\alpha\Omega}\sum_{\mathbf{k}}^{k<Q}\frac{(\cos(\mathbf{k}\cdot\mathbf{x}_1)-\cos(\mathbf{k}\cdot\mathbf{x}_2))^2+(\sin(\mathbf{k}\cdot\mathbf{x}_1)-\sin(\mathbf{k}\cdot\mathbf{x}_2))^2}{k^2}\right\}, \quad (8)$$

where $\alpha=4\beta c\Psi_0^2$. The cutoff Q is introduced to eliminate the short-wavelength fluctuations which are not included in the Ginzburg-Landau theory. The integrand in (8) reduces at once to give

$$I_\varphi\sim\exp\left\{-\frac{1}{\alpha\Omega}\sum_{\mathbf{k}}^{k<Q}\frac{2\{1-\cos[\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)]\}}{k^2}\right\}. \quad (9)$$

This can be evaluated in a straightforward manner in the limit $QX\gg1$, where $X=|\mathbf{x}_1-\mathbf{x}_2|$, and then we get

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[13] L. P. Gorkov, Zh. Eksperim. i Teor. Fiz. **36**, 1918 (1959) [English transl.: Soviet Phys.—JETP **9**, 1364 (1959)].

$$
\begin{aligned}
I_\varphi(X) &\sim \exp\{-X/\alpha\} \quad \text{1 dimension} \\
&\sim \exp\{-[\ln(2QX)]/\pi\alpha\} \quad \text{2 dimensions} \\
&\sim \exp\{-(Q-\pi/2X)/\pi^2\alpha\} \quad \text{3 dimensions.}
\end{aligned} \tag{10}
$$

The integration over the modulus of $\Psi(\mathbf{x})$ can also be done in a straightforward manner by expanding in a Fourier series and we obtain

$$
I_\Psi \sim \Psi_0^2 + \frac{1}{2\beta\Omega}\sum_{\mathbf{k}} \frac{\exp\{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)\}}{-2a+ck^2}. \tag{11}
$$

This integral can be done immediately and combining the result with that for the phase integration we get

$$
\begin{aligned}
G(X) &\sim [\Psi_0^2 + \exp\{-qX\}/4\beta cq]\exp\{-X/\alpha\} \quad \text{1 dimension} \\
&\sim [\Psi_0^2 + \exp\{-qX\}/4\beta c(2\pi qX)^{1/2}] \\
&\quad \times \exp\{-\ln(2QX)/\pi\alpha\} \quad \text{2 dimensions} \\
&\sim [\Psi_0^2 + \exp\{-qX\}/8\beta c\pi X] \\
&\quad \times \exp\{-(Q-\pi/2X)/\pi^2\alpha\}, \quad \text{3 dimensions}
\end{aligned} \tag{12}
$$

where $q=(-2a/c)^{1/2}$ is the reciprocal of the usual correlation length.

Examining the behavior of $G(X)$ as $X\to\infty$, we see at once that in three dimensions $G(X)\to$ a finite constant as $X\to\infty$ while in one and two dimensions the behavior of the off-diagonal elements of the two-particle correlation function is inconsistent with Yang's criterion for ODLRO.

For a charged system the foregoing considerations must be supplemented by a study of the effects of charge fluctuations of the free energy. Preliminary work indicates that these effects do not alter our main conclusions.

3. CONCLUSIONS

We have shown that if we start with the assumption of ODLRO and a Ginzburg-Landau order parameter, and if we calculate the off-diagonal elements of the two-particle correlation function allowing for the thermodynamic fluctuations in the phase of the order parameter, then only in three dimensions is Yang's criterion of ODLRO satisfied. In one and two dimensions the behavior of the off-diagonal elements of the two-particle correlation function is inconsistent with Yang's criterion for ODLRO. This strongly suggests that there can be no ODLRO in one and two dimensions.

It is of some interest to examine the temperature dependence of the function $\alpha(T)=4\beta c\Psi_0^2$ which enters the theory. Note that whereas for a one-dimensional system α is a characteristic length, for a two-dimensional system α is dimensionless, and for a three-dimensional system α is a reciprocal length. As a function of temperature $\alpha(T)\to 0$ as $T\to T_c$ and $\alpha(T)\to\infty$ as $T\to 0°$K. We also note that for a two-dimensional system $\alpha(T)$ is very large at low temperatures ($\sim\beta\epsilon_F$) and that $G(X)$ will fall off very slowly. Since the results were derived using the Ginzburg-Landau theory they are strictly true only in the neighborhood of the critical temperature, however an examination of a more general formulation of the theory of superconductivity in terms of functional integrals suggests that this restriction is unnecessary.

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